Closure homomorphisms and interior mappings

by

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This paper is a continuation of my paper *Closure Algebras* [1]) cited hereafter as CA.

A *closure algebra* $\boldsymbol{A}$ is a Boolean σ-algebra with a closure operation satisfying the well known axioms of Kuratowski:

$$\text{I.}\quad \overline{A+B}=\bar{A}+\bar{B}, \qquad \text{II.}\quad A\subset\bar{A},$$

$$\text{III.}\quad \overline{(\bar{A})}=\bar{A}, \qquad \text{IV.}\quad \bar{0}=0.$$

Every closure algebra is thus an "abstract algebra" [2]) with the following fundamental operations:

(a) Boolean enumerable addition $\sum_{n=1}^{\infty}A_n$,

(b) Boolean complementation A',

(c) closure operation $\bar{A}$.

By a *closure homomorphism* we shall understand a homomorphism (in the sense of the Modern Algebra) [3]) with respect to the fundamental operations (a), (b), (c), *i.e.* a transformation h (of a closure algebra $\boldsymbol{A}$ into another closure algebra $\boldsymbol{B}$) preserving all the operations (a), (b), (c):

(a′) $h(\sum_{n=1}^{\infty}A_n)=\sum_{n=1}^{\infty}h(A_n)$;

(b′) $h(A')=h(A)'$;

(c′) $h(\bar{A})=\overline{h(A)}$.

The conditions (a′) and (b′) mean that h is a Boolean σ-homomorphism. Thus a closure homomorphism is a Boolean σ-homomorphism satisfying the condition (c′).

[1]) See [9]. Terminology and notation in this paper are the same as in CA. In particular, the letters $\boldsymbol{A}, \boldsymbol{B}$ denote closure algebras (or Boolean algebras), the letters $A, B, \ldots$ — their elements.

[2]) See [12], p. 212.

[3]) Cf. [3] and [12], p. 212.

A *closure isomorphism* is a one-to-one closure homomorphism, *i. e.* a Boolean σ-isomorphism h satisfying the condition (c′).

Closure isomorphisms have been examined in CA under the name *homeomorphisms* [4]) since they are a generalization of the notion of homeomorphism from the Topology of Point Spaces.

Another class of homomorphisms examined in CA is the class of *continuous homomorphisms* [4]), *i. e.* Boolean σ-homomorphisms h of a closure algebra $\boldsymbol{A}$ into another $\boldsymbol{B}$, satisfying the condition

(c″) $\overline{h(A)}\subset h(\bar{A})$ for each $A\in\boldsymbol{A}$.

Continuous homomorphisms are a natural generalization of the notion of a continuous point mapping from the Topology of Point Spaces. Clearly every closure homomorphism is continuous. The converse statement is not true.

The subject of the first part of this paper is the study of closure homomorphisms. It will be shown that closure homomorphisms are a generalization of the notion of interior mapping from the Topology of Point Spaces (see (ii)).

The second part contains some representation theorems for closure algebras with an enumerable basis. The representation problem for such closure algebras is not completely solved. It can be reduced to the question whether every Hausdorff space with an enumerable basis is an interior image of a separable metric space.

Incidentally I shall show that the dimension of a closure subalgebra $\boldsymbol{S}$ of a C-algebra [5]) $\boldsymbol{A}$ can be greater than the dimension of $\boldsymbol{A}$.

§ 1. The relation between closure homomorphisms and interior mappings

By a *topological space* we shall mean a set $\mathfrak{X}$ with a closure operation defined for all $A\subset\mathfrak{X}$ such that I-IV hold (it is not assumed that $\bar{A}=A$ if A is a one-point set). The closure algebra of all subsets of $\mathfrak{X}$ will be denoted by $\mathfrak{S}(\mathfrak{X})$. The closure algebra of all Borel subsets of $\mathfrak{X}$ will be denoted by $\mathfrak{B}(\mathfrak{X})$.

The letters $\mathfrak{X}$ and $\mathcal{Y}$ will always denote topological spaces.

