On the definition of computable functionals

by

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The notion of computable function has been defined in several different ways (cf. for example S. C. Kleene [2], Part II). The equivalence of all those definitions implies that the notion of computability is well chosen. There are two main ways of defining this notion: mathematical and meta-mathematical. In this paper I shall prove that also the mathematical definition of computable functional is equivalent to the meta-mathematical.

Let $\mathfrak{N}$ be the set of natural numbers (positive integers with zero). Let $\mathfrak{F}$ be the set of all functions of one argument defined over the set $\mathfrak{N}$ and assuming values from the set $\mathfrak{N}$ ($\mathfrak{F}=\mathfrak{N}^{\mathfrak{N}}$).

The functionals considered in this paper are defined on the n-tuples of functions belonging to $\mathfrak{F}$ and on the k-tuples of natural numbers and assume the natural value:

$$\Phi\langle\varphi_1,\dots,\varphi_n\rangle(x_1,\dots,x_k)=y\in\mathfrak{N}.$$

We start from the following definition of the *class* $\mathfrak{K}$ *of computable functionals*: $\mathfrak{K}$ is the smallest class of functionals containing the following initial functionals:

$$\text{(1)}\quad \begin{array}{ll} \boldsymbol{U}\langle\varphi\rangle(x)=\varphi(x), & \boldsymbol{S}\langle\varphi_1,\dots,\varphi_n\rangle(x)=S(x)=x+1,\\ \boldsymbol{M}\langle\varphi_1,\dots,\varphi_n\rangle(x,y)=x\dot{-}y, & \boldsymbol{P}\langle\varphi_1,\dots,\varphi_n\rangle(x,y)=x^y, \end{array}$$

and closed under the operations of substitution and of effective minimum.

Substitution. If $\Phi,\Psi\in\mathfrak{K}$ and

$$\text{(2)}\quad \Omega\langle\varphi_1,\dots,\varphi_n\rangle(x_1,\dots,x_k,y_1,\dots,y_s) =\Phi\langle\varphi_1,\dots,\varphi_n\rangle\big(x_1,\dots,x_k,\Psi\langle\varphi_1,\dots,\varphi_n\rangle(y_1,\dots,y_s)\big),$$

then $\Omega\in\mathfrak{K}$.

Effective minimum. If $\Phi\in\mathfrak{K}$,

$$\text{(3)}\quad \Omega\langle\varphi_1,\dots,\varphi_n\rangle(x_1,\dots,x_k)=(\mu y)[\Phi\langle\varphi_1,\dots,\varphi_n\rangle(x_1,\dots,x_k,y)=0]$$

and the following condition is satisfied:

$$\text{(4)}\quad \prod_{\varphi_1,\dots,\varphi_n\in\mathfrak{F}}\prod_{x_1,\dots,x_k\in\mathfrak{N}}\sum_{y\in\mathfrak{N}}\Phi\langle\varphi_1,\dots,\varphi_n\rangle(x_1,\dots,x_k,y)=0,$$

then $\Omega\in\mathfrak{K}$ [1]).

In order to formulate the meta-mathematical definition we shall consider the system (S) of arithmetics of natural numbers. Let $x_1,\dots,x_n,\dots$ are the number variables. Let 0, $+$, $\cdot$, S, be the constants denoting zero, addition, multiplication, and successor. Parenthesis, connectives and the "μ" symbol are the constants of the system, $f_1,\dots,f_n,\dots$ are the function-variables.

The *well formed formulas* (wff) of the system (S) are the finite sequences of these symbols. We distinguish the *well formed number formulas* (wfnf) and the *well formed sentential formulas* (wfsf). The number-variables and 0 are wfnf. If α and β are wfnf, and A and B are wfsf, then $S(\alpha)$, $\alpha+\beta$, $\alpha\cdot\beta$, $f_n(\alpha)$, and $(\mu x_i)[A]$ are wfnf, $\alpha=\beta$, $A\to B$, $\sim A$ are wfsf. The function-variables are also always free in any wff. Thus we can say that (S) is an elementary arithmetics. The axioms of arithmetics as well as the axiom schemes of the calculus of propositions and of the calculus of quantifiers (the calculus of μ-operation), are assumed to be axioms of (S) [2]).

