ACTA ARITHMETICA
V (1958)

The construction of perfect and extreme forms I

by

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1. Introduction. Let

$$f(\boldsymbol{x}) = \sum_{1}^{n} a_{ij} x_i x_j \qquad (a_{ij} = a_{ji})$$

be a positive definite quadratic form of determinant D, and let M be the minimum of $f(\boldsymbol{x})$ for integral $\boldsymbol{x} \neq \boldsymbol{O}$. Then $f(\boldsymbol{x})$ assumes the value M for a finite number of integral $\boldsymbol{x} = \pm \boldsymbol{m}_1, \ldots, \pm \boldsymbol{m}_s$, called its *minimal vectors*.

$f(\boldsymbol{x})$ is said to be *perfect* if the s relations

$$f(\boldsymbol{m}_k) = \sum a_{ij} m_{ik} m_{jk} = M \qquad (k = 1, \ldots, s)$$

uniquely determine the $\frac{1}{2}n(n+1)$ distinct coefficients a_{ij} of f; i. e. if the equations

$$\sum b_{ij} m_{ik} m_{jk} = 0 \qquad (k = 1, \ldots, s) \qquad (b_{ij} = b_{ji})$$

have only the trivial solution $b_{ij} = 0$.

All classes of perfect forms are now known for $n \leqslant 6$, and a few classes are known for larger n. Their interest lies mainly in the facts that (i) they lead to a method of reduction of quadratic forms under integral unimodular transformation; and (ii) they include all extreme forms, i. e. those for which $M/D^{1/n}$ is a local maximum, and hence all absolutely extreme forms, for which $M/D^{1/n}$ assumes its greatest value γ_n.

Most known perfect and extreme forms are listed in Coxeter [5]; these include nearly all those previously published and several new types. All others, for $n \geqslant 7$, are: J_{12}, K_{12} given in [6]; T_9 of [7] which is equivalent to Φ_9 of [4]; Φ_{10} of [4]; the forms discussed in [1], which we shall denote here by M_n^r; and the forms discussed in [3], which we shall denote here by P_n.

The known methods of constructing perfect or extreme forms have shown themselves to be prohibitively laborious for large n. In particular we may mention Voronoi's algorithm [10] by which all perfect forms in a given number of variables may be found and which has been successfully applied for $n \leqslant 6$; and Minkowski's reduction [8], whereby all extreme forms appear as edges of a fundamental region in the coefficient-space.

With existing methods, therefore, one can at best hope to obtain a selection of the perfect forms for $n > 6$. It is for this reason that I present, in this and a succeeding paper, two new methods which yield large numbers of perfect forms with relatively little labour. Each method proceeds from a known perfect or extreme form and produces a new form, either by extending the range of values or by increasing the dimension of the known form.

In this article I treat only the first method, which will be described in detail in § 2. In §§ 3, 4 and 5 I describe briefly, for completeness, the forms A_n, B_n, L_n^r, M_n^r, P_n, Q_n (of which only L_n^r and Q_n do not appear in previous literature). Application of the method to some of these produces several new general classes of forms, of which $A_n^{t,q,r}$, $B_n^{t,q,r}$, A_n^t, B_n^t, $M_n^{r,2}$ are discussed in §§ 6-9.

The applications here make no pretence of completeness, and I have preferred to obtain classes of forms describable in terms of parameters rather than make a complete analysis of any particular form. However, even these forms considerably extend the table of extreme forms given in [5]; thus, for the early values of n, we find:

for $n = 7$, 7 extreme and 1 perfect (non-extreme) form;
for $n = 8$, 9 extreme and 2 perfect forms;
for $n = 9$, 13 extreme and 2 perfect forms;
for $n = 10$, 13 extreme and 3 perfect forms;
for $n = 11$, 18 extreme and 5 perfect forms;
for $n = 12$, 19 extreme and 6 perfect forms.

These numbers do not include Φ_{10} of [4], or K_{12} of [6]; these forms will be discussed in part II, where the above lists will be considerably extended.