A mapping φ of $\mathfrak{X}$ into $\mathcal{Y}$ is said to be *open* if $\varphi(G)$ is open in $\mathcal{Y}$ for every open set $G\subset\mathfrak{X}$ (in particular, $\varphi(\mathfrak{X})$ is an open subset of $\mathcal{Y}$). The mapping φ is said to be *interior* if it is open and continuous.

(i) *A mapping φ of $\mathfrak{X}$ into $\mathcal{Y}$ is open if and only if $\varphi^{-1}(\bar{Y})\subset\overline{\varphi^{-1}(Y)}$ for every set $Y\subset\mathcal{Y}$.*

[4]) See CA, p. 175.

[5]) A closure algebra $\boldsymbol{A}$ is called a *C-algebra*, if there is a sequence (R_n) of open elements of $\boldsymbol{A}$ such that each open $G\in\boldsymbol{A}$ is the sum of all R_n such that $\bar{R}_n\subset G$. See CA, p. 182. C-algebras are a generalization of separable metric space.

The proof is based on the following true statements [6]):

(α) if $X\subset\mathcal{X}$, $Y\subset\mathcal{Y}$, then $\varphi(X)\cdot Y=\varphi(X\cdot\varphi^{-1}(Y))$;

(β) if U is open, then $U\overline{Z}\subset\overline{UZ}$.

Suppose φ is open. Let $Y\subset\mathcal{Y}$ and $G=\mathcal{X}-\overline{\varphi^{-1}(Y)}$. The set $\varphi(G)$ being open, we have by (α) and (β)

$$\varphi(\varphi^{-1}(\overline{Y})\cdot G)=\varphi(G)\cdot\overline{Y}\subset\overline{\varphi(G)\cdot Y}$$
$$=\overline{\varphi(\mathcal{X}-\overline{\varphi^{-1}(Y)})\cdot Y}\subset\overline{\varphi(\mathcal{X}-\varphi^{-1}(Y))\cdot Y}=\overline{0}=0.$$

Hence $\varphi^{-1}(\overline{Y})\cdot G=0$, *i. e.* $\varphi^{-1}(\overline{Y})\subset\overline{\varphi^{-1}(Y)}$.

Suppose now that $\varphi^{-1}(\overline{Y})\subset\overline{\varphi^{-1}(Y)}$ for each $Y\subset\mathcal{Y}$. Let G be any open subset of $\mathcal{X}$. Put $Y=\mathcal{Y}-\varphi(G)$. We have by (α) and (β)

$$\varphi(G)\cdot\overline{Y}=\varphi(G\cdot\varphi^{-1}(\overline{Y}))\subset\varphi(G\cdot\overline{\varphi^{-1}(Y)})\subset\varphi(\overline{G\cdot\varphi^{-1}(Y)})$$
$$=\varphi(\overline{G\cdot\varphi^{-1}(\mathcal{Y}-\varphi(G))})=\varphi(0)=0.$$

Hence $\overline{\mathcal{Y}-\varphi(G)}=\overline{Y}\subset\mathcal{Y}-\varphi(G)$, *i. e.* $\varphi(G)$ is open.

It follows immediately from (i) that

(ii) *A mapping φ of $\mathcal{X}$ into $\mathcal{Y}$ is interior if and only if $\varphi^{-1}(\overline{Y})=\overline{\varphi^{-1}(Y)}$ for every $Y\subset\mathcal{Y}$, i. e. if the Boolean σ-homomorphism h of $\mathfrak{S}(\mathcal{Y})$ into $\mathfrak{S}(\mathcal{X})$:*

(*) $$h(Y)=\varphi^{-1}(Y)\quad \textit{for all}\quad Y\in\mathfrak{S}(\mathcal{Y})$$

is a closure homomorphism.

