[5] P. Alexandroff, *Untersuchungen über Gestalt und Lage abgeschlossener Mengen beliebiger Dimension*, Annals of Math. (2), 30 (1928), S. 101-187.

[6] E. Čech, *Théorie générale de l'Homologie*, Fund. Math. 19 (1932), p. 149-183.

[7] П. Александров, а) *О размерности бикомпактных пространств*, Доклады Академии Наук СССР 26 (1940), стр. 627-630.

— б) *On the dimension of normal spaces*, Proceed. of the Royal Society of London, A., 189 (1947), p. 11-39.

[8] P. Alexandroff, *Dimensionstheorie*, Math. Annalen 106 (1932), S. 161-238, или *О размерности замкнутых множеств*, Успехи Матем. Наук 4, вып. 6 (1950), стр. 17-88.

[9] S. Kaplan, *Homology properties of arbitrary subsets of euclidean spaces*, Transactions of the Amer. Math. Soc. 62 (1947), p. 248-271.

[10] Г. С. Чогошвили, а) *Закон двойственности для ретрактов*, Доклады Академии Наук СССР 51, № 2 (1946), стр. 90-97.

— б) *О гомологических аппроксимациях и законах двойственности множеств*, Мат. Сборн. 28 (1951), стр. 89-118 (и цитированные там более ранние работы — 1945 и последующих лет).

[11] C. H. Dowker, *Mapping theorems for non compact spaces*, Amer. Journ. of Math. 69 (1947), p. 200-242.

[12] E. Hemmingsen, *Some theorems in dimension theory for normal Hausdorff spaces*, Duke Math. Journ. 13 (1946), p. 495-504.

[13] C. Kuratowski, *Sur un théorème fondamental concernant le nerf d'un système d'ensembles*, Fund. Math. 20 (1933), p. 191-196 (см. также C. Kuratowski, *Topologie I*, Warszawa 1952 (Edition troisième), p. 202 и след.).

[14] Е. Ф. Мищенко, а) *К гомологической теории незамкнутых множеств*, Мат. Сборн. 29 (1951), стр. 587-592.

— б) *О некоторых вопросах комбинаторной топологии незамкнутых множеств*, Мат. Сборн. 32 (1953), стр. 219-224.

Reçu par la Rédaction le 5. 9. 1953

Boolean representation through propositional calculus [1])

by

L. Henkin (Los Angeles)

By the *boolean representation theorem* we mean the proposition, first proved by Marshall Stone [1], that every boolean algebra is isomorphic to a boolean algebra of sets. By the *Gödel-Malcev (functional) theorem* we shall mean the metamathematical result stating that every formally consistent set of sentences of a first-order functional calculus is simultaneously satisfiable [2]). Each of these two theorems has been proved with the aid of the axiom of choice [3]), but neither one seems to be as strong as that axiom.

Although the boolean representation theorem and the Gödel-Malcev theorem appear to deal with very different subjects, the two have recently been shown to be quite closely related. Rasiowa and Sikorski [7] have given a proof of the Gödel-Malcev theorem using some of the same techniques which Stone employed in establishing the boolean representation theorem. By a slight change in their argument, it can be turned into a proof that the boolean representation theorem *implies* the Gödel-Malcev theorem. On the other hand, it has also been shown [4]) that the boolean representation theorem follows from the Gödel-Malcev theorem. In short, the two theorems are equivalent. Furthermore, although it appears necessary to employ the axiom of choice in order to establish

[1]) This work was supported in part by a grant from the National Science Foundation.

[2]) Gödel [2] and Malcev [3].

[3]) The Gödel-Malcev theorem is often stated for a first-order functional calculus containing only a countable number of symbols. This is the form in which it was first established by Gödel, and in this form it is not necessary to use the axiom of choice. It was Malcev, in the case of formulas of the propositional calculus, who first considered this theorem in connection with formal systems containing a non-denumerable number of symbols. The first statement or proof of the Gödel-Malcev theorem for first-order functional calculi with a non-denumerable number of symbols seems to occur in Henkin [4]. Subsequently, and independently, the theorem appeared and was proved in Robinson [6].

