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Note on the mean value theorem

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This paper deals with the generalization of the mean value theorem. For strong topology in Banach spaces the mean value theorems have been proved by T. Ważewski [4]. In the case of weak topology a suitable theorem has been given by A. Alexiewicz [1].

By E we designate a linear topological locally convex space, *i. e.* a linear space in which a topology is introduced in such a fashion that the operations of addition and multiplication by real numbers are continuous in the topology, moreover the fundamental system of neighbourhoods of 0 is formed by convex sets. E^* denotes the class of all linear (additive and continuous) functionals defined on E.

For $f \epsilon E^*$ and real a we define the right (left) half-space $H^+(f,a)$ ($H^-(f,a)$) as a set of all $x \epsilon E$ for which we have the inequality $f(x) \geqslant a$ ($f(x) \leqslant a$). We have $H^-(f,a) = H(-f,-a)$. It suffices therefore to investigate the right half-spaces only.

Now we formulate the following lemma:

Lemma 1 [1]). *Suppose that A is a closed and convex subset of E. Then A is a common part of all right half-spaces including A.*

By $\psi(t)$ and $\varphi(t)$ we shall denote real valued functions. We assume $\varphi(t)$ to be increasing and continuous in the given interval Δ.

Lemma 2 [2]). *Let the function $\psi(t)$ be continuous in Δ and let*

$$\overline{\lim_{\tau\to 0+}}\{[\psi(t+\tau)-\psi(t)]/[\varphi(t+\tau)-\varphi(t)]\} = D_\varphi^+\psi(t) \geqslant a \qquad (a \textit{ real})$$

except an at most denumerable set of points of Δ. Then for $t_1, t_2 \epsilon \Delta$ and $t_1 \neq t_2$ we have the inequality $[\psi(t_1)-\psi(t_2)]/[\varphi(t_1)-\varphi(t_2)] \geqslant a$.

Let us assume that the function $x(t)$ with values lying in E, defined on Δ is weakly continuous in Δ, *i. e.* for every $f \epsilon E^*$ the real valued function $f[x(t)]$ is continuous in Δ. We formulate the mean value theorem.

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[1]) See for instance [2], livre V, chapt. II, p. 73, col. 1.

[2]) This lemma may be proved in the same manner as in the case of $\varphi(t) = t$ and $a = 0$. See [3], p. 203.

THEOREM 1. *Suppose that A is a closed and convex subset of E. For every $f \in E^*$ there is an at most denumerable set $\Delta_f \subset \Delta$ such that for every $t \in \Delta - \Delta_f$ there exists a sequence $y_n \in A$ and a sequence of reals $\tau_n \to 0+$ such that*

$$f\{[x(t+\tau_n)-x(t)]/[\varphi(t+\tau_n)-\varphi(t)]-y_n\} \to 0. \tag{1}$$

Under the assumptions given above for $t_1, t_2 \in \Delta$ and $t_1 \neq t_2$

$$[x(t_1)-x(t_2)]/[\varphi(t_1)-\varphi(t_2)] \in A.$$

Proof. Denote by $H^+(f,a)$ an arbitrary right half-space including A. Relation (1) implies $D_\varphi^+ f[x(t)] \geqslant a$ for $t \in \Delta - \Delta_f$. Applying lemma 2 we get

$$f\{[x(t_1)-x(t_2)]/[\varphi(t_1)-\varphi(t_2)]\} \geqslant a \quad \text{for} \quad t_1, t_2 \in \Delta, \quad t_1 \neq t_2.$$

In other words

$$[x(t_1)-x(t_2)]/[\varphi(t_1)-\varphi(t_2)] \in H^+(f,a) \quad \text{for} \quad t_1, t_2 \in \Delta, \quad t_1 \neq t_2. \tag{2}$$

But (2) holds for every $H^+(f,a)$ including A. Then by lemma 1 the element $[x(t_1)-x(t_2)]/[\varphi(t_1)-\varphi(t_2)]$ belongs to A.

The essential moment in the proof given above is the fact that the relation of inclusion may be expressed by certain inequalities between numbers. On the same idea the following theorem is based:

THEOREM 2. *Suppose that for every $f \in E^*$ the function $f[x(t)]$ is absolutely continuous in the interval Δ. Let A be a closed and convex subset of E. For every $f \in E^*$ and for almost every $t \in \Delta$ there exists a sequence $y_n \in A$ and a sequence of reals $\tau_n \to 0$ such that*

$$f\{[x(t+\tau_n)-x(t)]/\tau_n - y_n\} \to 0. \tag{3}$$

Under the assumptions given above for $t_1, t_2 \in \Delta$, $t_1 \neq t_2$

$$[x(t_1)-x(t_2)]/(t_1-t_2) \in A.$$

Proof. For fixed f and a such that $A \subset H^+(f,a)$ the relation (3) implies the inequality $\omega_f'(t) \geqslant a$ for almost all $t \in \Delta$ $(\omega_f(t) = f[x(t)])$. The function $\omega_f(t)$ is absolutely continuous in Δ. Therefore

$$[\omega_f(t_1)-\omega_f(t_2)]/(t_1-t_2) = f\{[x(t_1)-x(t_2)]/(t_1-t_2)\} \geqslant a$$

for $t_1, t_2 \in \Delta$, $t_1 \neq t_2$. Now we apply lemma 1.

Now the role of lemma 1 and the classical theorems concerning the simple differential inequalities is quite clear. Different forms of the mean value theorems, some of which have been mentioned in this paper, may be used to generalize l'Hôpital's rule, just as it has been done in the case of Banach spaces ([1] and [4]).

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