(iii) *Suppose $\overline{\overline{\mathcal{Y}}}$ is less than the first aleph inaccessible in the strict sense* [7]). *Then $\mathcal{Y}$ is an interior image of $\mathcal{X}$ (i. e. $\mathcal{Y}=\varphi(\mathcal{X})$ where φ is interior) if and only if $\mathfrak{S}(\mathcal{Y})$ is homeomorphic to a closure subalgebra of $\mathfrak{S}(\mathcal{X})$.*

In fact, if $\mathcal{Y}=\varphi(\mathcal{X})$ and φ is interior, then h defined by (*) is a homeomorphism of $\mathfrak{S}(\mathcal{Y})$ onto a closure subalgebra of $\mathfrak{S}(\mathcal{X})$. Conversely, if h is an homeomorphism of $\mathfrak{S}(\mathcal{Y})$ onto a closure subalgebra of $\mathfrak{S}(\mathcal{X})$, then there is a mapping [8]) φ such that (*) holds. Then φ is an interior mapping and $\mathcal{Y}=\varphi(\mathcal{X})$.

Kolmogoroff, Každan and Anderson [9]) have given examples of compact metric spaces $\mathcal{X}_0$ and $\mathcal{Y}_0$ such that $\mathcal{Y}_0$ is an interior image

[6]) See [7], p. 17 and p. 25.

[7]) An aleph $\aleph_\lambda$ is said to be *inaccessible* in the strict sense if 1° $\lambda>0$; 2° $\mathfrak{m}_t<\aleph_\lambda$ and $\overline{\overline{T}}<\aleph_\lambda$ imply $\sum_{t\in T}\mathfrak{m}_t<\aleph_\lambda$; 3° if $\mathfrak{m}<\aleph_\lambda$, then $2^{\mathfrak{m}}<\aleph_\lambda$.

[8]) See [10], p. 12.

[9]) [6], [5], [2].

of $\mathcal{X}_0$ and $\dim\mathcal{Y}_0>\dim\mathcal{X}_0$. By (iii), $\mathfrak{S}(\mathcal{Y}_0)$ is homeomorphic to a closure subalgebra $\boldsymbol{S}$ of $\mathfrak{S}(\mathcal{X}_0)$. Clearly $\boldsymbol{S}$ and $\mathfrak{S}(\mathcal{X}_0)$ are C-algebras [5]) and $\dim\boldsymbol{S}>\dim(\mathfrak{S}\mathcal{X}_0)$. This example shows that the dimension of a C-subalgebra of a C-algebra can be greater than the dimension of the entire C-algebra.

(iv) *Let $\mathcal{Y}$ fulfil one of the following conditions:*

(1) *$\mathcal{Y}$ is a T_0-space* [10]) *satisfying the second axiom of countability (i. e. $\mathcal{Y}$ has an enumerable basis)* [11]);

(2) *$\mathcal{Y}$ is a T_1-space* [10]) *satisfying the first axiom of countability.*

Then the following statements are equivalent for any mapping φ of $\mathcal{X}$ into $\mathcal{Y}$:

(α) *φ is interior;*

(β) $\varphi^{-1}(\overline{Y})=\overline{\varphi^{-1}(Y)}$ *for each* $Y\in\mathfrak{B}(\mathcal{Y})$;

(γ) *φ is continuous and* $\varphi^{-1}(\overline{Y})=\overline{\varphi^{-1}(Y)}$ *for every at most enumerable set $Y\subset\mathcal{Y}$.*

Consequently, *φ is interior if and only if the homomorphism h defined by* (*) *and restricted to Borel subsets of $\mathcal{Y}$ is a closure homomorphism of $\mathfrak{B}(\mathcal{Y})$ into $\mathfrak{S}(\mathcal{X})$* (or: *into* $\mathfrak{B}(\mathcal{X})$).

The implication (α)→(β) follows from (ii). The implication (β)→(γ) is trivial since, under our assumptions about $\mathcal{Y}$, each enumerable set $Y\subset\mathcal{Y}$ is a Borel set.

Suppose that the condition (γ) is satisfied. In order to prove (α) it suffices to show that $\varphi^{-1}(\overline{Y})\subset\overline{\varphi^{-1}(Y)}$ for every $Y\subset\mathcal{Y}$ (see (i)). Let $x\in\varphi^{-1}(\overline{Y})$. Then $\varphi(x)\in\overline{Y}$. Each of the conditions (1) and (2) implies that there is an at most enumerable set $Y_0\subset Y$ such that $\varphi(x)\in\overline{Y}_0$. Hence $x_0\in\varphi^{-1}(\overline{Y}_0)=\overline{\varphi^{-1}(Y_0)}\subset\overline{\varphi^{-1}(Y)}$, q. e. d.