[4]) Henkin [5]. This paper contains some discussion of the relative strength of the axiom of choice and the Gödel-Malcev theorem.

either of these theorems, the equivalence of the two theorems can be established without recourse to the axiom of choice.

In this paper we shall show that the boolean representation theorem follows, without the axiom of choice, from the Gödel-Malcev *propositional* theorem, *i. e.* from the metamathematical result that every formally consistent set of formulas of a *propositional calculus* is simultaneously satisfiable. On the surface, this appears to be a weaker statement than the Gödel-Malcev functional theorem. That, conversely, the Gödel-Malcev propositional theorem follows from the boolean representation theorem, can easily be demonstrated by the methods of Rasiowa and Sikorski, as they themselves have indicated. In the last sections we consider the question of representing a boolean algebra as a boolean algebra of sets using fewer points than there are elements in the given algebra.

1. A boolean algebra $\boldsymbol{a}$ is a system $\langle A, -, +, \circ, \leqslant, 0, I\rangle$, where A is a set, 0 and I are elements of A, $-$ is a unary operation on A, $+$ and $\circ$ are binary operations on A, and $\leqslant$ is a binary relation on A. We assume that the reader is familiar with a set of axioms for boolean algebras, as well as with the principal laws which relate the basic notions [5]). A boolean algebra of sets is a boolean algebra in which I is a set, A is a family of subsets of I, 0 is the empty set, $-$ is the operation of complementation (relative to I), $+$ and $\circ$ are respectively the operations of union intersection, and $\leqslant$ is the relation of inclusion.

A propositional calculus is a formal system containing among its primitive symbols a set of propositional symbols, parentheses, and the further symbols $\sim$, $\vee$, $\wedge$, $\supset$, and $\equiv$, of which the first is a unary connective and the other four are binary connectives. We assume that the reader is familiar with a set of formal axioms for such calculi, as well as with the principal formal theorems which can be derived [6]). We shall assume that the calculus is formalized with a single formal rule of inference, *modus ponens*, which permits the inference of a formula β from the formulas α and $\alpha\supset\beta$.

Let Γ be any set of formulas of the propositional calculus. The class of formulas *formally derivable* from Γ is the smallest class of formulas containing Γ and the formal axioms, and closed under modus ponens. To indicate that a formula α is derivable from Γ, we write $\Gamma\vdash\alpha$. Thus, a set Γ of formulas is *formally consistent* if and only if there is no formula α such that $\Gamma\vdash\alpha$ and also $\Gamma\vdash\sim\alpha$.

[5]) See, for example, Stone [1].

[6]) See, for example, Hilbert and Bernays [8].

Now consider an arbitrary boolean algebra $\boldsymbol{a}=\langle A, -, +, \circ, \leqslant, 0, I\rangle$. Our object is to find a boolean algebra of sets, $\boldsymbol{a}_1=\langle A_1, -_1, +_1, \circ_1, \leqslant_1, 0_1, I_1\rangle$, which is isomorphic to the given $\boldsymbol{a}$. To this end, we consider a propositional calculus which contains a propositional symbol p_x corresponding to each element x of A (distinct symbols corresponding to distinct elements). And we let Γ be the set containing all of the following formulas:

(i) $\sim p_x\equiv p_{-x}$ for all $x\in A$,

(ii) $(p_x\vee p_y)\equiv p_{x+y}$ for all $x, y\in A$,

(iii) $(p_x\wedge p_y)\equiv p_{x\circ y}$ for all $x, y\in A$,

(iv) $(p_x\supset p_y)$ for all $x, y\in A$ such that $x\leqslant y$,

(v) $\sim p_0$,

(vi) p_I.

LEMMA. *Let a be any element of A other than 0. Then the set Γ_a, consisting of the above mentioned set Γ together with the additional formula p_a, is formally consistent.*

Proof. We define a function ψ which assigns to each formula of the propositional calculus an element of A, as follows:

(ψi) $\psi(p_x)=x$ for each $x\in A$,

(ψii) $\psi(\sim\alpha)=-\psi(\alpha)$ for each formula α,

(ψiii) $\psi(\alpha\vee\beta)=\psi(\alpha)+\psi(\beta)$ for each α,β,

(ψiv) $\psi(\alpha\wedge\beta)=\psi(\alpha)\circ\psi(\beta)$ for each α,β,

(ψv) $\psi(\alpha\supset\beta)=-\psi(\alpha)+\psi(\beta)$ for each α,β,

(ψvi) $\psi(\alpha\equiv\beta)=(\psi(\alpha)\circ\psi(\beta))+(-\psi(\alpha)\circ-\psi(\beta))$ for each α,β.

