D'après (20), (21) et (15) il résulte

$$|\varphi_1-\varphi_0| < |\lambda| A^2 \frac{|z-z_0|^2}{2} + |\lambda| A^2 \frac{|\zeta-\zeta_0|^2}{2} < |\lambda| A^2 \frac{[|z-z_0|+|\zeta-\zeta_0|]^2}{2!},$$

$$|\varphi_2-\varphi_1| < 2|\lambda|^2 A^2(A+B) \frac{[|z-z_0|+|\zeta-\zeta_0|]^3}{3!},$$

$$|\varphi_3-\varphi_2| < 2^3|\lambda|^3 A^2(A+B)B \frac{[|z-z_0|+|\zeta-\zeta_0|]^4}{4!}$$

où en général

$$|\varphi_{n+1}-\varphi_n| < 2^{2n-1}|\lambda|^{n+1}(A+B)A^2B^{n-1} \frac{[|z-z_0|+|\zeta-\zeta_0|]^{n+2}}{(n+2)!}.$$

L'expression $[|z-z_0|+|\zeta-\zeta_0|]^{n+2}/(n+2)!$ est le terme du developpement en série de la fonction $\exp(|z-z_0|+|\zeta-\zeta_0|)$ convergente pour toute valeur de $|z-z_0|+|\zeta-\zeta_0|$. Nous pouvons tirer de ce qui précéde la conclusion suivante: La série $\sum_{n=0}^{\infty}(\varphi_{n+1}-\varphi_n)$ est uniformement convergente, et la suite $\{\varphi_n\}$ est convergente. De la même façon nous pouvons montrer la convergence des séries

$$\sum_{n=0}^{\infty}(u_{n+1}-u_n), \quad \sum_{n=0}^{\infty}(v_{n+1}-v_n), \quad \sum_{n=0}^{\infty}(w_{n+1}-w_n)$$

et des suites: $\{u_n\}$, $\{v_n\}$, $\{w_n\}$. Les fonctions limites

$$\varphi(z,\zeta) = \lim \varphi_n(z,\zeta), \qquad v(z,\zeta) = \lim v_n(z,\zeta),$$
$$u(z,\zeta) = \lim u_n(z,\zeta), \qquad w(z,\zeta) = \lim w_n(z,\zeta)$$

donnent la solution du système (16). La preuve que la solution précédente du système (16) est unique, sera la même, que dans la méthode des approximations successives en général. La fonction $\varphi(z,\zeta)$ satisfaisant au système (16) satisfait de même aux équations (13), (12) et (11). La fonction $\Psi(x,y)=\varphi(z,\zeta)$ donne la solution de l'équation proposée.

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Some properties of plane sets with positive transfinite diameter

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The main object of this work is to strenghten Kellog's lemma, which states that if the boundary of the domain containing ∞ has the positive transfinite diameter [1], then there exists on this boundary a point which is regular for its Green's function. The first part contains some theorems from the general theory of the integral. The theorems in the second part are not new, but the method of the proofs seems to be new. It is shown that a Green's function constructed by the extreme points-method [5] is equal to that constructed by Frostman's "masse du balayage" [2]. The third part considers the so called "polynomial condition" ([4], [6]) and, as its consequence, the above lemma of Kellog.

I. The following two theorems are given without proof. The proofs are to be found in Frostman's paper [2].

Let X be any set in the Cartesian space, $\mathfrak{X}$ denote the class of Borel sets. Let $\{\mu_n\}$ be a sequence of measures on $\mathfrak{X}$. Let $\{f_n\}$ be a sequence of continuous functions converging to any function f. Then we have

THEOREM 1. *If the set $\{\mu_n(X)\}$ is bounded, there exists a subset $\{\mu_{n_k}\}$ converging*[1] *to some measure μ.*

THEOREM 2. *If the sequences $\{\mu_n\}$ and $\{f_n\}$ are uniformly bounded on X, $\mu_n\to\mu$, $f_n\to f$ uniformly, then*

$$\lim_{n\to\infty}\int_X f_n\,d\mu_n = \int_X f\,d\mu. \tag{1}$$

I shall prove

THEOREM 3. *If the functions f_n are lower uniformly bounded on X, $f_n\geqslant M$, continuous, and for every $\varepsilon>0$ there exists a set $e=e_\varepsilon$ such that*

(i) $\mu(e)=\lim\limits_{n\to\infty}\mu_n(e)<\varepsilon$ *and*

(ii) *all $|f_n|$, except a finite number, are uniformly bounded on $X-e$, then*

$$\varliminf_{n\to\infty}\int_X f_n\,d\mu_n \geqslant \int_X f\,d\mu. \tag{2}$$

[1]) For the definition of this convergence cf. [2].