Notice that in the case (2) (in particular for metric spaces) the condition (γ) may be weakened. It suffices to require that the equation $\varphi^{-1}(\overline{Y})=\overline{\varphi^{-1}(Y)}$ should hold if Y is the set of all terms of a convergent sequence. If X satisfies one of the conditions (1) or (2), we can omit in (γ) the hypothesis that φ is continuous. In fact, if $x\in\overline{\varphi^{-1}(Y)}$, then there is an at most enumerable set $X_0\subset\varphi^{-1}(Y)$ such that $x\in\overline{X}_0$. We have $x\in\overline{\varphi^{-1}(\varphi(X_0))}=\varphi^{-1}(\overline{\varphi(X_0)})\subset\varphi^{-1}(\overline{Y})$, which proves the continuity of φ.

As an application we shall prove the following generalization of the well known theorem of Eilenberg [12]):

[10]) See [1], p. 58-59.

[11]) An enumerable basis is a sequence $\{R_n\}$ of open subsets such that each open subset G is the sum of all R_n such that $R_n\subset G$. More generally, an enumerable basis of a closure algebra A is a sequence $\{R_n\}$ of open elements of A such that each open $G\in A$ is the sum of all R_n such that $R_n\subset G$.

[12]) [4], p. 174. Eilenberg's assumptions that $\mathcal{X}$ is compact and $\mathcal{Y}=\varphi(X)$ are superfluous.

(v) *Let φ be a mapping of a metric space $\mathfrak{X}$ into another metric space $\mathcal{Y}$. The following conditions are equivalent:*

(α) *φ is interior;*

(β) *$\varphi^{-1}(\lim y_n)=\operatorname{Lim}\varphi^{-1}(y_n)$ for every convergent sequence $y_n \in \mathcal{Y}$;*

(γ) *$\varphi^{-1}(\lim y_n)=\operatorname{Ls}\varphi^{-1}(y_n)$ for every convergent sequence $y_n \in \mathcal{Y}$* [13].

If $\varphi(\mathfrak{X})=\mathcal{Y}$, the condition (β) means that $\{\varphi^{-1}(y)\}_{y\in\mathcal{Y}}$ is a continuous decomposition of $\mathfrak{X}$.

Suppose that φ is interior. Let $y_0=\lim y_n$. We have $(y_0)=\prod_{n=1}^{\infty}\overline{\sum_{k=0}^{\infty}(y_{n+k})}$. Hence

$$\varphi^{-1}(y_0)=\prod_{n=1}^{\infty}\varphi^{-1}\Big(\overline{\sum_{k=0}^{\infty}(y_{n+k})}\Big)=\prod_{n=1}^{\infty}\overline{\varphi^{-1}\Big(\sum_{k=0}^{\infty}(y_{n+k})\Big)}=\prod_{n=1}^{\infty}\overline{\sum_{k=0}^{\infty}\varphi^{-1}(y_{n+k})},$$

that is [14]

$$\varphi^{-1}(y_0)=\operatorname{Ls}\varphi^{-1}(y_n).$$

Since the last equation holds also for every subsequence $\{y_{k_n}\}$ of $\{y_n\}$, we obtain [15]

$$\varphi^{-1}(y_0)=\operatorname{Lim}\varphi^{-1}(y_n),$$

which proves the implication (α)→(β).

The implication (β)→(γ) is trivial.

