et par η la quantité $\eta=[\log C]^{-1}\log[(a+2\varepsilon)/(a+\varepsilon)]>0$. Alors $m\eta-1<k_m\leqslant m\eta$ et le quotient k_m/m tend vers η lorsque $m\to\infty$. Posons dans l'identité

$$A_{m+k}=\left(\sqrt[m]{a_1a_2\dots a_m}\right)^{m/(m+k)}(a_{m+1}\dots a_{m+k})^{1/(m+k)}$$

$k=k_m$ et $m=m_1,m_2,\dots$ et appliquons (33). Il vient

$$A_{m+k_m}\leqslant A_m^{m/(m+k_m)}(a+2\varepsilon)^{k_m/(m+k_m)}$$

d'où l'on déduit en vertu de (31) l'inégalité

$$g\leqslant g^{1/(1+\eta)}(a+2\varepsilon)^{\eta/(1+\eta)}.$$

Il en résulte que $g\leqslant a+2\varepsilon$ et comme d'après (32) $\beta\leqslant g$ on a $\beta\leqslant a+2\varepsilon$ quel petit que soit $\varepsilon>0$, ce qui prouve que la limite $\lim a_n=g$ existe.

Par conséquent, dans le cas où les ensembles E_1 et E_2 sont fermés, il existe la limite

$$\lim_{n\to\infty}k_{2n}(E_1,E_2)=v(E_1,E_2). \tag{34}$$

Ce résultat reste vrai lorsque les ensembles E_1 et E_2 sont disjoints quelconques, ce qu'on peut prouver par la méthode appliquée dans la remarque du numéro 2.

Ajoutons que le résultat (34) peut être généralisé au cas considéré dans le numéro 3, où E est la somme de $p\geqslant 2$ ensembles disjoints.

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The epidemic effect for partial differential inequalities of the first order

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The purpose of this note is to investigate, for partial differential inequalities of the first order, the epidermic effects which are known in the case of ordinary differential inequalities [5]. The problem discussed in this paper has been suggested to me by J. Szarski.

§ 1. To begin with we formulate the following lemma (see [1], proof of theorem 1; also [3], theorem 1.1).

Lemma K. *Suppose that the function $f(x,y_1,\dots,y_n,u,q_1,\dots,q_n)$ is defined in an open set Ω and satisfies the following Lipschitz condition:*

$$|f(x,y_1,\dots,y_n,u,\bar{q}_1,\dots,\bar{q}_n)-f(x,y_1,\dots,y_n,u,\bar{\bar{q}}_1,\dots,\bar{\bar{q}}_n)|\leqslant M\sum_{i=1}^{n}|\bar{q}_i-\bar{\bar{q}}_i|.$$

We assume that the projection of Ω on the space of points $(x,y_1,\dots,y_n)$ covers entirely the following Haar's pyramid

$$x^0\leqslant x<x^0+a,\quad |y_i-y_i^0|\leqslant a_i-M(x-x^0)\quad (i=1,2,\dots,n), \tag{1.1}$$
$$a>0,\quad M>0,\quad a_i>aM.$$

We assume that the functions $u(x,y_1,\dots,y_n)$, $v(x,y_1,\dots,y_n)$ are continuous in (1.1) and the inequality

$$u(x^0,y_1,\dots,y_n)<v(x^0,y_1,\dots,y_n)$$

holds for $|y_i-y_i^0|\leqslant a_i$ $(i=1,2,\dots,n)$.

We suppose that if, for a point $P(x,y_1,\dots,y_n)$ of (1.1), the equality

$$u(x,y_1,\dots,y_n)=v(x,y_1,\dots,y_n)$$

is satisfied, then the functions u and v possess the total differentials in P and the inequalities

$$\left(\frac{\partial v}{\partial x}\right)_P\geqslant f\left(x,y_1,\dots,y_n,v,\frac{\partial v}{\partial y_1},\dots,\frac{\partial v}{\partial y_n}\right)_P,$$

$$\left(\frac{\partial u}{\partial x}\right)_P< f\left(x,y_1,\dots,y_n,u,\frac{\partial u}{\partial y_1},\dots,\frac{\partial u}{\partial y_n}\right)_P$$

hold.