Proof. Let ε be a positive number and e a suitable set, $\mu(e)<\varepsilon$.

$$\int_X f_n d\mu_n = \int_{X-e} f_n d\mu_n + \int_e f_n d\mu_n \geqslant \int_{X-e} f_n d\mu_n + M\mu_n(e).$$

Since $|f_n|$ for all sufficiently large n are uniformly bounded on $X-e$, we have

$$\lim_{n\to\infty} \int_{X-e} f_n d\mu_n = \int_{X-e} f d\mu.$$

Therefore

$$\underline{\lim}_{n\to\infty} \int_X f_n d\mu_n \geqslant \int_{X-e} f d\mu + M\varepsilon.$$

In view of ε being arbitrarily small, we obtain our theorem.

THEOREM 4. *If by the conditions of the above theorem there exists a set e such that* (i) *and* (ii) (of Theorem 3) *hold, and*

(iii) $\overline{\lim}_{n\to\infty} \int_e f_n d\mu_n < \varepsilon$, *then*

(3) $$\lim_{n\to\infty} \int_X f_n d\mu_n = \int_X f d\mu.$$

Proof.

$$\overline{\lim}_{n\to\infty} \int_X f_n d\mu_n \leqslant \overline{\lim}_{n\to\infty} \int_{X-e} f_n d\mu_n + \overline{\lim}_{n\to\infty} \int_e f_n d\mu_n \leqslant \int_{X-e} f d\mu + \varepsilon.$$

In view of the opposite inequality (2) and of ε being arbitrary small, we obtain our theorem.

II. Let E be a set closed and bounded in z-plane. Denote by M_E a class of all measures μ for which the following properties hold:

(i) $\mu(E)=1$, $\mu(e)=0$ if $eE=0$,

(ii) the Borel sets (in z-plane) are measurable.

In the Cartesian product $E\times E$ introduce the measure $\mu\times\mu$ [3]. We consider with Frostman

(4) $$I(\mu) = \int_E\int_E \log|z-\xi|^{-1} d\mu(z)\, d\mu(\xi) = \int_{E\times E} \log|z-\xi|^{-1} d(\mu\times\mu)(z,\xi).$$

It is proved [2] that in M_E there exists a measure, denoted by η, which realises

(5) $$\inf_{\mu\epsilon M_E} I(\mu)$$

and η is unique. We shall write $I(\eta) = \inf_{\mu\epsilon M_E} I(\mu) = \gamma_E$.

Further we assume that $\gamma_E<\infty$.

Let $\eta^n = \{\eta_0^n, \eta_1^n, \ldots, \eta_n^n\}$ be the nth extreme system of the set E (*i. e.* a system of points of E such that the product

$$\prod_{\substack{i<k\\ 0\leqslant i,k\leqslant n}} |\eta_i^n - \eta_k^n|$$

is as large as possible). Then the transfinite diameter $d(E)$ of E is

(6) $$d(E) = \lim_{n\to\infty} \Big(\prod_{i<k} |\eta_i^n - \eta_k^n|\Big)^{2/n(n+1)}.$$

It has been proved by G. Szegö [7] that $d(E)=e^{-\gamma_E}$. Hence

(7) $$\gamma_E = \log\frac{1}{d(E)} = \lim_{n\to\infty}\Big(\frac{2}{n(n+1)} \sum_{i<k} \log|\eta_i^n - \eta_k^n|^{-1}\Big)$$
$$= \lim_{n\to\infty} \sum_{i\neq k} \log|\eta_i^n - \eta_k^n|^{-1}(n+1)^{-2}.$$

Let μ_n be a measure which is equal to $(n+1)^{-1}$ at the points $\eta_0^n, \ldots, \eta_n^n$, 0 except. Evidently $\mu_n \epsilon M_E$. In $E\times E$ we introduce the measure $\mu_n\times\mu_n$. Then $\mu_n\times\mu_n$ is $(n+1)^{-2}$ for every pair (η_i^n, η_k^n) of extreme points, 0 except.