The rules of inference are the well known logical rules of inference with the rule of substitution of a wfnf for free number-variables. Sb$A(x_j/\alpha_j)$ symbolizes the substitution in the formula A, of the wfnf α_j for the number-variable x_j. A formula obtained from a finite number of axioms by means of the application of the rules of inference a finite number of times is a theorem of (S).

For each finite set of functions $\varphi_1,\dots,\varphi_n\in\mathfrak{F}$ we shall consider a set of postulates $\mathbf{P}(\varphi_1,\dots,\varphi_n)$ of the form

$$\text{(5)}\quad f_i\big(S^m(0)\big)=S^{\varphi_i(m)}(0)$$

for any $m\in\mathfrak{N}$ and $0<i\leqslant n$, where $S^n(0)$ is the formula of the form $\overbrace{SSS\dots S}^{n}(0)$; ($S^0(0)=0$, $S^1(0)=S(0)$ etc.). The postulates $\mathbf{P}(\varphi_1,\dots,\varphi_n)$ define the function-variables $f_1,\dots,f_n$ as function-constants representing the functions $\varphi_1,\dots,\varphi_n$.

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[1]) Some theorems on the functionals defined above are proved in the paper of A. Grzegorczyk [1]. Especially the applications to the computable analysis are considered there.

[2]) A system of arithmetics with the minimum operation instead of the quantifiers was described in the paper of A. Mostowski [3]. We assume that our system (S) is similar to the system (S) of Mostowski.

We shall say that a formula A is a consequence of a set $\mathbf{P}$ if there exists a finite set of formulae $B_1, \ldots, B_s \in \mathbf{P}$ such that the implication

$$(B_1 \wedge \ldots \wedge B_s) \to A \tag{6}$$

is a theorem of (S).

Let $\mathbf{C}(\varphi_1, \ldots, \varphi_n)$ be the set of consequences of the set of postulates $\mathbf{P}(\varphi_1, \ldots, \varphi_n)$.

In this paper we prove the following

THEOREM. *If for any $\varphi_1, \ldots, \varphi_n$ the set $\mathbf{C}(\varphi_1, \ldots, \varphi_n)$ is consistent, then for each functional $\Phi\langle\varphi_1, \ldots, \varphi_n\rangle(z_1, \ldots, z_k)$, $\Phi \in \mathcal{K}$ if and only if there exists a formula $\alpha \in$ wfnf with n function-variables $f_1, \ldots, f_n$ and k free number-variables $x_1, \ldots, x_k$, such that for any functions $\varphi_1, \ldots, \varphi_n \in \mathfrak{F}$ and for any $z_1, \ldots, z_k \in \mathcal{N}$ the formula*

$$\mathrm{Sb}\,\alpha\big(x_1/S^{z_1}(0), \ldots, x_k/S^{z_k}(0)\big) = S^{\Phi\langle\varphi_1, \ldots, \varphi_n\rangle(z_1, \ldots, z_k)}(0) \tag{7}$$

is a consequence of the set of postulates $\mathbf{P}(\varphi_1, \ldots, \varphi_n)$ [3]).

Proof. The first implication we prove by induction. For the identity functional $\boldsymbol{U}$ we set $\alpha =$ "$f_1(x_1)$", hence formulae (7) are identical with postulates (5), and obviously belong to $\mathbf{C}(\varphi_1, \ldots, \varphi_n)$. For the constant functionals $\boldsymbol{S}, \boldsymbol{M}, \boldsymbol{P}$, it is evident that the functions $x+1$, $x \dot{-} y$, x^y are representable in the system (S) and the corresponding formulae (7) are theorems of (S) and therefore belong to $\mathbf{C}(\varphi_1, \ldots, \varphi_n)$.