Under these assumptions the inequality $u(x,y_1,\ldots,y_n)<v(x,y_1,\ldots,y_n)$ *is fulfilled in* (1.1).

Now we introduce the following assumptions:

Assumptions A. The function $f(x,y_1,\ldots,y_n,z,q_1,\ldots,q_n)$ and its derivatives of the first and of the second order with respect to the variables $y_1,\ldots,y_n,z,q_1,\ldots,q_n$ are continuous in the cube

(1.2) $$|x-x^0|\leqslant C,\quad |y_i-y_i^0|\leqslant C,\quad |z-z^0|\leqslant C,\quad |q_i-q_i^0|\leqslant C$$

their absolute values being less than M in (1.2).

The function $\omega(y_1,\ldots,y_n)$ is of class C^2 in the cube $|y_i-y_i^0|\leqslant C$ $(i=1,2,\ldots,n)$ and the following inequalities:

$$\left|\frac{\partial\omega}{\partial y_i}\right|<M,\quad \left|\frac{\partial^2\omega}{\partial y_i\partial y_j}\right|<M\quad (i,j=1,2,\ldots,n),$$

$$|\omega_{y_i}(y_1^0,\ldots,y_n^0)-q_i^0|<C/4,\quad |q_i^0|<M\quad (i=1,2,\ldots,n),$$

$$|\omega(y_1^0,\ldots,y_n^0)-z^0|<C/4$$

are satisfied.

Remark 1. Under the assumptions A there exists (see [4]) in the set

(1.3) $$|x-x^0|<\delta,\quad |y_i-y_i^0|\leqslant C[4n(M+1)]^{-1}-M|x-x^0|$$

where $\delta=C^2[(n+1)(M+C+1)]^{-5}$ a unique solution $v(x,y_1,\ldots,y_n)$ of the equation

(1.4) $$\partial z/\partial x=f(x,y_1,\ldots,y_n,z,\partial z/\partial y_1,\ldots,\partial z/\partial y_n),$$

satisfying the initial condition

(1.5) $$v(x^0,y_1,\ldots,y_n)\equiv\omega(y_1,\ldots,y_n)$$

for $|y_i-y_i^0|\leqslant C[4n(M+1)]^{-1}$ $(i=1,2,\ldots,n)$ and possessing continuous derivatives of the first order. The solution depends continuously on the initial condition.

Remark 2. Let us consider the equation

(1.6) $$\partial z/\partial x=f(x,y_1,\ldots,y_n,z,\partial z/\partial y_1,\ldots,\partial z/\partial y_n)+1/\nu\quad (\nu=1,2,3,\ldots),$$

assumptions A being fulfilled.

From remark 1 one can easily conclude that for ν sufficiently large there exists a solution $v_\nu(x,y_1,\ldots,y_n)$ of (1.6), determined in the set

(1.7) $$x^0\leqslant x<x^0+\delta,\quad |y_i-y_i^0|\leqslant C[4n(M+1)]^{-1}-M(x-x^0),$$

satisfying the initial condition $v_\nu(x,y_1,\ldots,y_n)\equiv\omega(y_1,\ldots,y_n)+1/\nu$ in the cube $|y_i-y_i^0|\leqslant C[4n(M+1)]^{-1}$ and possessing continuous derivatives of the first order.

Using the same notation as in remark 1 we note furthermore that the following relations are satisfied:

(1.8) $$\lim_{\nu\to\infty} v_\nu=v\ \text{uniformly in (1.7)},$$

(1.9) $$v(x,y_1,\ldots,y_n)<v_\nu(x,y_1,\ldots,y_n)\ \text{in (1.7)}.$$

The last inequality follows from lemma K.