Let us put

(8) $$L(z,\xi) = \begin{cases} \log|z-\xi|^{-1} & \text{if} \quad z\neq\xi, \\ 0 & \text{if} \quad z=\xi. \end{cases}$$

Because of the assumption $\gamma_E = \int_E\int_E \log|z-\xi|^{-1} d\eta(\xi)\, d\eta(z) < \infty$, the measure η of every single point is 0 and (for $L(z,\xi) = \log|z-\xi|^{-1}$ except a set of η-measure 0)

(9) $$I(\eta) = \int_E\int_E L(z,\xi)\, d\eta(\xi)\, d\eta(z) = \int_{E\times E} L(z,\xi)\, d(\eta\times\eta).$$

It is evident that

(10) $$\sum_{i\neq k} \log|\eta_i^n - \eta_k^n|^{-1}(n+1)^{-2} = \int_{E\times E} L(z,\xi)\, d(\mu_n\times\mu_n).$$

We shall prove

THEOREM 5. *The sequence of measures μ_n defined above is convergent to the measure η realising the lower bound* (5).

Proof. By theorem 1 $\{\mu_n\}$ contains the convergent subsequence $\{\mu_{n_k}\}$. Write $\lim_{k\to\infty} \mu_{n_k} = \mu$. By (10) and (7) we have

$$\gamma_E = \lim_{n\to\infty} \int_{E\times E} L(z,\xi)\, d(\mu_n\times\mu_n).$$

By theorem 3, in view of μ of every single point being 0, we have

$$\gamma_E = \lim_{k\to\infty} \int_{E\times E} L(z,\xi)\, d(\mu_{n_k}\times\mu_{n_k}) \geqslant \int_{E\times E} L(z,\xi)\, d(\mu\times\mu)$$

$$\geqslant \int_{E\times E} L(z,\xi)\, d(\eta\times\eta) = I(\eta) = \gamma_E.$$

Hence $I(\mu)=I(\eta)$.

Since η is unique, $\mu=\eta$ and μ $(=\lim_k \mu_{n_k})$ is independent of the choice of the convergent subsequence. Therefore μ_n is convergent and $\lim \mu_n=\eta$.

Denote by F the set of accumulation points of the set $\{\eta_j^n\}$ $(n=1, 2, \ldots,\ 0\leqslant j\leqslant n)$. It is known that F is the boundary of the domain D_∞, consisting of CE and containing the point ∞. By the above theorem $\eta(F)=1$ and $\eta(E-F)=0$.

THEOREM 6. *Let $\{\eta_{j_k}^{n_k}\}$ $(n_{k+1}>n_k)$ be a subsequence of extreme points converging to any point $z_0 \epsilon F$. Then*

$$\lim_{k\to\infty} \int_E L(\eta_{j_k}^{n_k},\xi)\, d\mu_{n_k}(\xi) = \int_E \log|z_0-\xi|^{-1} d\eta(\xi). \tag{11}$$

Proof. It is sufficient to prove that the conditions of theorem 4 hold, *i. e.* that for every $\varepsilon>0$ there exists a circle K_ϱ $\big(=\underset{z}{E}(|z-z_0|<\varrho)\big)$ such that

$$\overline{\lim_{k\to\infty}} \int_{K_\varrho} L(\eta_{j_k}^{n_k},\xi)\, d\mu_{n_k}(\xi) < \varepsilon.$$

Suppose the contrary: then by any $\{\eta_{j_k}^{n_k}\}$ converging to any $\varepsilon_0>0$, and every $\varrho>0$,

$$\overline{\lim_{k\to\infty}} \int_{K_\varrho} L(\eta_{j_k}^{n_k},\xi)\, d\mu_{n_k}(\xi) \geqslant \varepsilon_0.$$

I shall show that any subsequence of μ_{n_k} is not convergent to η. Put $\nu_k^{(\varrho)}(e)=\mu_{n_k}(e-K_\varrho)/\big(1-\eta(K_\varrho)\big)$ and

$$I(\mu)=\int_E\int_E L\, d\mu d\mu, \qquad I_{11}(\mu)=\int_{E-K_\varrho}\int_{E-K_\varrho} L\, d\mu d\mu,$$

$$I_{12}(\mu)=\int_{E-K_\varrho}\int_{K_\varrho} L\, d\mu d\mu, \qquad I_{22}=\int_{K_\varrho}\int_{K_\varrho} L\, d\mu d\mu.$$