2. Forms, lattices and refinements. If T is a regular $n \times n$ matrix, the points

$$\xi = Tx, \quad x \text{ integral}, \tag{2.1}$$

form a lattice Λ of determinant $d(\Lambda) = |\det T|$. A positive quadratic form $f(x)$ is said to have lattice $\Lambda = \Lambda(f)$, given by (2.1), if $f(x) = \xi'\xi = x'T'Tx$. Every positive form is representable as a sum of squares of linear forms, and may thus be associated (in infinitely many ways) with a lattice.

If U is an integral unimodular matrix, T and TU yield the same lattice (2.1) and correspond to the equivalent forms $x'T'Tx$ and $x'U'T'TUx$; thus a lattice determines a class of equivalent forms.

We shall sometimes find it convenient to extend this idea as follows. We say that f is the form g with lattice Λ, (2.1), if $f(x) = g(Tx)$. Then the values of f (or of any form equivalent to f) for integral x are precisely the values of $g(\xi)$ for $\xi \in \Lambda$. In this way, our forms will usually be representable by a simple form g, with Λ a sublattice of the integra lattice.

A form f with lattice Λ has minimum M if and only if Λ is admissible for the sphere S: $\xi'\xi \leqslant M$ (i. e. if Λ has no point other than the origin in the interior of the sphere) while some point of Λ lies on the boundary of the sphere. The minimal vectors of f, in ξ-coordinates, are simply the lattice points $\pm\xi_i = \pm Tm_i$ lying on the boundary of S.

With these notations, it follows easily that (i) f is *perfect* if there exists no quadratic cone containing all the minimal vectors of f; (ii) f is *extreme* if any sufficiently near lattice $\bar{\Lambda}$ which is admissible for S has $d(\bar{\Lambda}) \geqslant d(\Lambda)$.

If a lattice Λ is contained in a lattice Λ', we say that Λ is a sublattice of Λ' and Λ' a refinement of Λ. Correspondingly, we say that a form f' is a *refinement* of f if $\Lambda(f')$ is a refinement of $\Lambda(f)$. We restrict ourselves here to the case when Λ and Λ' are both n-dimensional (and the case when Λ has lower dimension than Λ' will be taken up in Part II). The basic results on which the method of this paper rests are now easily proved:

THEOREM 2.1. *Let f' be a refinement of f with the same minimum M. Then if f is perfect, so is f'; and if f is extreme, so is f'.*

Proof. The lattice Λ' of f' is a refinement of Λ, the lattice of f, and both Λ' and Λ are admissible for the sphere S: $\xi'\xi \leqslant M$. Also, since $M(f') = M(f) = M$, the minimal vectors of f' clearly include those of f.

If now f is perfect, there is no quadratic cone containing the minimal vectors of f, and *a fortiori* none containing those of f'; hence f' is perfect.

Suppose next that f is extreme, and consider any neighbouring lattice $\bar{\Lambda}'$ of Λ' which is admissible for S. The points of $\bar{\Lambda}'$ which correspond to points of Λ form a lattice $\bar{\Lambda}$; thus $\bar{\Lambda}$ is a neighbour of Λ and a sublattice of $\bar{\Lambda}'$. Also, if k is the density of Λ in Λ', then

$$k = \frac{d(\Lambda)}{d(\Lambda')} = \frac{d(\bar{\Lambda})}{d(\bar{\Lambda}')}. \tag{2.2}$$

Since f is extreme and $\bar{\Lambda}$ is admissible for S (since $\bar{\Lambda}'$ is), we have

$$d(\bar{\Lambda}) \geqslant d(\Lambda)$$

if $\bar{\Lambda}$ is sufficiently close to Λ. From (2.2), it follows that then $\bar{d}(\Lambda') \geqslant d(\Lambda')$, so that f' is extreme, as asserted.

We note that a refinement f' of a perfect non-extreme form f may well be extreme and not merely perfect. For this, it is easy to see that f' must have more minimal vectors than f, but there is no simple sufficient condition. In such cases we can apply Voronoi's direct criterion:

THEOREM 2.2 (Voronoi). *A perfect form $f(\boldsymbol{x})$ is extreme if and only if it is eutactic, i. e. if its adjoint $F(\boldsymbol{x})$ is expressible as*

$$F(\boldsymbol{x}) = \sum_{k=1}^{s} \varrho_k (\boldsymbol{m}_k' \boldsymbol{x})^2, \quad \varrho_k > 0$$

(*where $\boldsymbol{m}_1, \ldots, \boldsymbol{m}_s$ are the minimal vectors of $f(\boldsymbol{x})$*).