From the elementary laws of boolean algebra, together with (ψv), we see that $\psi(\alpha\equiv\beta)=I$ if and only if $\psi(\alpha)=\psi(\beta)$, while from (ψiv) we infer that $\psi(\alpha\supset\beta)=I$ if and only if $\psi(\alpha)\leqslant\psi(\beta)$. From these facts, together with formulas (i)-(vi), we easily conclude that $\psi(\alpha)=I$ for every formula α in Γ. Since $\psi(p_a)=a$, $a\leqslant I$, and $I\leqslant I$, we see that $a\leqslant\psi(\alpha)$ for every $\alpha\in\Gamma_a$.

Now we also have $a\leqslant\psi(\alpha)$ for each formal axiom α of the propositional calculus, for in fact $\psi(\alpha)=I$ for each such α as we easily show by the elementary laws of boolean algebras. Furthermore, modus ponens preserves the property of formulas to be mapped by ψ into an element of A which includes a. That is, if $a\leqslant\psi(\alpha)$ and $a\leqslant\psi(\alpha\supset\beta)$, then also $a\leqslant\psi(\beta)$. (This follows from the boolean law that $z\leqslant x$ and $z\leqslant(-x+y)$ imply $z\leqslant y$). Hence we see that $a\leqslant\psi(\alpha)$ for every formula α *derivable* from Γ_a.

Suppose, now, that Γ_a were not formally consistent. Then there would be a formula α such that both $\Gamma_a\vdash\alpha$ and $\Gamma_a\vdash\sim\alpha$. Hence $a\leqslant\psi(\alpha)$

and $a \leqslant \psi(\sim a) = -\psi(a)$. Using the boolean law that $y \leqslant x$ and $y \leqslant -x$ imply $y = 0$, we conclude that $a = 0$. But this contradicts the assumption of the lemma. Hence Γ_a must be formally consistent, and the lemma is proved.

Having demonstrated that Γ_a is formally consistent (for each $a \neq 0$), we can apply the Gödel-Malcev (propositional) theorem and conclude that Γ_a is simultaneously satisfiable. This means that there exists a function φ_a, assigning to each formula of the propositional calculus a truth-value, T or F, such that

(φi) $\varphi_a(\sim\alpha)$ is T or F according as $\varphi_a(\alpha)$ is F or T, for each formula α,
(φii) $\varphi_a(\alpha \vee \beta) = T$ if and only if $\varphi_a(\alpha) = \varphi_a(\beta) = T$,
(φiii) $\varphi_a(\alpha \wedge \beta) = F$ if and only if $\varphi_a(\alpha) = \varphi_a(\beta) = F$,
(φiv) $\varphi_a(\alpha \supset \beta) = F$ if and only if $\varphi_a(\alpha) = T$ and $\varphi_a(\beta) = F$,
(φv) $\varphi_a(\alpha \equiv \beta) = T$ if and only if $\varphi_a(\alpha) = \varphi_a(\beta)$,
(φvi) $\varphi_a(\alpha) = T$ for every $\alpha \in \Gamma_a$.

In general, of course, there will be many such functions φ_a for each $a \neq 0$, and without the axiom of choice we do not know of any way to select a unique φ_a corresponding to each a.

Let I_1 be the set of all these functions φ_a (for all $a \in A$, $a \neq 0$). We associate to each element x of A a subset $S(x)$ of I_1 as follows: For every $\varphi_a \in I_1$,

(*) $$\varphi_a \in S(x) \quad \text{if and only if} \quad \varphi_a(p_x) = T.$$

Let A_1 be the family of all sets $S(x)$, $x \in A$. Let $-_1$, $+_1$, $\circ_1$, $\leqslant_1$, 0_1 be the standard boolean set-theoretic concepts: complementation, union, intersection, inclusion, null set. We assert that $\boldsymbol{a}_1 = \langle A_1, -_1, +_1, \circ_1, \leqslant_1, 0_1, I_1 \rangle$ is a boolean algebra of sets, and that the mapping S of A onto A_1 is an isomorphism between $\boldsymbol{a}$ and $\boldsymbol{a}_1$.