Since $\eta=\lim \mu_{n_k}$ realises the lower bound (5), we have

$$\underline{\lim_{k\to\infty}} \big(I(\mu_{n_k})-I(\nu_k^{\varrho})\big)<0. \tag{12}$$

By ordinary computation we find

$$\overline{\lim_{k\to\infty}} \big(I(\mu_{n_k})-I(\nu_k^{(\varrho)})\big) \tag{13}$$

$$=\overline{\lim_{k\to\infty}} \left(\eta(K_\varrho)\frac{2-\eta(K_\varrho)}{\big(1-\eta(K_\varrho)\big)^2} I_{11}(\mu_{n_k}) + 2\frac{2-\eta(K_\varrho)}{1-\eta(K_\varrho)} I_{12}(\mu_{n_k}) - I_{22}(\mu_{n_k})\right)$$

$$=\eta(K_\varrho)\frac{2-\eta(K_\varrho)}{1-\eta(K_\varrho)} I_{11}(\eta) + 2\frac{2-\eta(K_\varrho)}{1-\eta(K_\varrho)} I_{12}(\eta) - I_{22}(\eta).$$

The first and the third members tend to 0 with $\eta(K_\varrho)\to 0$. Since

$$I_{12}(\eta)=\lim_{k\to\infty}\int_{E-K_\varrho}\int_{K_\varrho} L(z,\xi)\, d\mu_{n_k}(\xi)\, d\mu_{n_k}(z) \geqslant \big(1-\eta(K_\varrho)\big)\varepsilon_0 \tag{14}$$

(by hypothesis), if for ϱ sufficiently small the difference (13) is $>\varepsilon_0/2$, which contradicts to (12).

THEOREM 7. *The constant γ_E is exactly the upper bound of the function*

$$u(z)=\int_E \log|z-\xi|^{-1} d\eta(\xi). \tag{15}$$

Proof. Write

$$\Delta^j(\eta^n)=\prod_{\substack{i\neq j\\ 0\leqslant i\leqslant n}} |\eta_j^n-\eta_i^n| \quad\text{and}\quad \Delta_n^0=\min_{0\leqslant j\leqslant n}\Delta^j(\eta^n)=\prod_{i=1}^n |\eta_0^n-\eta_i^n|.$$

It is known [4] that $\lim_{n\to\infty}\sqrt[n]{\Delta_n^0}=d(E)$. Hence

$$\lim_{n\to\infty}\left(\frac{1}{n+1}\sum_{i=1}^n \log|\eta_0^n-\eta_i^n|^{-1}\right) = -\log d(E)=\gamma_E. \tag{16}$$

The points η_0^n have on F a point of accumulation z_0. Let $\eta_0^{n_k}\underset{k\to\infty}{\to} z_0$. By theorem 6 and formula (16)

$$\gamma_E=\lim_{k\to\infty}\frac{1}{n_k+1}\sum_{i=1}^{n_k}\log|\eta_0^{n_k}-\eta_i^{n_k}|^{-1}=\lim_{k\to\infty}\int_E L(\eta_0^{n_k},\xi)\, d\mu_{n_k}(\xi)$$

$$=\int_E L(z_0,\xi)\, d\eta(\xi)=\int_E \log|z_0-\xi|^{-1} d\eta(\xi).$$

For any $z'\epsilon F$ we take any sequence $\{\eta_{j_k}^{n_k}\}$ converging to z'. By (14) we have

$$\int L(\eta_{j_k}^{n_k},\xi)\, d\mu_{n_k}(\xi) \leqslant \int_E L(\eta_0^{n_k},\xi)\, d\mu_{n_k}(\xi) \underset{k\to\infty}{\to} \gamma_E.$$

Hence, in the limit $\int_E \log|z'-\xi|^{-1} d\eta(\xi)\leqslant\gamma_E$, q. e. d.

Denote by F_0 a subset of F such that for $z\in F_0$ we have

$$u(z)=\int_E \log|z-\xi|^{-1}d\eta(\xi)=\gamma_E.$$

THEOREM 8. *F_0 is a G_δ-set and $\eta(F-F_0)=0$.*

Proof. By Fatou's lemma if any sequence of points $z_k \underset{k\to\infty}{\to} z_0$, then

$$\varliminf_{k\to\infty} u(z_k)=\varliminf_{k\to\infty}\int_E \log|z_k-\xi|^{-1}d\eta(\xi)\geqslant u(z_0).$$

Hence $u(z)$ is lower semi-continuous, and thus F_0 is a G_δ-set.