Suppose now that (γ) is true. We shall show that φ is interior. By (iv) and by the remarks below the proof of (iv) it is sufficient to prove that

$$\varphi^{-1}\big(\overline{(y_1,y_2,\ldots)}\big)=\overline{\varphi^{-1}\big((y_1,y_2,\ldots)\big)}$$

for every convergent sequence $y_n \in Y$. Let $y_0=\lim y_n$. Clearly (γ) implies that $\varphi^{-1}(y)$ is closed for every $y\in\mathcal{Y}$. We have [14]

$$\varphi^{-1}\big(\overline{(y_1,y_2,\ldots)}\big)=\varphi^{-1}\big((y_0,y_1,y_2,\ldots)\big)=\varphi^{-1}(y_0)+\sum_{n=1}^{\infty}\varphi^{-1}(y_n)$$
$$=\operatorname{Ls}\varphi^{-1}(y_n)+\sum_{n=1}^{\infty}\overline{\varphi^{-1}(y_n)}=\overline{\sum_{n=1}^{\infty}\varphi^{-1}(y_n)}=\overline{\varphi^{-1}\big((y_1,y_2,\ldots)\big)},$$

which completes the proof of (v).

Let $\mathbf{A}$ be a closure algebra and let $E\in\mathbf{A}$. The symbol $E\mathbf{A}$ will denote the closure algebra [16] formed of all elements $A\subset E$ with the following closure operation: $\bar{A}_E=E\bar{A}$ for $A\in E\mathbf{A}$. For instance, if $\mathbf{A}=\mathfrak{S}(\mathfrak{X})$, then $E\mathfrak{S}(\mathfrak{X})=\mathfrak{S}(E)$.

(vi) *The continuous homomorphism h of $\mathbf{A}$ onto $E\mathbf{A}$ defined by the equation*

$$h(A)=EA \quad \textit{for} \quad A\in\mathbf{A}$$

is a closure homomorphism if and only if E is open in $\mathbf{A}$.

If E is open, then [17] $\overline{h(A)}=E\cdot\overline{EA}=E\cdot\overline{E\bar{A}}=E\bar{A}=h(\bar{A})$, thus h is a closure homomorphism. Conversely, if h is closure homomorphism, then $E\cdot\overline{E'}=h(\overline{E'})=\overline{h(E')}=\bar{0}=0$, *i. e.* E is open.

The following theorem explains the structure of closure homomorphisms.

(vii) *Let $\mathbf{A}$ be a closure algebra with an enumerable basis, and let h be a closure homomorphism of $\mathbf{A}$ into a closure algebra $\mathbf{B}$. Then there is an open element $G\in\mathbf{A}$ such that*

(α) *$h(A)=0$ if and only if $AG=0$;*

(β) *the homomorphism h restricted to elements $A\in G\mathbf{A}$ is a homeomorphism of $G\mathbf{A}$ onto a closure subalgebra of $\mathbf{B}$.*

If $\{F_\xi\}_{\xi<\eta}$ is an transfinite strictly increasing sequence of closed elements in $\mathbf{A}$ such that $h(F_\xi)=0$, then $\eta<\Omega$ by CA 3.3, and $h(\sum_{\xi<\eta}F_\xi)=0$ since the sequence $\{F_\xi\}$ is enumerable. Consequently $h(\overline{\sum_{\xi<\eta}F_\xi})=\overline{h(\sum_{\xi<\eta}F_\xi)}=0$.

Using Zorn's lemma we infer that there is a closed element $F_0\in\mathbf{A}$ such that $h(F_0)=0$, and the conditions $h(F)=0$, $F=\bar{F}$ imply $F\subset F_0$. Since $h(A)=0$ implies $h(\bar{A})=\overline{h(A)}=0$, we infer that $h(A)=0$ if and only if $A\subset F_0$. Put $G=F_0'$ (*i. e.* the complement of F_0). Clearly (α) is fulfilled. If $A\in G\mathbf{A}$, then $h(\bar{A}_G)=h(\bar{A}G)=h(\bar{A})h(G)=\overline{h(A)}$ which proves (β).

If $\mathbf{I}$ is a σ-ideal of a Boolean algebra $\mathbf{A}$ and $A\in\mathbf{A}$, then $[A]$ denotes the element (coset) of $\mathbf{A}/\mathbf{I}$ determined by A [18].

If $\mathbf{A}$ is a closure algebra with an enumerable basis, and $\mathbf{I}$ is a σ-ideal of $\mathbf{A}$, then the closure operation in $\mathbf{A}$ induces a closure operation in $\mathbf{A}/\mathbf{I}$ such that $\mathbf{A}/\mathbf{I}$ forms a closure algebra [19].