To show that $S(-x) = -_1 S(x)$, we proceed as follows. By (*), we have $\varphi_a \in S(-x)$ if and only if $\varphi_a(p_{-x}) = T$. From (φv), (φvi), and (i) we conclude that $\varphi_a(p_{-x}) = \varphi_a(\sim p_x)$, so that from (φi) and (*) it follows that $\varphi_a \in S(-x)$ if and only if $\varphi_a \notin S(x)$. Hence, by definition of $-_1$, $S(-x) = -_1 S(x)$.

In an entirely similar manner we can show

$$S(x+y) = S(x) +_1 S(y), \qquad S(x \circ y) = S(x) \circ_1 S(y),$$
$$x \leqslant y \quad \text{if and only if} \quad S(x) \leqslant_1 S(y),$$
$$S(0) = 0_1, \qquad S(I) = I_1.$$

In short, S is a homomorphism, and hence $\boldsymbol{a}_1$ is a boolean algebra of sets. It remains only to show that S is a one-one mapping, and for

this it suffices to show that $S(x) \neq 0_1$ for every $x \neq 0$, $x \in A$. But this is clear, for if $x \neq 0$ then by (*), (φvi), and the fact that $p_x \in \Gamma_x$, we see that $\varphi_x \in S(x)$.

This concludes the proof of the boolean representation theorem from the Gödel-Malcev (propositional) theorem.

2. Since the proof of the Gödel-Malcev (propositional) theorem from the boolean representation theorem follows closely the line of argument of the Rasiowa-Sikorski theorem, we give it here only in brief outline.

We suppose, then, that we are given a propositional calculus and a formally consistent set Γ of its formulas. Our object is to show that Γ is simultaneously satisfiable.

To this end we consider the binary relation $\simeq$ defined on the formulas of the calculus, such that $\alpha \simeq \beta$ holds if and only if $\Gamma \vdash \alpha \equiv \beta$. This is easily shown to be an equivalence relation, so that the formulas of the calculus are partitioned into disjoint equivalence classes $E(\alpha)$, where $E(\alpha)$ is the set of all formulas β such that $\beta \simeq \alpha$.

Let A be the set of all these equivalence classes. It can be shown that there is an operation $-$ defined on A by the rule $-E(\alpha) = E(\sim\alpha)$. Similarly we define $E(\alpha) + E(\beta) = E(\alpha \vee \beta)$, $E(\alpha) \circ E(\beta) = E(\alpha \wedge \beta)$, $E(\alpha) \leqslant E(\beta)$ if and only if $\Gamma \vdash \alpha > \beta$, $0 = E(\alpha \wedge \sim\alpha)$, $I = E(\alpha \vee \sim\alpha)$. Then from the elementary theorems of propositional calculus we can show that $\boldsymbol{a} = \langle A, -, +, \circ, \leqslant, 0, I \rangle$ satisfies the axioms for boolean algebras.

Applying the boolean representation theorem, we conclude that there exists a boolean algebra of sets, $\boldsymbol{a}_1 = \langle A_1, -_1, +_1, \circ_1, \leqslant_1, 0_1, I_1 \rangle$, and a mapping S of A onto A_1, such that S is an isomorphism between $\boldsymbol{a}$ and $\boldsymbol{a}_1$. Choose any point j in I_1. Let φ be the function which assigns to each formula of the propositional calculus a truth-value, T or F, according to the rule: $\varphi(\alpha) = T$ if and only if $j \in S(E(\alpha))$. This assignment φ can be shown to satisfy simultaneously all formulas of Γ.