In view of theorem 5 and using Fubini's theorem, we have

$$0=\int_F\int_F \log|z-\xi|^{-1}d\eta(\xi)\,d\eta(z)-\gamma_E=\int_F\big(u(z)-\gamma_E\big)d\eta(z)$$

$$=\int_{F-F_0}\big(u(z)-\gamma_E\big)d\eta(z).$$

Since $u(z)-\gamma_E<0$ on $F-F_0$, $\eta(F-F_0)$ must be 0.

COROLLARY 1. *The transfinite diameter of F_0 (defined by the integral* (5) *and formula $d(E)=e^{-\gamma_E}$) is equal to that of E.*

Since $F_0\subset F\subset E$, we have $d(F_0)\leqslant d(E)$. But for $\eta(E-F_0)=\eta(E-F)+ +\eta(F-F_0)=0$

$$\log\frac{1}{d(E)}=\int_{E\times E}\log|z-\xi|^{-1}d(\eta\times\eta)=\int_{F_0\times F_0}\log|z-\xi|^{-1}d(\eta\times\eta)\geqslant\log\frac{1}{d(F_0)};$$

therefore $d(E)=d(F_0)$.

COROLLARY 2. *The power of F_0 is greater than denumerable, for in the contrary case its transfinite diameter would be* 0.

Denote by μ_n^j a measure which is equal $1/n$ at the points $\{\eta_i^n\}$ $(i\neq j)$ (*i. e.* excluding the point η_j^n) and 0 except. Evidently also $\mu_n^j\to\eta$ as $\mu_n\to\eta$, with $n\to\infty$. Write

$$L^j(z,\eta^n)=\prod_{j\neq i}(z-\eta_i^n)/(\eta_j^n-\eta_i^n),$$

which is Lagrange's interpolatory polynomial. It is known [4] that for $z\in D_\infty$

$$\lim_{n\to\infty}\sqrt[n]{|L^0(z,\eta^n)|}\overset{\mathrm{df}}{=}\Phi(z)$$

exists, and $\log\Phi(z)$ *is the Green's function of the domain D_∞ generalized in the sense of its being continuous and equal to* 0 *at every point z_0 which belongs, to any boundary continuum.*

Evidently

$$\log\sqrt[n]{|L^0(z,\eta^n)|}=-\int_E\log|z-\xi|^{-1}d\mu_n^0(\xi)+\log(\Delta_n^0)^{-1/n}.$$

In the limit, for $n\to\infty$, we obtain

$$\log\Phi(z)=\gamma_E-\int_E\log|z-\xi|^{-1}d\eta(\xi)$$

and this last is the *Green's function constructed by Frostman's "masse du balayage"*, which is equal to the Green's function of D_0 generalized in Wiener's sense. We have obtained

THEOREM 9. *The function $\log\Phi(z)$ of Leja is the Green's function $G(z,\infty)$ of the domain D_∞ generalized in Wiener's sense.*

III. Let $\{P_n(z)\}$ be any set of polynomials such that the degree of P_n is not larger than n. A denotes any set in the z-plane. Assume that all $|P_n(z)|$ are uniformly bounded on A, $|P_n(z)|\leqslant M$ $(n=1,2,\ldots)$.

We say that *for any point $z_0\in A$ the polynomial condition holds if and only if for every $\varepsilon>0$ the inequality $|P_n(z)|\leqslant M$ $(z\in A,\ n=1,2,\ldots)$ implies*

$$\varlimsup_{n\to\infty}|P_n(z)|^{1/n}\leqslant 1+\varepsilon$$

in some neighbourhood K_δ of z_0.

THEOREM 10. *Let E be a set with a positive transfinite diameter. For every point $z_0\in F_0$ (F_0 defined above) the polynomial condition holds.*

Proof. Let $\eta^n=\{\eta_0^n,\eta_1^n,\ldots,\eta_n^n\}$ denote the nth system of extreme points of the set E. Put

$$L^j(z,\eta^n)=\prod_{i=j}(z-\eta_i^n)(\eta_j^n-\eta_i^n)^{-1}. \tag{17}$$

These are the Lagrange interpolatory polynomials. To begin with I shall prove the theorem for these polynomials. It is easy to show [4] that on E

$$|L^j(z,\eta^n)|\leqslant 1\quad \text{for}\quad n=1,2,\ldots,\quad 0\leqslant j\leqslant n.$$