Now suppose that for two functionals $\Phi, \Psi \in \mathcal{K}$ there exist such $\alpha_\Phi, \alpha_\Psi \in$ wfnf that the following formulae belong to $\mathbf{C}(\varphi_1, \ldots, \varphi_n)$:

$$\mathrm{Sb}\,\alpha_\Phi\big(x_1/S^z(0),\ x_2/S^v(0)\big) = S^{\Phi\langle\varphi_i\rangle(z,v)}(0), \tag{8}$$

$$\mathrm{Sb}\,\alpha_\Psi\big(x_1/S^t(0)\big) = S^{\Psi\langle\varphi_i\rangle(t)}(0). \tag{9}$$

Setting $v = \Psi\langle\varphi_i\rangle(t)$ and using the theorems of extensionality we can deduce from formulae (8) and (9) the following formula:

$$\mathrm{Sb}\,\alpha_\Phi\Big(x_1/S^z(0), x_2/\mathrm{Sb}\,\alpha_\Psi\big(x_1/S^t(0)\big)\Big) = S^{\Phi\langle\varphi_i\rangle\left(z, \Psi\langle\varphi_i\rangle(t)\right)}(0). \tag{10}$$

Hence formula (10) belongs to $\mathbf{C}(\varphi_1, \ldots, \varphi_n)$. The theorem is also true for the functionals obtained by substitution (4). To prove that this property

[3]) This theorem imitates the definition of computable functions formulated by A. Mostowski [3], p. 74. The supposition of the consistency of the set $\mathbf{C}(\varphi_1, \ldots, \varphi_n)$ is unessential because it is easy to prove the consistency of this set in the well known semantic manner of Tarski.

is hereditary with respect to the operation of minimum suppose that formulae (8) belong to $\mathbf{C}(\varphi_1, \ldots, \varphi_n)$ and that

$$\Omega\langle\varphi_i\rangle(z) = (\mu v)[\Phi\langle\varphi_i\rangle(z, v) = 0], \tag{11}$$

$$\prod_{\varphi_i \in \mathfrak{F}} \prod_{z \in \mathcal{N}} \sum_{v \in \mathcal{N}} \Phi\langle\varphi_i\rangle(z, v) = 0. \tag{12}$$

From (11) and (12) it follows that

$$\Phi\langle\varphi_i\rangle\big(z, \Omega\langle\varphi_i\rangle(z)\big) = 0, \tag{13}$$

$$\Phi\langle\varphi_i\rangle(z, v) \neq 0 \quad \text{for} \quad v < \Omega\langle\varphi_i\rangle(z). \tag{14}$$

It is evident that $S^u(0) = 0$ if and only if $u = 0$. Hence from (13) it follows that formulae (8) imply the following:

$$\mathrm{Sb}\,\alpha_\Phi\big(x_1/S^z(0), x_2/S^{\Omega\langle\varphi_i\rangle(z)}(0)\big) = 0. \tag{15}$$

Similarly (14) involves that formulae (8) imply the following:

$$\mathrm{Sb}\,\alpha_\Phi\big(x_1/S^z(0), x_2/S^v(0)\big) \neq 0 \tag{16}$$

for any $v < \Omega\langle\varphi_i\rangle(z)$. From (15) and (16) it is easy to deduce in (S) the formula of the form

$$(\mu x_2)\Big[\mathrm{Sb}\,\alpha_\Phi\big(x_1/S^z(0)\big) = 0\Big] = S^{\Omega\langle\varphi_i\rangle(z)}(0). \tag{17}$$

Hence according to the deduction-theorem the formulae of the form (17) belong to the set $\mathbf{C}(\varphi_1, \ldots, \varphi_n)$.

To prove the inverse implication we need an arithmetization of the syntax of the system (S).

Obviously it is easy to define an arithmetical recursive function $Th(n)$, such that the set of values of the function Th is identical with the set of numbers representing the theorems of the system (S). Each recursive function is identical with a constant functional of the class $\mathcal{K}$. Hence $Th \in \mathcal{K}$.

On the other hand, it is easy to define a computable functional $\Theta\langle\varphi_1, \ldots, \varphi_n\rangle(z)$ such that the set of values of Θ it identical with the set of numbers representing the formulae of the set $\mathbf{P}(\varphi_1, \ldots, \varphi_n)$. Indeed, we can assume that: if $z = mn + i$, where $0 < i \leqslant n$, then $\Theta\langle\varphi_1, \ldots, \varphi_n\rangle(z)$ is the number representing formula (5). For $z = mn$ we assume that $\Theta\langle\varphi_1, \ldots, \varphi_n\rangle(z) = \Theta\langle\varphi_1, \ldots, \varphi_n\rangle(z+1)$. It is evident that the arithmetical description of a formula of form (5) can be obtained in a computable manner. The first symbol of (5) is the function-variable f_i, the second a parenthesis, then come m-times the symbols of successor, parenthesis, zero, and two parentheses, later the identity symbol, $\varphi_i(m)$-times the successor-symbol, parenthesis, zero and two parentheses. Such a description expresses an arithmetical computable relation.