We shall prove the following theorem:

Theorem 1. *Suppose assumptions* A *to be fulfilled and let* $v(x,y_1,\ldots,y_n)$ *be the solution of* (1.4), *possessing continuous derivatives in* (1.7) *and satisfying* (1.5). *We assume that the functions* $u(x,y_1,\ldots,y_n)$ *and* $\varepsilon(x,y_1,\ldots,y_n)$ *are continuous in* (1.7) *and* $\varepsilon(x,y_1,\ldots,y_n)>0$. *We suppose furthermore that the inequality*

$$u(x^0,y_1,\ldots,y_n)\leqslant\omega(y_1,\ldots,y_n)$$

holds in the cube $|y_i^0-y_i|\leqslant C[4n(M+1)]^{-1}$ *and the following epidermic condition is true:*

If, for a point $P(x,y_1,\ldots,y_n)$ *of* (1.7), *the inequality*

$$v(P)<u(P)<v(P)+\varepsilon(P)$$

is satisfied, then there exists in P the total differential of u and the inequality

$$(\partial u/\partial x)_P\leqslant f(x,y_1,\ldots,y_n,u,\partial u/\partial y_1,\ldots,\partial u/\partial y_n)_P$$

holds.

Under these assumptions the inequality

$$u(x,y_1,\ldots,y_n)\leqslant v(x,y_1,\ldots,y_n)$$

is fulfilled in (1.7).

Proof. We use the notation introduced in remark 2. From (1.8) and (1.9) it follows that there is such a ν_0 that for $\nu\geqslant\nu_0$ the inequality

(1.10) $$0<v_\nu(P)-v(P)<\varepsilon(P)$$

is fulfilled in (1.7). Let us consider $\nu\geqslant\nu_0$. Suppose that, for a point $P(x,y_1,\ldots,y_n)$ belonging to (1.7) the equality $v_\nu(P)=u(P)$ is satisfied. Then by (1.10) we get

$$v(P)<u(P)<v(P)+\varepsilon(P),$$

and from the epidermic assumption we have

$$(\partial u/\partial x)_P\leqslant f(x,y_1,\ldots,y_n,u,\partial u/\partial y_1,\ldots,\partial u/\partial y_n)_P.$$

But $u(x^0, y_1, \ldots, y_n) < v_\nu(x^0, y_1, \ldots, y_n)$ in the cube $|y_i - y_i^0| \leqslant C[4n(M+1)]^{-1}$ and

$$\partial v_\nu/\partial x = f(x, y_1, \ldots, y_n, v_\nu, \partial v_\nu/\partial y_1, \ldots, \partial v_\nu/\partial y_n) + 1/\nu.$$

Therefore the functions $f+1/\nu$, v_ν and u satisfy the assumptions of lemma K. Then $u(P) < v_\nu(P)$ for P belonging to (1.7). Because of $\lim_{\nu\to\infty} v_\nu = v$ we have $u(P) \leqslant v(P)$.

Now we investigate the epidermic theorem in the case of a system of differential inequalities for one function. Let us consider the system of partial equations

$$(1.11)\quad \frac{\partial z}{\partial x_\mu} = f_\mu\left(x_1, \ldots, x_k, y_1, \ldots, y_n, z, \frac{\partial z}{\partial y_1}, \ldots, \frac{\partial z}{\partial y_n}\right) \quad (\mu = 1, 2, \ldots, k).$$

We introduce the following assumptions:

Assumptions A_1. Suppose that the functions f_μ $(\mu = 1, 2, \ldots, k)$ possess continuous derivatives of the first and of the second order with respect to the last $2n+1$ variables in the cube

$$|x_\mu - x_\mu^0| \leqslant C \quad (\mu = 1, 2, \ldots, k), \qquad |y_i - y_i^0| \leqslant C \quad (i = 1, 2, \ldots, n),$$
$$|z - z^0| \leqslant C, \qquad |q_i - q_i^0| \leqslant C \quad (i = 1, 2, \ldots, n).$$