3. In part 1 of this paper we started with an arbitrary boolean algebra $\boldsymbol{a}$ and associated with it a certain formally consistent set Γ of formulas of a propositional calculus. Let us call this set $\Gamma^*(\boldsymbol{a})$, to show that it is dependent on and determined by $\boldsymbol{a}$. In part 2 we started with a formally consistent set Γ of formulas, and associated with it a boolean algebra $\boldsymbol{a}$, which we will now call $\boldsymbol{a}^*(\Gamma)$. These two constructions are related by the following theorems, whose proof we leave to the interested reader:

Any boolean algebra $\boldsymbol{a}$ is isomorphic to $\boldsymbol{a}^*(\Gamma^*(\boldsymbol{a}))$.

If Γ is any formally consistent set of formulas of a propositional calculus, and α any formula, then $\Gamma \vdash \alpha$ if and only if $\Gamma^*(\boldsymbol{a}^*(\Gamma)) \vdash p_{E(\alpha)}$.

4. In part 1 we have shown how, starting with a boolean algebra $\boldsymbol{a}$ we can find an isomorphic algebra of sets $\boldsymbol{a}_1$. The unit element I_1 of the latter has as its elements certain functions φ_a, and it may well happen that the cardinality of I_1 is greater than that of the set A of elements of the given algebra $\boldsymbol{a}$ [7]).

If we permit ourselves the use of the axiom of choice (as we shall freely do in the remainder of this paper), we can easily modify the construction of part 1 to ensure that in the representing algebra of sets, $\boldsymbol{a}_1$, the unit element I_1 has a cardinality not exceeding that of the given A. Namely, for each $a \in A$, $a \neq 0$, we select a *single* function φ_a which simultaneously satisfies all formulas of Γ_a. We let I_1 be the set of non-zero elements of A. $S(x)$ is then defined to be the subset of I_1 consisting of those elements a such that $\varphi_a(p_x) = T$. Clearly, under this modified construction, the element I_1 will have exactly as many elements as there are elements in A different from 0. Hence the cardinal of I_1 will be less than or equal to the cardinal of A according as A is finite or infinite.

Now there are certainly boolean algebras $\boldsymbol{a}$ for which no representing algebra $\boldsymbol{a}_1$ can have a unit element I_1 whose cardinality is less than that of the given A — for example, this is true whenever A is denumerably infinite. On the other hand, for other algebras $\boldsymbol{a}$ such a representation is possible — for example, if $\boldsymbol{a}$ is itself a boolean algebra of sets in which A consists of *all* subsets of I. We may, therefore, seek some condition on the algebraic structure of a boolean algebra $\boldsymbol{a}$ which will determine whether or not $\boldsymbol{a}$ admits a representation in which I_1 has smaller cardinality than A.

As the theorem below shows, such a condition can be described in terms of the concept of *finite intersection property*. A set u of elements of a boolean algebra is said to have the finite intersection property if, whenever $x_1, x_2, \ldots, x_n$ are a finite number of elements of u, $x_1 \circ x_2 \circ \ldots \circ x_n \neq 0$.

THEOREM. *Let $\boldsymbol{a} = \langle A, -, +, \circ, \leqslant, 0, I\rangle$ be a given boolean algebra. A necessary and sufficient condition for $\boldsymbol{a}$ to be isomorphic to some boolean algebra of sets, $\boldsymbol{a}_1$, whose unit element has cardinality less than that of A, is the existence of a class U satisfying the following conditions:*

[7]) The close relation between our method of representing a boolean algebra and that of Stone will be appreciated if it is observed that the set of functions φ_a and the set of maximal ideals of $\boldsymbol{a}$ are in one-one correspondence. In fact, for each function φ_a the set of all elements x of A such that $\varphi_a(p_x) = F$, is a maximal ideal not containing a. Now it is well known that there are denumerable boolean algebras with a non-denumerable number of prime ideals; hence for such a denumerable algebra there will be a non-denumerable number of functions φ_a associated with the corresponding propositional calculus.

(Ui) *Each element u of U is a subset of A which has the finite intersection property.*

(Uii) *Every element x of A, other than 0, is in at least one element u of U.*

(Uiii) *The cardinal of U is less than the cardinal of A.*

Proof. Assume first the existence of a class U satisfying (Ui)-(Uiii). As in part 1, we construct a propositional calculus which contains a propositional symbol p_x corresponding to each element x of A; and we form the same set Γ of formulas which is described there.