If $b_1,\ldots,b_s$ are the numbers of the expressions $B_1,\ldots,B_s$, then $(b_1\dot{\wedge}\ldots\dot{\wedge}b_s)$ is the number of the conjunction $B_1\wedge\ldots\wedge B_s$ and $(b_1\dot{\rightarrow}b_2)$ is the number of the implication $B_1\rightarrow B_2$. Let $[\mathrm{Sb}\,\alpha(z_1,\ldots,z_k)\dot{=}u]$ be the number of the formula

$$\mathrm{Sb}\,\alpha\big(x_1/S^{z_1}(0),\ldots,x_k/S^{z_k}(0)\big)=S^u(0).$$

Proving the inverse implication we assume that for a functional Φ there exists a formula $\alpha\in\mathrm{wfnf}$ such that for any $\varphi_1,\ldots,\varphi_n\in\mathfrak{F}$ and $z_1,\ldots,z_k\,\varepsilon\,\mathcal{N}$ the formulae of form (7) belong to $\mathbf{C}(\varphi_1,\ldots,\varphi_n)$. In the arithmetical language this condition can be expressed as follows:

$$(18)\qquad \prod_{\varphi_1,\ldots,\varphi_n\in\mathfrak{F}}\prod_{z_1,\ldots,z_k\in\mathcal{N}}\sum_{s,t\in\mathcal{N}}$$
$$Th(t)=\Big(\big(\Theta\langle\varphi_1,\ldots,\varphi_n\rangle(0)\,\dot{\wedge}\ldots\dot{\wedge}\,\Theta\langle\varphi_1,\ldots,\varphi_n\rangle(s)\big)$$
$$\dot{\rightarrow}[\mathrm{Sb}\,\alpha(z_1,\ldots,z_k)\dot{=}\Phi\langle\varphi_1,\ldots,\varphi_n\rangle(z_1,\ldots,z_k)]\Big).$$

This follows from (6) and from the definitions of the set $\mathbf{C}(\varphi_1,\ldots,\varphi_n)$ and of functionals Th and Θ. Setting $U=\Phi\langle\varphi_1,\ldots,\varphi_n\rangle(z_1,\ldots,z_k)$ and using the pairing functions J,K,L (for example $J(x,y)=(x+y)^2+x$, $Kz=z\dot{-}[\sqrt{z}]^2$, $Lz=[\sqrt{z}]\dot{-}Kz$) we can represent the triplet $\langle s,t,u\rangle$ by a number v such that $v=J\big(J(s,t),u\big)$ and hence

$$s=KKv,\qquad t=LKv,\qquad u=Lv.$$

Thus from (18) we obtain the following condition:

$$(19)\qquad \prod_{\varphi_1,\ldots,\varphi_n\in\mathfrak{F}}\prod_{z_1,\ldots,z_k\in\mathcal{N}}\sum_{v\in\mathcal{N}}$$
$$Th(LKv)=\Big(\big(\Theta\langle\varphi_1,\ldots,\varphi_n\rangle(0)\,\dot{\wedge}\ldots\dot{\wedge}\,\Theta\langle\varphi_1,\ldots,\varphi_n\rangle(KKv)\big)$$
$$\dot{\rightarrow}[\mathrm{Sb}\,\alpha(z_1,\ldots,z_k)\dot{=}Lv]\Big).$$