The absolute values of these derivatives and of f_μ are less than M in that cube. The function $\omega(y_1, \ldots, y_n)$ is of class C^2 in the cube $|y_i - y_i^0| \leqslant C$. The following inequalities are satisfied:

$$|\partial\omega/\partial y_i| < M, \qquad |\partial^2\omega/\partial y_i \partial y_j| < M \quad \text{in} \quad |y_i - y_i^0| \leqslant C,$$
$$|\omega_{y_i}(y_1^0, \ldots, y_n^0) - q_i^0| < C/4, \qquad |q_i^0| < M \quad (i = 1, 2, \ldots, n),$$
$$|\omega(y_1^0, \ldots, y_n^0) - z^0| < C/4.$$

THEOREM 2. *We suppose the assumptions A_1 to be satisfied. Let the function $v(x_1, \ldots, x_k, y_1, \ldots, y_n)$ be a solution of the system* (1.11) *possessing continuous derivatives of the first order in the set*

$$(1.12)\quad \begin{gathered} x_\mu^0 \leqslant x_\mu < x_\mu^0 + \delta/k \quad (\mu = 1, 2, \ldots, k), \\ |y_i - y_i^0| \leqslant C[4n(M+1)]^{-1} - \sum_{\mu=1}^{k} M(x_\mu - x_\mu^0) \quad (i = 1, 2, \ldots, n) \end{gathered}$$

where δ is defined by (1.3), *and satisfying the initial conditions*

$$v(x_1^0, \ldots, x_k^0, y_1, \ldots, y_n) \equiv \omega(y_1, \ldots, y_n)$$

in the cube $|y_i - y_i^0| \leqslant C[4n(M+1)]^{-1}$.

Let the functions $u(x_1, \ldots, x_k, y_1, \ldots, y_n)$ and $\varepsilon(x_1, \ldots, x_k, y_1, \ldots, y_n)$ be continuous in (1.12) *and $\varepsilon > 0$ in* (1.12).

Suppose that the inequality

$$u(x_1^0, \ldots, x_k^0, y_1, \ldots, y_n) \leqslant v(x_1^0, \ldots, x_k^0, y_1, \ldots, y_n)$$

holds in the cube $|y_i - y_i^0| \leqslant C[4n(M+1)]^{-1}$.

We assume that the following epidermic condition is true:

If, for a point $P(x_1, \ldots, x_k, y_1, \ldots, y_n)$ of (1.12), *the inequality*

$$v(P) < u(P) < v(P) + \varepsilon(P)$$

is satisfied, then there exists in P the total differential of u and the inequalities

$$\left(\frac{\partial u}{\partial x_\mu}\right)_P \leqslant f_\mu\left(x_1, \ldots, x_k, y_1, \ldots, y_n, u, \frac{\partial u}{\partial y_1}, \ldots, \frac{\partial u}{\partial y_n}\right)_P \quad (\mu = 1, 2, \ldots, k)$$

hold.

Under these assumptions the inequality $u(P) \leqslant v(P)$ holds for every P belonging to (1.12).

Proof. Suppose that the point $(\bar{x}_1, \ldots, \bar{x}_k, y_1, \ldots, y_n)$ of (1.12) is different from $(x_1^0, \ldots, x_k^0, y_1, \ldots, y_n)$. We introduce the following Mayer's transformation

$$x_\mu = x_\mu^0 + \lambda_\mu t \quad \text{where} \quad \lambda_\mu = (\bar{x}_\mu - x_\mu^0)/\sum_{\mu=1}^{k} (\bar{x}_\mu - x_\mu^0) \quad (\mu = 1, 2, \ldots, k).$$

We have $\sum_{\mu=1}^{k} \lambda_\mu = 1$ and $0 \leqslant \lambda_\mu \leqslant 1$. Put

$$U(t, y_1, \ldots, y_n) = u(x_1^0 + \lambda_1 t, \ldots, x_k^0 + \lambda_k t, y_1, \ldots, y_n),$$
$$V(t, y_1, \ldots, y_n) = v(x_1^0 + \lambda_1 t, \ldots, x_k^0 + \lambda_k t, y_1, \ldots, y_n),$$
$$E(t, y_1, \ldots, y_n) = \varepsilon(x_1^0 + \lambda_1 t, \ldots, x_k^0 + \lambda_k t, y_1, \ldots, y_n).$$