Under the above assumptions:

(viii) *The natural homomorphism h of $\mathbf{A}$ onto $\mathbf{A}/\mathbf{I}$:*

$$h(A)=[A] \quad \textit{for} \quad A\in\mathbf{A}$$

[13]) The superior limit $\operatorname{Ls}X_n$ of a sequence of sets $X_n\in\mathfrak{X}$ is the set of all points $x\in\mathfrak{X}$ such that $x=\lim x_n$, $x_n\in X_{k_n}$, $k_1<k_2<\ldots$ The inferior limit $\operatorname{Li}X_n$ of a sequence of sets $X_n\subset\mathfrak{X}$ is the set of all points $x\in\mathfrak{X}$ such that $x=\lim x_n$, $x_n\in X_n$ for $n=1,2,\ldots$ If $\operatorname{Ls}X_n=\operatorname{Li}X_n=X$, we write $X=\operatorname{Lim}X_n$ and we say that the sequence $\{X_n\}$ converges to X. See C. Kuratowski [7], p. 245 and p. 243.

[14]) See C. Kuratowski [7], p. 243, IV, 8.

[15]) See C. Kuratowski [7], p. 244, V, 1.

[16]) See CA, p. 171.

[17]) See the statement (β) on p. 14, which holds also for every closure algebra.

[18]) See CA, p. 168.

[19]) See CA, p. 180.

is a closure homomorphism if and only if the ideal $\boldsymbol{I}$ *is principal and generated by a closed element* $F_0 \in \boldsymbol{A}$ *(i. e.* $B \in I$ *if and only if* $B \subset F_0$*).*

This remark follows immediately from (vii).

§ 2. The representation problem for closure algebras with an enumerable basis

(ix) *Suppose that the closure algebra* $\boldsymbol{A}$ *is of the form* $\boldsymbol{A}=\boldsymbol{B}/\boldsymbol{I}$ *where* $\boldsymbol{B}$ *is a Boolean* σ*-algebra and* $\boldsymbol{I}$ *is a* σ*-ideal of* $\boldsymbol{B}$*. If* $\boldsymbol{A}$ *has an enumerable basis, it is possible to define a closure operation in* $\boldsymbol{B}$ *in such a way that*

(α) $\boldsymbol{B}$ *is a closure algebra with an enumerable basis;*

(β) *the closure algebra* $\boldsymbol{A}$ *is identical with the closure algebra which we obtain by the division of the closure algebra* $\boldsymbol{B}$ *by the ideal* $\boldsymbol{I}$ *using the method described in* CA 9 (p. 180).

The proof of this theorem is analogous to the proof of CA 14.1. It is even simpler in the above case.

If a closure algebra is a σ-field of sets, it is called a *closure field.*

(x) *Every closure field is weakly homeomorphic* [20]) *to a* T_0*-space. Every closure field with an enumerable basis is weakly homeomorphic to a* T_0*-space with an enumerable basis.*

The proof is similar to that of CA 13.1.

Since every Boolean σ-algebra is isomorphic to a quotient algebra [21]) $\boldsymbol{X}/\boldsymbol{I}$ where $\boldsymbol{X}$ is a σ-field of sets and $\boldsymbol{I}$ is a σ-ideal of $\boldsymbol{X}$, we find from (ix) that

(xi) *Every closure algebra* $\boldsymbol{A}$ *with an enumerable basis is isomorphic to a quotient closure algebra* $\boldsymbol{X}/\boldsymbol{I}$ *where* $\boldsymbol{X}$ *is a closure field with an enumerable basis, and* $\boldsymbol{I}$ *is a* σ*-ideal of* $\boldsymbol{X}$*.*

Combining (ix) and (x) we find that

(xii) *Every closure algebra* $\boldsymbol{A}$ *with an enumerable basis is weakly homeomorphic to a closure quotient algebra* $\mathfrak{S}(\mathcal{X})/\boldsymbol{I}$ *where* $\mathcal{X}$ *is a* T_0*-space with an enumerable basis (i. e.* $\mathfrak{B}(\boldsymbol{A})$ *is homeomorphic to* $\mathfrak{B}(\mathcal{X})/\boldsymbol{I}_0$*,* $\boldsymbol{I}_0=\boldsymbol{I}\cdot\mathfrak{B}(\mathcal{X})$*).*