LEMMA 1. *If we put*

$$\sigma_j^n=\log[\Delta^j(\eta^n)]^{-1/n}=\int_E\log|\eta_j^n-\xi|^{-1}d\mu_n^j(\xi),$$

then for every j, $0\leqslant j\leqslant n$, and for all sufficiently large j and n we have

$$\sigma_j^n<\gamma_E+\varepsilon. \tag{18}$$

Proof of the lemma 1. Clearly

$$\sigma_j^n=\int_E L(\eta_j^n,\xi)\,d\mu_n(\xi)(n+1)n^{-1}.$$

Suppose the existence of some $\varepsilon_0>0$ such that for any sequence $\eta_{j_k}^{n_k}$ $(0\leqslant j_k\leqslant n_k)$ we have $\sigma_{j_k}^{n_k}\geqslant\gamma_E+\varepsilon_0$. We choose from $\eta_{j_k}^{n_k}$ any convergent subsequence $\eta_{j_l}^{n_l}$, say $\eta_{j_l}^{n_l}\to z'$. Then, by theorem 6,

$$\lim_{l\to\infty}\sigma_{j_l}^{n_l}=\lim_{l\to\infty}(n_l+1)n_l^{-1}\int_E L(\eta_{j_l}^{n_l},\xi)\,d\mu_{n_l}(\xi)=\int_E L(z',\xi)\,d\eta(\xi)$$

and by theorem 7 it is $\leqslant\gamma_E$. This contradicts the hypothesis (18).

Lemma 2. *If we put*

$$\lambda_j^n(z)=n^{-1}\sum_{i\neq j}\log|z-\eta_i^n|,$$

then for every $\varepsilon>0$ and for every $z_0\epsilon F_0$ there exists a $\delta=\delta(\varepsilon,z_0)$ such that, in the circle K_δ $\big(=\underset{z}{E}(|z-z_0|<\delta)\big)$, for all sufficiently large n we have $\lambda_j^n<-\gamma_E+\varepsilon$.

Proof of the lemma 2. In the contrary case there would exist a sequence of points z_k converging to z_0 and two sequences of indices $\{n_k\}$, $\{j_k\}$ $(0\leqslant j_k\leqslant n_k)$ and $\varepsilon_0>0$ such that

$$\lambda_{j_k}^{n_k}(z_k)\geqslant-\gamma_E+\varepsilon_0. \tag{19}$$

Clearly

$$\overline{\lim_{k\to\infty}}\,\lambda_{j_k}^{n_k}(z_k)=\overline{\lim_{k\to\infty}}\int_E -L(z_k,\xi)\,d\mu_{n_k}^{j_k}(\xi)\leqslant\int_E -L(z_0,\xi)\,d\eta(\xi)=-\gamma_E$$

by theorem 3. (It should be noted that here the inequality must be opposite to that in theorem 3.) This is a contradiction of the above hypothesis (19).

Assuming both lemmas we obtain:

For every $\varepsilon'>0$ and every $z_0\epsilon F_0$ there exist some $N_{\varepsilon'}$ and $\delta>0$ such that for $|z-z_0|<\delta$ and $n>N_{\varepsilon'}$

$$\lambda_j^n(z)+\sigma_j^n<\varepsilon' \quad\text{ or }\quad n^{-1}\sum_{i\neq j}\log\left(|z-\eta_i^n||\eta_j^n-\eta_i^n|^{-1}\right)<\varepsilon'.$$

Hence

$$|L^j(z,\eta^n)|^{1/n}<e^{\varepsilon'}\quad\text{ if }\quad |z-z_0|<\delta,\quad n>N_{\varepsilon'},\quad 0\leqslant j\leqslant n.$$

Now, if we have any positive ε, we can take $\varepsilon'>0$ such that $e^{\varepsilon'}\leqslant1+\varepsilon$ and take the appropriate N and δ such that with $|z-z_0|<\delta$ and $n>N$ and for $0\leqslant j\leqslant n$

$$|L^j(z,\eta^n)|^{1/n}<1+\varepsilon,$$

which directly gives the assertion of the theorem for the extreme polynomials (17).