The functions U, V and E are defined in the set

$$(1.13)\quad 0 \leqslant t \leqslant \sum_{\mu=1}^{k} (\bar{x}_\mu - x_\mu^0) < \delta, \qquad |y_i - y_i^0| \leqslant C[4n(M+1)]^{-1} - Mt \quad (i = 1, 2, \ldots, n).$$

V is the solution of the equation

$$(1.14)\quad \partial z/\partial t = \sum_{\mu=1}^{k} \lambda_\mu f_\mu(x_1^0 + \lambda_1 t, \ldots, x_k^0 + \lambda_k t, y_1, \ldots, y_n, z, \partial z/\partial y_1, \ldots, \partial z/\partial y_n)$$

valid in (1.13) and satisfies the initial condition

$$V(0, y_1, \ldots, y_n) \equiv \omega(y_1, \ldots, y_n).$$

Because of the assumptions A_1 and $\sum_{\mu=1}^{k}\lambda_\mu=1$, $\lambda_\mu\geqslant 0$ we find that, for the right member of (1.14), the derivatives of the first and of the second order with respect to the variables $y_1,\ldots,y_n,z,q_1,\ldots,q_m$ are continuous and their absolute values are less than M in the cube

$$|y_i-y_i^0|\leqslant C,\quad |z-z^0|\leqslant C,\quad |q_i-q_i^0|\leqslant C,\quad |t|\leqslant C$$

Therefore the assumptions A are satisfied for the right member of (1.14) considered in the cube $|t|\leqslant C$, $|y_i-y_i^0|\leqslant C$, $|z-z^0|\leqslant C$, $|q_i-q_i^0|\leqslant C$. The hypotheses of theorem 2 imply that U,V,E satisfy the assumptions of theorem 1. From theorem 1 we conclude that the inequality $U(t,y_1,\ldots,y_n)\leqslant V(t,y_1,\ldots,y_n)$ holds in (1.13). Putting $t=\sum_{\mu=1}^{k}(\bar{x}_\mu-x_\mu^0)$ we get

$$u(\bar{x}_1,\ldots,\bar{x}_k,y_1,\ldots,y_n)\leqslant v(\bar{x}_1,\ldots,\bar{x}_k,y_1,\ldots,y_n)$$

This completes the proof.

§ 2. It remains to discuss the epidermic effects for differential inequalities of the form

$$\partial u_\mu/\partial x\leqslant f_\mu(x,y_1,\ldots,y_n,u_1,\ldots,u_k,\partial u_\mu/\partial y_1,\ldots,\partial u_\mu/\partial y_n)\qquad(\mu=1,2,\ldots,k)$$

for the system of functions $u_1,\ldots,u_k$, and for the differential inequalities of the form

$$\partial u_\mu/\partial x_a\leqslant f_a^\mu(x_1,\ldots,x_s,y_1,\ldots,y_n,u_1,\ldots,u_k,\partial u_\mu/\partial y_1,\ldots,\partial u_\mu/\partial y_n)\tag{2.1}$$
$$(\mu=1,2,\ldots,k,\ a=1,2,\ldots,s).$$

To investigate the first case it is convenient to introduce the following assumptions:

Assumptions S. The functions f_μ $(\mu=1,2,\ldots,k)$ are of class C^3 in the set

$$x^0\leqslant x<x^0+C,\quad |y_i-y_i^0|\leqslant C,\quad |z_\mu-z_\mu^0|\leqslant C,\quad |q_i^\mu-q_i^{0\mu}|\leqslant C\tag{2.2}$$

and the absolute values of f_μ and of their derivatives of the first, of the second and of the third order are less than M in (2.2).