Theorems (ix), (x), (xi), (xii) are analogous to Theorems CA 14.1, 13.1, 14.2 and 15.1 proved for C-algebras. The question arrises whether

[20]) If A is a closure algebra, then $\mathfrak{B}(A)$ denotes the closure algebra of all Borel elements of A. Two closure algebras A and B are *weakly homeomorphic* if $\mathfrak{B}(A)$ and $\mathfrak{B}(B)$ are homeomorphic. See CA, p. 171 and p. 176.

[21]) See [8] and [11], p. 256.

a theorem analogous to CA 15.2 is true for closure algebras with an enumerable basis. I hope that the following non-proved statement is true:

(H_0) For every closure algebra $\boldsymbol{A}$ with an enumerable basis there is a closure subalgebra $\boldsymbol{S}$ of the closure algebra $\mathfrak{S}(\mathcal{H})$ of all subsets of the Hilbert cube $\mathcal{H}$, and a σ-ideal $\boldsymbol{I}$ of $\boldsymbol{S}$ such that $\boldsymbol{A}$ is weakly homeomorphic to $\boldsymbol{S}/\boldsymbol{I}$.

The difficulty of the above problem lies in the fact that we know no characterization of closure subalgebras of closure algebras $\mathfrak{S}(\mathcal{X})$ where $\mathcal{X}$ is a separable metric space. By CA 4.3 each such subalgebra has an enumerable basis and therefore is weakly homeomorphic to a T_0-space. A simple example given in CA (p. 184, footnote [38])) shows that a complete four-element closure subalgebra of $\mathfrak{S}(\mathcal{R})$, where $\mathcal{R}=$ the set of all real numbers, need not be weakly homeomorphic to a T_1-space. It seems probable that

(H) If $\mathcal{X}$ is a T_0-space with an enumerable basis, then $\mathfrak{S}(\mathcal{X})$ is homeomorphic to a complete closure subalgebra of $\mathfrak{S}(\mathcal{Y})$ where $\mathcal{Y}$ is a separable metric space.

On account of (iii), hypothesis (H) may be formulated in the following equivalent form:

(H′) Each T_0-space with an enumerable basis is an interior image of a separable metric space [22]).

We can only prove that

(xiii) *Each* T_0*-space* $\mathcal{X}$ *with an enumerable basis is an interior image of a totally disconnected* [23]) *Hausdorff space with an enumerable basis.*

Let $\mathcal{R}$ denote the set of all real numbers. Consider the Cartesian product $\mathcal{X}\times\mathcal{R}$ with the usual topology. $\mathcal{X}\times\mathcal{R}$ has an enumerable basis. Since $\overline{\overline{\mathcal{X}}}\leqslant 2^{\aleph_0}$, we can associate with every $x \in \mathcal{X}$ a set R_x of irrational numbers such that

(1) R_x is dense in $\mathcal{R}$ for every $x \in \mathcal{X}$;

(2) if $x_1 \neq x_2$ $(x_1, x_2 \in \mathcal{X})$, then $R_{x_1}\cdot R_{x_2}=0$.

The space $\mathcal{Y}=\sum\limits_{x\in X}(x)\times R_x \subset \mathcal{X}\times\mathcal{R}$ (with the topology induced by $\mathcal{X}\times\mathcal{R}$) has an enumerable basis. $\mathcal{Y}$ is a totally disconnected Hausdorff space. In fact, let $(x_1,r_1)\neq(x_2,r_2)$ be two points in $\mathcal{Y}$. If $x_1=x_2$, then $r_1\neq r_2$. If $x_1\neq x_2$, then $r_1\neq r_2$ also since $r_1\in R_{x_1}$, $r_2 \in R_{x_2}$ and $R_{x_1}\cdot R_{x_2}=0$. Suppose, for instance, that $r_1<r_2$. Let r_0 be a rational number such that $r_1<r_0<r_2$. The sets $U_1=\mathcal{Y}\cdot \underset{(r,x)}{E}(r<r_0)$, $U_2=\mathcal{Y}\cdot\underset{(x,r)}{E}(r>r_0)$ are disjoint neighbourhoods of (x_1,r_1) and (x_2,r_2) respectively, and $\mathcal{Y}=U_1+U_2$.