For typographical convenience, we shall adopt throughout the convention

$$m = n+1.$$

The n-dimensional integral lattice will be denoted by Γ_n, and the unit vectors in n-space by $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$. We shall follow as far as possible the notation for forms used in [5]: a capital roman letter for a form, with a suffix denoting the number of variables and superscripts denoting various integral parameters. Following [5], we also set

$$\Delta = \Delta(f) = \left(\frac{2}{M}\right)^n D$$

(where f has minimum M and determinant D); and

$$\Delta_n = \min_f \Delta(f),$$

so that $\Delta(f) = \Delta_n$ when f is any absolutely extreme form in n variables.

3. The forms A_n, B_m. We may represent A_n and B_m (with $m = n+1$) by

$$f(\boldsymbol{x}) = \sum_1^m x_i^2$$

with lattices the sublattices of Γ_m given by:

$$\Lambda(A_n): \quad \sum_1^m x_i = 0;$$

$$\Lambda(B_m): \quad \sum_1^m x_i \equiv 0 \pmod 2.$$

Each form then has minimum $M = 2$ and

$$\Delta(A_n) = n+1, \quad \Delta(B_m) = 4.$$

A_n has the $\frac{1}{2}n(n+1)$ minimal vectors $\boldsymbol{e}_i - \boldsymbol{e}_j$ $(1 \leqslant i < j \leqslant m)$ and is perfect for all n. B_m has the $m(m-1)$ minimal vectors $\boldsymbol{e}_i \pm \boldsymbol{e}_j$ $(1 \leqslant i < j \leqslant m)$ and is perfect for $m \geqslant 4$ (and $B_3 \sim A_3$).

These forms are all extreme, as is well known. In fact,

$$\sum_1^n y_i^2 + \sum_{i<j} (y_i - y_j)^2 = n \sum_1^n y_i^2 - 2 \sum_{i<j} y_i y_j$$

is the adjoint of $A_n(\boldsymbol{x}) = \sum_1^n x_i^2 + \left(\sum_1^n x_i\right)^2$; and

$$\sum_{i<j} (y_i - y_j)^2 + \sum_{i<j} (y_i + y_j)^2 = 2(m-1) \sum_1^m y_i^2$$

is a multiple of the adjoint $\sum_1^m y_i^2$ of $\sum_1^m x_i^2$; so that both forms are eutactic.

4. The forms L_m^r, M_n^r. We define L_m^r as the form

(4.1) $$f(\boldsymbol{x}) = \sum_{i=1}^{r} (x_i^2 - x_i x_{i+r} + x_{i+r}^2) + \sum_{k=2r+1}^{m} x_k^2 \quad (m \geqslant 2r)$$

with lattice the sublattice of Γ_m given by

(4.2) $$\sum_{i=1}^{m} x_i \equiv 0 \pmod 3.$$

M_n^r (with $m = n+1$) has the same definition, save that (4.2) is replaced by

(4.3) $$\sum_{i=1}^{m} x_i = 0.$$

Since f has determinant $(\frac{3}{4})^r$, we see that

$$D(L_m^r) = \frac{3^{r+2}}{2^{2r}}, \quad D(M_n^r) = \frac{3^r}{2^{2r}}(n+2r+1).$$

We shall also show that

(4.4) $$M(L_m^r) = 2, \quad \text{with} \quad \Delta = \frac{3^{r+2}}{2^{2r}}, \; s = \tfrac{1}{2}m(m-1) + \tfrac{1}{2}r(2m+r-7),$$

(4.5) $$M(M_n^r) = 2, \text{ with } \Delta = \frac{3^r}{2^{2r}}(n+2r+1), \; s = \tfrac{1}{2}n(n+1) + \tfrac{1}{2}r(r-3);$$

and that L_m^r is perfect if $r \geqslant 3$ or if $r = 2$ and $m \geqslant 5$, and that M_n^r is perfect if $r \geqslant 3$.

These results for M_n^r, with $n \geqslant 2r$, are established in [1], where it is also shown that M_n^r is eutactic only if $n \leqslant 4r-2$.