According to the operation of effective minimum the functional

$$(20)\qquad \Xi\langle\varphi_1,\ldots,\varphi_n\rangle(z_1,\ldots,z_k)$$
$$=(\mu v)\Big[Th(LKv)=\Big(\big(\Theta\langle\varphi_1,\ldots,\varphi_n\rangle(0)\,\dot{\wedge}\ldots\dot{\wedge}\,\Theta\langle\varphi_1,\ldots,\varphi_n\rangle(KLv)\big)$$
$$\dot{\rightarrow}[\mathrm{Sb}\,\alpha(z_1,\ldots,z_k)\dot{=}Lv]\Big)$$

belongs to $\mathcal{K}$. Setting

$$(21)\qquad \Phi^*\langle\varphi_1,\ldots,\varphi_n\rangle(z_1,\ldots,z_k)=L\big(\Xi\langle\varphi_1,\ldots,\varphi_n\rangle(z_1,\ldots,z_k)\big)$$

we find that $\Phi^*\in\mathcal{K}$. From (19), (20) and (21) it follows that for any $\varphi_1,\ldots,\varphi_n\in\mathfrak{F}$ and $z_1,\ldots,z_k\in\mathcal{N}$, formula (7*) obtained from (7) by replacing Φ by Φ^* is a consequence of $\mathbf{P}(\varphi_1,\ldots,\varphi_n)$. If (7) and (7*) belong to $\mathbf{C}(\varphi_1,\ldots,\varphi_n)$ and the set $\mathbf{C}(\varphi_1,\ldots,\varphi_n)$ is consistent, then it is possible only if $\Phi=\Phi^*$. Also $\Phi\in\mathcal{K}$.

Remarks. 1. It is easy to verify that the functionals Th and Θ can be defined by using only the simple induction scheme (even by using only the operation of limited minimum). Hence we can prove that each computable functional can be presented in a canonical form

$$L(\mu v)[\Psi\langle\varphi_1,\ldots,\varphi_n\rangle(z_1,\ldots,z_k,v)=0]$$

where the functional Ψ can be said to be recursive or even elementarily recursive.

2. If formula (7) follows from the postulates $\mathbf{P}(\varphi_1,\ldots,\varphi_n)$, then it follows from a finite number of postulates: $P_1,\ldots,P_u$. Hence we find that, if other functions $\psi_1,\ldots,\psi_n$ satisfy the postulates $P_1,\ldots,P_u$, then the functional Φ assumes the same value for the functions $\varphi_1,\ldots,\varphi_n$ as for $\psi_1,\ldots,\psi_n$. The postulates $P_1,\ldots,P_u$ determine only a finite set of values of functions $\varphi_1,\ldots,\varphi_n$. Thus for each functional $\Phi\in\mathcal{K}$

$$(22)\qquad \prod_{z_1,\ldots,z_k\in\mathcal{N}}\prod_{\varphi_1,\ldots,\varphi_n\in\mathfrak{F}}\sum_{l\in\mathcal{N}}\prod_{\psi_1,\ldots,\psi_n\in\mathfrak{F}}$$
$$\textit{if}\quad \varphi_i(v)=\psi_i(v)\quad \textit{for}\quad v<l,\quad \textit{then}\quad \Phi\langle\varphi_1,\ldots,\varphi_n\rangle(z_1,\ldots,z_k)$$
$$=\Phi\langle\psi_1,\ldots,\psi_n\rangle(z_1,\ldots,z_k).$$

This means that for given numbers $z_1,\ldots,z_n$, the functional Φ depends only upon a finite set of values of the function-arguments.

These two properties can also be obtained by induction from the inductive definition of the class $\mathcal{K}$ (without the supposition of the consistency of the system (S)) [4]).

3. From (22) we can deduce

$$(23)\qquad \prod_{z_1,\ldots,z_k\in\mathcal{N}}\prod_{\chi\in\mathfrak{F}}\sum_{l\in\mathcal{N}}\prod_{\varphi_1,\ldots,\varphi_n,\psi_1,\ldots,\psi_n\in\mathfrak{F}}$$
$$\textit{if}\quad \varphi_i(v)=\psi_i(v)\quad \textit{for}\quad v<l,\quad \textit{and}\quad \varphi_i(v)\leqslant\chi(v),\ \psi_i(v)\leqslant\chi(v)\quad \textit{for any}\ v\in\mathcal{N},$$
$$\textit{then}\quad \Phi\langle\varphi_1,\ldots,\varphi_n\rangle(z_1,\ldots,z_k)=\Phi\langle\psi_1,\ldots,\psi_n\rangle(z_1,\ldots,z_k).$$