The functions $\omega_\mu(y_1,\ldots,y_n)$ are of class C^3 in the cube $|y_i-y_i^0|\leqslant C$ and the absolute values of their derivatives are less than M in that cube.

We have

$$|q_i^{0\mu}|<M\quad\text{for}\quad i=1,2,\ldots,n,$$
$$|\omega_\mu(y_1^0,\ldots,y_n^0)-z_\mu^0|<C/4,\quad \mu=1,2,\ldots,k,$$
$$\left|\frac{\partial\omega_\mu(y_1^0,\ldots,y_n^0)}{\partial y_i}-q_i^{0\mu}\right|<C/4,\quad \mu=1,2,\ldots,k,\ i=1,2,\ldots,n.$$

Remark 1. The assumptions S guarantee ([2], theorem (W) and [3], theorem 1.2) the existence of the unique solution $v_1,\ldots,v_k$ of the system of equations

$$\partial z_\mu/\partial x=f_\mu(x,y_1,\ldots,y_n,z_1,\ldots,z_k,\partial z_k/\partial y_1,\ldots,\partial z_\mu/\partial y_n)\tag{2.3}$$

satisfying the initial conditions

$$v_\mu(x^0,y_1,\ldots,y_n)\equiv\omega_\mu(y_1,\ldots,y_n)\tag{2.4}$$

in the cube $|y_i-y_i^0|\leqslant C[4n(N+1)]^{-1}$.

The solution $v_1,\ldots,v_k$ is valid in the pyramid

$$x^0\leqslant x<x^0+\delta,\quad |y_i-y_i^0|\leqslant C[4n(N+1)]^{-1}-N(x-x^0)\tag{2.5}$$
$$(i=1,2,\ldots,n)$$

where δ and N are suitable constants (see [3], theorem 1.2) depending only on C,n,M.

Using the same notation as in remark 1 we formulate the following

THEOREM 3. *Suppose the assumptions* S *to be fulfilled. Let* $v_1,\ldots,v_k$ *be the solution of* (2.3) *possessing continuous derivatives in* (2.5) *and satisfying* (2.4). *We assume that the functions* $u_1,\ldots,u_k$ *and* $\varepsilon_1,\ldots,\varepsilon_k$ *are continuous in* (2.5) *and* $\varepsilon_\mu>0$, *the inequalities*

$$u_\mu(x^0,y_1,\ldots,y_n)\leqslant v_\mu(x^0,y_1,\ldots,y_n)\qquad(\mu=1,2,\ldots,k)$$

being satisfied in the cube $|y_i-y_i^0|\leqslant C[4n(N+1)]^{-1}$. *We assume that the right members of* (2.3) *satisfy the following condition* (W):

If $z_i\geqslant\bar{z}_i$ *for* $i=1,\ldots,\mu-1,\mu+1,\ldots,k$ *and* $z_\mu=\bar{z}_\mu$, *then*

$$f_\mu(x,y_1,\ldots,y_n,z_1,\ldots,z_k,q_1,\ldots,q_n)\geqslant f_\mu(x,y_1,\ldots,y_n,\bar{z}_1,\ldots,\bar{z}_k,q_1,\ldots,q_n).$$

We suppose furthermore that the following epidermic condition is true: Let μ *be an arbitrary number of the sequence* $1,2,\ldots,k$. *If, for a point* $Q(x,y_1,\ldots,y_n)$ *of* (2.5), *the inequality* $v_\mu(Q)<u_\mu(Q)<v_\mu(Q)+\varepsilon_\mu(Q)$ *is satisfied, then there exists in* Q *the total differential of* $u_\mu(P)$ *and the inequality*

$$\partial u_\mu(Q)/\partial x\leqslant f_\mu\big(Q,u_1(Q),\ldots,u_k(Q),\partial u_\mu(Q)/\partial y_1,\ldots,\partial u_\mu(Q)/\partial y_n\big)$$

holds.