The forms L_m^r, M_n^r present a clear generalization of some known perfect forms in five and six variables. The forms A_5^3, and $\varphi_3, \varphi_4, \varphi_5$ of [2], [3] may in fact be represented as L_5^2, M_6^3, L_6^3 and L_6^2 respectively.

We set for convenience

$$\psi(x, y) = x^2 - xy + y^2.$$

Then $\psi \leqslant 2$ only when

$$\psi = 0, \quad x = y = 0,$$

or

$$\psi = 1, \quad \pm(x, y) \equiv (1,0), (0,1) \text{ or } (-1, -1);$$

and each of these last three sets satisfies $x+y \equiv 1 \pmod 3$.

If now $L_m^r(\boldsymbol{x}) \leqslant 2$, $\boldsymbol{x} \neq \boldsymbol{O}$, we must have *either*

(i) $\psi(x_i, x_{i+r}) = 0$ for $1 \leqslant i \leqslant r$, and then $L_m^r = 2$ when some two of $x_{2r+1}, \ldots, x_m$ are $1, -1$ and the rest are zero; *or*

(ii) $\psi(x_i, x_{i+r}) = 1$ for just one value of i, $1 \leqslant i \leqslant r$, say $i = 1$; then $\pm(x_1, x_{r+1}) = (1,0)$, $(0,1)$ or $(-1,-1)$ and so, by (4.2), some $x_k = \mp 1$ for $k > 2r$ and $L_m^r = 2$; *or*

(iii) $\psi(x_i, x_{i+r}) = 1$ for just two values of i, $1 \leqslant i \leqslant r$, say $i = 1, 2$; then $L_m^r = 2$ and $\pm(x_1, x_{r+1}) = (1,0)$, $(0,1)$ or $(-1,-1)$, $\mp(x_2, x_{r+2}) = (1,0), (0,1)$ or $(-1,-1)$, and all remaining x_i are zero.

This shows that $M(L_m^r) = 2$, as asserted, and the minimal vectors are:

$$\text{(4.6)} \qquad \boldsymbol{e}_i - \boldsymbol{e}_j \quad \big(1 \leqslant i < j \leqslant m,\ (i,j) \neq (1, r+1), \ldots, (r, 2r)\big),$$

$$\text{(4.7)} \qquad \boldsymbol{e}_i + \boldsymbol{e}_{i+r} - \boldsymbol{e}_j - \boldsymbol{e}_{j+r} \quad (1 \leqslant i < j \leqslant r),$$

$$\text{(4.8)} \qquad \boldsymbol{e}_i + \boldsymbol{e}_{i+r} + \boldsymbol{e}_j \quad (1 \leqslant i \leqslant r,\ j \neq i,\ i+r).$$

Thus

$$s = \binom{m}{2} - r + \binom{r}{2} + r(m-2),$$

which reduces to the result given in (4.4).

It now follows immediately, since the values assumed by M_n^r form a subset of those assumed by L_m^r, that also $M(M_n^r) = 2$, and that the minimal vectors of M_n^r are given by (4.6), (4.7). Thus the results (4.4), (4.5) are now established.

It is clear that L_m^r is not perfect if $r < 2$ or if $r = 2$, $m = 4$, since then $s < \frac{1}{2}m(m+1)$. Similarly M_n^r is not perfect if $0 < r \leqslant 2$ (while $M_n^0 = A_n$ and may be disregarded here).

To establish the perfection of these forms in all other cases, we begin by considering an arbitrary quadratic relation

$$\text{(4.9)} \qquad \sum_1^m p_{ij} x_i x_j = 0 \quad (p_{ij} = p_{ji})$$

satisfied by all the vectors (4.6), (4.7), which are minimal vectors of both L_m^r and M_n^r. We set

$$q_{ij} = q_{ji} = 2p_{ij} - p_{ii} - p_{jj} \quad (i \neq j).$$

Then, from the vectors (4.6), we have

$$q_{ij} = 0 \quad \text{for} \quad j \neq i+r, \quad 1 \leqslant i \leqslant r.$$

From (4.7), assuming as we may that $r \geqslant 2$, we obtain

$$q_{i,i+r} + q_{j,j+r} = 0 \quad (1 \leqslant i < j \leqslant r);$$

it follows that $q_{i,i+r} = 0$ for $1 \leqslant i \leqslant r$, provided that $r \geqslant 3$.