Now for each $u \in U$, let Γ_u be the set of formulas obtained by adding to Γ all of the formulas p_a such that $a \in u$. We assert that Γ_u is formally consistent.

For suppose (for some $u \in U$) that Γ_u is inconsistent, so that $\Gamma_u \vdash \alpha$ and $\Gamma_u \vdash \sim\alpha$ for some formula α. From the definition of *formally derivable* it follows that $\Gamma_u' \vdash \alpha$ and $\Gamma_u' \vdash \sim\alpha$, where Γ_u' is obtained from Γ by adding some *finite number* of formulas $p_{a_1}, \ldots, p_{a_n}$ ($a_1, \ldots, a_n$ being elements of u). Now we construct the same function ψ described in the proof of the lemma of part 1, and following the argument presented there we show that $a_1 \circ \ldots \circ a_n \leqslant \psi(\beta)$ for each β formally derivable from Γ_u'. Hence, in particular, $a_1 \circ \ldots \circ a_n \leqslant \psi(\alpha)$ and $a_1 \ldots a_n \leqslant \psi(\sim\alpha) = -\psi(\alpha)$, so that we must have $a_1 \circ \ldots \circ a_n = 0$. This, however, contradicts (Ui), since $a_1, \ldots, a_n$ are all elements of u. This contradiction establishes the formal consistency of Γ_u.

We then apply the Gödel-Malcev (propositional) theorem to infer the existence of a function φ_u, assigning a truth-value, T or F, to each formula of the propositional calculus, which satisfies simultaneously all formulas of Γ_u; and we use the axiom of choice to select one such function φ_u corresponding to each $u \in U$. Finally, we take I_1 to be U, and for each $x \in A$ we let $S(x)$ be the set of those elements u of I_1 such that $\varphi_u(x) = T$. As in part 1, we let A_1 be the family of all these sets $S(x)$, we let $-_1, +_1, \circ_1, \leqslant_1, 0_1$ be the standard boolean set-theoretic concepts, and we then show that S is a homomorphism of $\boldsymbol{a}$ onto the system $\boldsymbol{a}_1 = \langle A_1, -_1, +_1, \circ_1, \leqslant_1, 0_1, I_1\rangle$, which is thereby shown to be a boolean algebra of sets. Finally, S is one-one because if $x \in A$, $x \neq 0$, then by (Uii) there is a $u \in U$ such that $x \in u$, and this u is in $S(x)$ since p_x is in Γ_u. The fact that I_1 has cardinality less than that of A is simply the condition (Uiii). This completes the proof of the sufficiency of the condition described in our theorem.

To show the necessity of our condition, we suppose that we have a boolean algebra of sets $\boldsymbol{a}_1$ whose unit element I_1 has cardinality less than that of A, and that S is an isomorphism between $\boldsymbol{a}$ and $\boldsymbol{a}_1$. Our object is to show the existence of a class U satisfying conditions (Ui)-(Uiii).

To this end, associate with each element j of I_1 the set u_j consisting of all elements x of A such that $j \in S(x)$, and let U be the class of all such sets u_j, $j \in I_1$. We assert that this class U satisfies the required conditions.

First let $x_1, \ldots, x_n$ be any finite number of elements from one of the sets u_j. Now $S(x_1) \circ_1 \ldots \circ_1 S(x_n) \neq 0_1$ since $j \in S(x_1) \circ_1 \ldots \circ_1 S(x_n)$. But S is an isomorphism, so $x_1 \ldots x_n \neq 0$. Thus condition (Ui) is satisfied.

Next consider any $x \in A$, $x \neq 0$. Since S is an isomorphism, $S(x) \neq 0_1$, and so there exists a j in $S(x)$. But then $x \in u_j$, so condition (Uii) is satisfied.

Finally, (Uiii) is an immediate consequence of our assumption on the cardinality of I_1, since the cardinality of U clearly does not exceed that of I_1.

This completes the proof of our theorem.