In my paper [1] there is an effective but very long proof of (23) (Uniformity Theorem). Now I shall sketch another proof, non effective but short. To simplify the notation let us consider the functional $\Phi\langle\varphi\rangle$ than one function-argument. For example, let χ be a constant function, $\chi(u)=1$ for any $u\in\mathcal{N}$. In that case theorem (23) is reduced to

$$(24)\qquad \sum_{k\in\mathcal{N}}\prod_{\varphi,\psi\in\mathfrak{F}}\quad \textit{if}\quad \varphi,\psi\in(0,1)\quad \textit{and}\quad \varphi(v)=\psi(v)\quad \textit{for}\quad v<k,$$
$$\textit{then}\quad \Phi\langle\varphi\rangle=\Phi\langle\psi\rangle$$

where $(0,1)$ *is the set of functions assuming only two values:* 0 *and* 1.

[4]) The property (22) was proved by induction in the paper of A. Grzegorczyk [1].

Suppose that (24) is not true. Hence we define the function

$$(25)\qquad \varrho(0)=\begin{cases}0 & \textit{if } \prod_k \sum_{\varphi,\psi\in(0,1)} \varphi(v)=\psi(v) \quad \textit{for} \quad v\leqslant k \\ & \quad \textit{and} \quad \varphi(0)=\psi(0)=0 \quad \textit{and} \quad \Phi\langle\varphi\rangle\neq\Phi\langle\psi\rangle, \\ 1 & \textit{in the contrary case.}\end{cases}$$

$$(26)\qquad \varrho(n+1)=\begin{cases}0 & \textit{if } \prod_k \sum_{\varphi,\psi\in(0,1)} \varphi(v)=\varphi(v) \quad \textit{for} \quad v\leqslant k \\ & \quad \textit{and} \quad \varphi(i)=\psi(i)=\varrho(i) \textit{ for } i\leqslant n, \\ & \quad \textit{and } \varphi(n+1)=\psi(n+1)=0, \quad \textit{and} \quad \Phi\langle\varphi\rangle\neq\Phi\langle\psi\rangle, \\ 1 & \textit{in the contrary case.}\end{cases}$$

From (25) and from the negation of (24) it follows that if $\varrho(0)=1$ then for each $k\in\mathcal{N}$ there exist $\varphi,\psi\in(0,1)$ such that $\varphi(0)=\psi(0)=1$ and $\varphi(v)=\psi(v)$ for $v\leqslant k$ and $\Phi\langle\varphi\rangle\neq\Phi\langle\psi\rangle$.

On the other hand, from (25) it follows that if $\varrho(0)=0$, then for each $k\in\mathcal{N}$ there exist $\varphi,\Psi\in(0,1)$ such that $\varphi(0)=\psi(0)=0$ and $\varphi(v)=\psi(v)$ for $v\leqslant k$ and $\Phi\langle\varphi\rangle\neq\Phi\langle\psi\rangle$. We can join the last two sentences into one saying that

$$(27)\qquad \prod_{k\in\mathcal{N}} \sum_{\varphi,\psi\in(0,1)} \varphi(0)=\psi(0)=\varrho(0) \quad \textit{and} \quad \varphi(v)=\psi(v) \quad \textit{for} \quad v\leqslant k, \quad \textit{and} \quad \Phi\langle\varphi\rangle\neq\Phi\langle\psi\rangle.$$

Now we whall prove by induction that for any $n\in\mathcal{N}$

$$(28)\qquad \prod_{k\in\mathcal{N}} \sum_{\varphi,\psi\in(0,1)} \varphi(i)=\psi(i)=\varrho(i) \quad \textit{for} \quad i\leqslant n \quad \textit{and} \quad \varphi(v)=\psi(v) \quad \textit{for} \quad v\leqslant k, \quad \textit{and} \quad \Phi\langle\varphi\rangle\neq\Phi\langle\psi\rangle.$$

If n=0, then we have (27). Now suppose that we have (28) and let us distinguish two cases: $\varrho(n+1)=0$ and $\varrho(n+1)=1$.