[22]) See Colloquium Mathematicum 2 (1951), p. 171, **P** 78.

[23]) A space $\mathcal{X}$ is *totally disconnected* if for every pair $x_1, x_2 \in \mathcal{X}$, $x_1\neq x_2$, there are open sets U_1 and U_2 such that $x_1 \in U_1$, $x_2\in U_2$, $\mathcal{X}=U_1+U_2$, $U_1\cdot U_2=0$.

To complete the proof we shall show that the projection π of $\mathcal{Y}$ onto $\mathcal{X}$ is open (clearly π is continuous). It suffices to prove that $\pi(\mathcal{Y}\cdot(G\times V))=G$ for every open $G\subset\mathcal{X}$ and for every open interval $V\neq 0$, $V\subset\mathcal{R}$.

Obviously $\pi(\mathcal{Y}\cdot(G\times V))\subset G$. If $x\in G$, then, by (1), there is an $r\in R_x\cdot V$. Consequently $x\in\pi(\mathcal{Y}\cdot(G\times V))$, which yields $G\subset\pi(\mathcal{Y}\cdot(G\times V))$, q. e. d.

The problem whether every Hausdorff space with an enumerable basis is an interior image of a separable metric space is unsolved [22]).

Notice that (H) implies easily (H_0).

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On existential theorems in non-classical functional calculi [1])

by

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Let $\mathcal{S}_\chi$ be the Heyting propositional calculus, and let $\mathcal{S}_\chi^*$ be the Heyting functional calculus. The individual variables of the system $\mathcal{S}_\chi^*$ will be denoted by $x_1, x_2, \ldots$, the quantifiers — by $\sum\limits_{x_p}$ and $\prod\limits_{x_p}$. The formulas from $\mathcal{S}_\chi^*$ will be denoted by the letters α, β. If α is a formula from $\mathcal{S}_\chi^*$, then $\alpha\binom{x_q}{x_p}$ denotes the formula obtained from α by replacing each free occurrence of x_p by x_q (each bound occurrence of x_q should be replaced earlier by x_l which does not appear in α, $l\neq q$).

Gödel [2]) formulated (without proof) the following theorem:

(χ_0) *Let σ, τ be two formulas from the Heyting propositional calculus $\mathcal{S}_\chi$. If the disjunction $\sigma+\tau$ is a theorem of $\mathcal{S}_\chi$, then either σ or τ is a theorem of $\mathcal{S}_\chi$.*

Theorem (χ_0) was later proved by McKinsey and Tarski [2] by an algebraical method. Another algebraical proof was given by Rieger [3]).

The purpose of this paper is to prove the following theorem (χ) which is an extension of (χ_0) over the Heyting functional calculus $\mathcal{S}_\chi^*$. The second part of Theorem (χ) shows that the Heyting functional calculus is the well formalization of Brouwer's ideas concerning existential theorems.

(χ) *If the formula $\alpha+\beta$ is provable in $\mathcal{S}_\chi^*$, then either α or β are provable in $\mathcal{S}_\chi^*$. If the formula $\sum\limits_{x_p}\alpha$ is provable in $\mathcal{S}_\chi^*$, then there is a positive integer q such that the formula $\alpha\binom{x_q}{x_p}$ is provable in $\mathcal{S}_\chi^*$.*

Clearly if the sequence $x_{i_1},\ldots,x_{i_n}$ contains all the free variables which appear in α, the integer q can be chosen among the numbers $i_1,\ldots,i_n$. If α contains no free variable, then q is an arbitrary integer, *e. g.* $q=p$.

[1]) Presented at the Seminar on Foundations of Mathematics in the Mathematical Institute of the Polish Academy of Sciences in November 1952.

[2]) See K. Gödel [1]. See also G. Gentzen [1].

[3]) See L. Rieger [1], p. 29.