5. We do not know whether the theorem of part 4 can be proven from the Gödel-Malcev (propositional) theorem without using the axiom of choice. However, without the axiom of choice we can show by Stone's method that the possibility of representing a given boolean algebra $\boldsymbol{a}$ by a boolean algebra of sets $\boldsymbol{a}_1$ whose unit element has smaller cardinality than that of A, is equivalent to the existence of a non-empty class V satisfying the following conditions:

(Vi) Every element v of V is a maximal ideal of $\boldsymbol{a}$.

(Vii) The intersection of all the elements v of V is empty.

(Viii) The cardinality of V is less than that of A.

Using the axiom of choice, one can give a direct proof that the existence of a class U satisfying (Ui)-(Uiii) is equivalent to the existence of a class V satisfying (Vi)-(Viii).

References

[1] M. H. Stone, *The theory of representations for boolean algebras*, Transactions of the American Mathematical Society 40 (1936), p. 37-111.

[2] K. Gödel, *Die Vollständigkeit der Axiome der logischen Funktionenkalküle*, Monatshefte für Mathematik und Physik 37 (1930), p. 349-360.

[3] A. Malcev, *Untersuchungen aus dem Gebiete der mathematischen Logik*, Recueil Mathématique, n. s. 1 (1936), p. 323-336.

[4] L. Henkin, *The completeness of the first-order functional calculus*, Journal of Symbolic Logic 14 (1949), p. 159-166.

[5] — *Some interconnections between modern algebra and mathematical logic*, Transactions of the American Mathematical Society 74 (1953), p. 410-427.

[6] A. Robinson, *On the Metamathematics of Algebra*, 1951.

[7] H. Rasiowa and R. Sikorski, *A proof of the Skolem-Löwenheim theorem*, Fundamenta Mathematicae 38 (1951), p. 230-232.

[8] D. Hilbert and P. Bernays, *Grundlagen der Mathematik*, erster Band. Berlin 1934.

Reçu par la Rédaction le 6. 7. 1953

On the existence of totally heterogeneous spaces

by

J. R. Büchi (Ann Arbor)

The main purpose of this note is to prove the existence of a set M of real numbers, which is heterogeneous in the sense that every Borel-function defined on a subset X of M into M is trivial. Some consequences and related facts are pointed out in notes at the end of the paper.

We first state the following fact:

(1) *Let f be a real valued measurable function defined on a measurable set X of real numbers. Then the set D of all y, for which $f^{-1}(y)$ is of positive measure, is at most of cardinality $\aleph_0$.*

Now we prove,

LEMMA 1. *Let F be a class of real valued measurable functions, defined on measurable sets of real numbers, and suppose the cardinality of F is $\aleph_1$. Then there exists a set M of real numbers, which is of cardinality $\aleph_1$, such that the sets $[f(x) | x \in M, f(x) \in M, f(x) \neq x]$ are at most of cardinality $\aleph_0$, for all members f of F.*

Proof. Let ω_1 be the first ordinal of cardinality $\aleph_1$. By hypothesis the class F can be arranged into a ω_1-series $[f_\xi | \xi < \omega_1]$. Let $D_\xi = [y | f_\xi^{-1}(y)$ of positive measure$]$ and define a ω_1-series of real numbers x_ξ by the following induction.

Choose any real number as x_1. If the x_η are already defined for all $\eta < \xi$, then choose x_ξ such that the following conditions are satisfied:

$$(\alpha) \qquad x_\xi \neq x_\eta \quad \text{for all} \quad \eta < \xi,$$

$$(\beta) \qquad x_\xi \neq f_\nu(x_\eta) \quad \text{for all} \quad \eta < \xi \quad \text{and} \quad \nu < \xi,$$

$$(\gamma) \qquad f_\nu(x_\xi) \neq x_\eta \quad \text{or} \quad f_\nu(x_\xi) \in D_\nu \quad \text{for all} \quad \eta < \xi \quad \text{and} \quad \nu < \xi.$$

That such an element x_ξ exists one shows as follows. To realize (α) and (β) one has to avoid a set of cardinality less than $\aleph_1$ only. As for the realization of (γ) note first that in case $x_\eta \in D_\nu$ the condition (γ) is void. In the alternative case the pair (η, ν) is such that $x_\eta \notin D_\nu$. Then, by definition of D_ν, $f_\nu^{-1}(x_\eta)$ is of measure 0. Therefore, for any pair (η, ν),