Yet, having an arbitrary set of polynomials $\{P_n(z)\}$, the degree of P_n being $\leqslant n$, we interpolate $P_n(z)$ by the extreme polynomials

$$P_n(z)=\sum_{j=0}^{n}P_n(\eta_j^n)L^j(z,\eta^n). \tag{20}$$

If there exists a constant M such that $|P_n(z)|\leqslant M$ on E, then by (20)

$$|P_n(z)|\leqslant\sum_{j=0}^{n}|P_n(\eta_j)||L^j(z,\eta^n)|\leqslant(n+1)M\max_j|L^j(z,\eta^n)|.$$

Assuming that $[(n+1)M]^{1/n}\to1$, we immediately obtain

$$\overline{\lim_{n\to\infty}}|P_n(z)|^{1/n}\leqslant1+\varepsilon\quad\text{ for }\quad|z-z_0|<\delta,$$

which proves our theorem.

Corollary 3. As has been shown above, *the function*

$$G(z_1)=\log\Phi(z)=\lim_{n\to\infty}\log|L^0(z,\eta^n)|^{1/n}\qquad(z\,\bar\epsilon\,F)$$

constructed by Leja [4] *is the generalized Green's function of D_∞ with the pole ∞.*

For this function F_0 is the set of regular points, because $G(z)$ is continuous and $=0$ at 0 if and only if $z_0\epsilon F_0$. This may be deduced directly from the semicontinuity of $G(z)$ and theorem 7. Then the sufficient and necessary condition for z_0 to be a regular point is that the polynomial condition hold for z_0. In view of the above and corollary 2 to theorem 8 we obtain as a consequence:

If the transfinite diameter of the plane set E is positive, E contains a set F_0 of regular points (with respect to $G(z)$) and the power of F_0 is greater than denumerable.

Denote the set of irregular points by H. Because of the semicontinuity of $G(z)$, H is an F_σ-set. The transfinite diameter of H (defined by $e^{-\gamma_H}$ where $\gamma_H=\inf_{\mu\epsilon M_H}I(\mu)$) is 0. In the contrary case, there would exist a closed set H_1 with positive distance from F_0. By theorem 7, corollary 2, $d(E)=d(F_0)$, and, as is well known, adjoining to F_0 any set of positive transfinite diameter would increase $d(F_0)$, which is absurd. Thus $d(H)=0$.

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Note on the mean value theorem

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This paper deals with the generalization of the mean value theorem. For strong topology in Banach spaces the mean value theorems have been proved by T. Ważewski [4]. In the case of weak topology a suitable theorem has been given by A. Alexiewicz [1].

By E we designate a linear topological locally convex space, *i. e.* a linear space in which a topology is introduced in such a fashion that the operations of addition and multiplication by real numbers are continuous in the topology, moreover the fundamental system of neighbourhoods of 0 is formed by convex sets. E^* denotes the class of all linear (additive and continuous) functionals defined on E.

For $f \epsilon E^*$ and real a we define the right (left) half-space $H^+(f,a)$ ($H^-(f,a)$) as a set of all $x \epsilon E$ for which we have the inequality $f(x) \geqslant a$ ($f(x) \leqslant a$). We have $H^-(f,a) = H(-f,-a)$. It suffices therefore to investigate the right half-spaces only.

Now we formulate the following lemma:

Lemma 1[1]. *Suppose that A is a closed and convex subset of E. Then A is a common part of all right half-spaces including A.*

By $\psi(t)$ and $\varphi(t)$ we shall denote real valued functions. We assume $\varphi(t)$ to be increasing and continuous in the given interval Δ.

Lemma 2[2]. *Let the function $\psi(t)$ be continuous in Δ and let*

$$\overline{\lim_{\tau\to 0+}}\{[\psi(t+\tau)-\psi(t)]/[\varphi(t+\tau)-\varphi(t)]\} = D^+_\varphi \psi(t) \geqslant a \qquad (a \textit{ real})$$

except an at most denumerable set of points of Δ. Then for $t_1, t_2 \epsilon \Delta$ and $t_1 \neq t_2$ we have the inequality $[\psi(t_1)-\psi(t_2)]/[\varphi(t_1)-\varphi(t_2)] \geqslant a$.

Let us assume that the function $x(t)$ with values lying in E, defined on Δ is weakly continuous in Δ, *i. e.* for every $f \epsilon E^*$ the real valued function $f[x(t)]$ is continuous in Δ. We formulate the mean value theorem.

[1]) See for instance [2], livre V, chapt. II, p. 73, col. 1.

[2]) This lemma may be proved in the same manner as in the case of $\varphi(t) = t$ and $a = 0$. See [3], p. 203.