If $\varrho(n+1)=0$, then from (26) it is evident that

$$(29)\qquad \prod_{k\in\mathcal{N}} \sum_{\varphi,\psi\in(0,1)} \varphi(i)=\psi(i)=\varrho(i) \quad \textit{for} \quad i\leqslant n+1 \quad \textit{and} \quad \varphi(v)=\psi(v) \quad \textit{for} \quad v\leqslant k, \quad \textit{and} \quad \Phi\langle\varphi\rangle\neq\Phi\langle\psi\rangle.$$

If $\varrho(n+1)=1$, then (26) implies that there exists k_1 such that for any $\varphi,\psi\in(0,1)$ if $\varphi(v)=\psi(v)$ for $v\leqslant k_1$ and $\varphi(i)=\psi(i)=\varrho(i)$ for $i\leqslant n$ and $\varphi(n+1)=\psi(n+1)=0$, then $\Phi\langle\varphi\rangle=\Phi\langle\psi\rangle$. Hence from (28) it follows that

$$(30)\qquad \prod_{k\in\mathcal{N}} \sum_{\varphi,\psi\in(0,1)} \varphi(i)=\psi(i)=\varrho(i) \quad \textit{for} \quad i\leqslant n, \quad \textit{and} \quad \varphi(v)=\psi(v) \quad \textit{for} \quad v\leqslant k, \quad \textit{and} \quad \Phi\langle\varphi\rangle\neq\Phi\langle\psi\rangle, \quad \textit{and} \quad \varphi(n+1)=\psi(n+1)=1.$$

Namely suppose that (30) is not true. Then there exists k_2 such that for any $\varphi,\psi\in(0,1)$ if $\varphi(i)=\psi(i)=\varrho(i)$ for $i\leqslant n$, and $\varphi(v)=\psi(v)$ for $v\leqslant k_2$ and $\varphi(n+1)=\psi(n+1)=1$, then $\Phi\langle\varphi\rangle=\Phi\langle\psi\rangle$. Hence there exists k_0 such that $k_0>k_1$, $k_0>k_2$, and $k_0>n+1$. Thus for any $\varphi,\psi\in(0,1)$ if $\varphi(i)=\psi(i)=\varrho(i)$ for $i\leqslant n$ and $\varphi(v)=\psi(v)$ for $v\leqslant k_0$, then $\Phi\langle\varphi\rangle=\Phi\langle\psi\rangle$. Indeed if $\varphi,\psi\in(0,1)$ and $\varphi(v)=\psi(v)$ for $v\leqslant k_0$, and $n+1<k_0$, then $\varphi(n+1)=\psi(n+1)$. If $\varphi(n+1)=\psi(n+1=0$ then from the fact that $k_0>k_1$ it follows that $\varphi(v)=\psi(v)$ for $v\leqslant k_1$ and $\Phi\langle\varphi\rangle=\Phi\langle\psi\rangle$ according to the above mentioned property of k_1. If $\varphi(n+1)=\psi(n+1)=1$ then from the fact that $k_0>k_2$ it follows that $\varphi(v)=\psi(v)$ for $v\leqslant k_2$ and $\Phi\langle\varphi\rangle=\Phi\langle\psi\rangle$ according to the property of k_2. But the existence of such a number k_0 contradicts condition (28).

Condition (30) means that (29) is likewise true when $\varrho(n+1)=1$. Also (28) implies (29) in any case. (28) is thus proved by induction for any $n\in\mathcal{N}$. Setting $n=k$ we find that

$$(31)\qquad \prod_{k\in\mathcal{N}} \sum_{\varphi,\psi\in(0,1)} \varphi(v)=\psi(v)=\varrho(v) \quad \textit{for} \quad v\leqslant k \quad \textit{and} \quad \Phi\langle\varphi\rangle\neq\Phi\langle\psi\rangle.$$

If $\Phi\langle\varphi\rangle\neq\Phi\langle\psi\rangle$, then $\Phi\langle\varphi\rangle\neq\Phi\langle\varrho\rangle$ or $\Phi\langle\psi\rangle\neq\Phi\langle\varrho\rangle$. Hence from (31) it follows that

$$(32)\qquad \prod_{k\in\mathcal{N}} \sum_{\chi\in(0,1)} \chi(v)=\varrho(v) \quad \textit{for} \quad v\leqslant k, \quad \textit{and} \quad \Phi\langle\chi\rangle\neq\Phi\langle\varrho\rangle.$$

And (32) contradicts (22), which completes the proof of (24). The general proof of (23) is similar to that of (24) but more complicated. Theorem (23) in the stronger effective form proved in [1] has many applications to the computable analysis.

References

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