Under the assumptions given above the inequalities $u_\mu(P)\leqslant v_\mu(P)$ $(\mu=1,2,\ldots,k)$ *are fulfilled for every point* P *of the set* (2.5).

The proof of this theorem is quite similar to the proof of theorem 1. A suitable lemma ([3], theorem 1.1), analogous to lemma K, is used. Just in that lemma condition (W) is necessary. The theorem of existence guarantees the solutions to depend continuously on the initial conditions in a suitable Haar's pyramid.

By means of theorem 3, one can easily obtain the epidermic theorem concerning the inequalities of the form (2.1). The idea is the same as in the case of theorem 2, it is based on the application of a suitable Mayer's transformation (see [3], proof of theorem 2.1). In this note we have discussed the case of the right-hand Haar's pyramids. Analogous theorems may be proved for left-hand pyramids the direction of the differential inequalities being changed.

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Teilweise Lösung eines verallgemeinerten Problems von K. Zarankiewicz

von K. Čulík (Brno)

Das Problem von K. Zarankiewicz[1]) kann man im allgemeinen Falle auf folgende Weise formulieren: Mit $A_n^m(p)$ bezeichnen wir einen Matrix, die m Reihen und n Spalten hat und die aus p Elementen die gleich 0 und aus $mn-p$ Elementen die gleich 1 sind, gebildet ist. Wir sagen, daß eine solche Matrix die Eigenschaft $\xi(i,j)$ hat, wenn ihre Submatrix $P_j^i(ij)$, wobei $1\leqslant i,j$ und $i\leqslant m$, $j\leqslant n$ ist, existiert. Für gegebene i,j und m,n besteht das verallgemeinerte Problem in der Bestimmung der minimalen natürlichen Zahl p, für welche jede Matrix $A_n^m(p)$ die Eigenschaft $\xi(i,j)$ besitzt. Damit aber ist eine Funktion $Z_{ij}(m,n)$ für alle natürliche Zahlen $i,j\geqslant 1$, $m\geqslant i$, $n\geqslant j$ definiert.

Wir wollen noch bemerken, daß die Eigenschaft $\xi(i,j)$ einer Matrix gegenüber dem Austausch ihrer Reihen untereinander, sowie auch ihrer Spalten invariant ist. Wenn die x-te Reihe (y-te Spalte) der Matrix $A_n^m(p)$ gerade $r_x(s_y)$ Nullen enthält, so kann man sich in der folgenden Untersuchung nur auf solche Matrizen beschränken, die die Bedingungen $r_1\geqslant r_2\geqslant\ldots\geqslant r_m$, $s_1\geqslant s_2\geqslant\ldots\geqslant s_n$ erfüllen. Diese Tatsache, daß die Reihen oder Spalten in einem gewissen Sinne geordnet sind, wird in wesentlicher Weise ausgenützt.

Für beliebige natürliche Zahlen i,j,m, $(1\leqslant i,j,\ i\leqslant m)$, und jedes genügend große n beweisen wir ein ganz allgemeines Resultat:

Satz. $Z_{ij}(m,n)=(i-1)n+(j-1)\binom{m}{i}+1$ *für* $n\geqslant(j-1)\binom{m}{i}$.

Beweis. Nach Definition der Funktion $Z_{ij}(m,n)$ genügt es den Satz nur für alle $n\geqslant\max\left[j,(j-1)\binom{m}{i}\right]$ zu beweisen. Man sieht leicht ein, daß $j>(j-1)\binom{m}{i}$ dann und nur dann ist, wenn wenigstens eine der folgenden zwei Bedingungen $j=1$, $m=i$ erfüllt ist. Für $i=1$ ist es klar, daß jede Matrix $A_n^m\{(j-1)m+1\}$ die Eigenschaft $\xi(1,j)$ hat und die Matrix $A_n^m\{(j-1)m\}$, die in jeder von ihren m Reihen gerade $j-1$ Nullen

[1]) Colloquium Mathematicum 2 (1951), S. 301, Problem 101.