Thus if $r \geqslant 3$ the relation (4.9) must be of the form

$$\text{(4.10)} \qquad \Big(\sum_1^m x_i\Big)\Big(\sum_1^m p_{ii} x_i\Big) = 0.$$

The perfection of M_n^r for $r \geqslant 3$ follows at once. For L_m^r, we have the further minimal vectors (4.8) satisfying (4.10). These yield

$$p_{ii} + p_{i+r,i+r} + p_{jj} = 0 \quad (1 \leqslant i \leqslant r,\ j \neq i,\ i+r),$$

whence all $p_{ii} = 0$. Hence L_m^r is perfect for $r \geqslant 3$. For $r = 2$, $m \geqslant 5$, a similar analysis shows that L_m^r is again perfect.

We note in conclusion that L_m^r is eutactic (and so extreme if it is perfect) if and only if $m = 2r$ or $2r+1$. For the inverse of (4.1) is

$$\text{(4.11)} \qquad f^*(\boldsymbol{y}) = \frac{4}{3}\sum_{i=1}^r (y_i^2 + y_i y_{i+r} + y_{i+r}^2) + \sum_{k=2r+1}^m y_k^2.$$

If now $m \geqslant 2r+2$, $f^*(\boldsymbol{y})$ has zero coefficient of $y_{2r+1}y_{2r+2}$. Of the linear forms $y_i - y_j$, $y_i + y_{i+r} - y_j - y_{j+r}$, $y_i + y_{i+r} + y_j$ associated with the minimal vectors (4.6), (4.7) and (4.8), only the square of $y_{2r+1} - y_{2r+2}$ involves a term in $y_{2r+1}y_{2r+2}$. Hence in any expression of $f^*(\boldsymbol{y})$ as $\sum_{k=1}^s \varrho_k \lambda_k^2(\boldsymbol{y})$, the coefficient ϱ_k of $(y_{2r+1} - y_{2r+2})^2$ must be zero.

Thus L_m^r is not eutactic for $m \geqslant 2r+2$. A simple calculation shows that L_m^r is eutactic for $m \leqslant 2r+1$.

5. The forms P_n, Q_m. We define Q_m to be

$$f(x) = \sum_1^m x_i^2 \tag{5.1}$$

with lattice the sublattice of Γ_m given by

$$\sum_1^m x_i \equiv 0 \pmod 4, \tag{5.2}$$

$$\sum_1^m i x_i \equiv 0 \pmod m. \tag{5.3}$$

P_n has the same definition, save that (5.2) is replaced by

$$\sum_1^m x_i = 0. \tag{5.4}$$

Hence

$$D(Q_m) = 16m^2, \quad D(P_n) = m^3 = (n+1)^3.$$

The form P_n is discussed in [3], where it appears as a generalization of the new extreme senary form found in [2]. It is stated there that P_n has

$$M = 4, \quad \Delta = \frac{(n+1)^3}{2^n}, \quad s = \begin{cases} \frac{1}{8}(n-1)^2(n+1) & \text{for } n \text{ odd}, \\ \frac{1}{8}n(n+1)(n-2) & \text{for } n \text{ even}, \end{cases} \tag{5.5}$$

and is perfect and extreme for all $n \geqslant 6$. The proof that P_n is perfect and extreme is carried through in [3] only for even n; a similar proof holds for odd n, though it is rather more intricate.

We shall show here also that Q_m has

$$M = 4, \quad \Delta = \frac{m^2}{2^{m-4}}, \quad s = \begin{cases} \frac{1}{12}(m-1)(m-3)(2m-1) & \text{if } 2 \nmid m, \\ \frac{1}{12}(m-1)(m-2)(2m-3) & \text{if } 2|m,\ 4 \nmid m, \\ \frac{1}{12}(m-3)(2m^2-3m+4) & \text{if } 4|m. \end{cases} \tag{5.6}$$

From the fact that P_n is perfect for $n \geqslant 6$ it is easy to prove that Q_m is perfect for $m \geqslant 8$.

Following the argument of [3], we see first that $f(x)$ takes only even values subject to (5.2) or (5.4), and takes the value 2 only for $x = e_i - e_j$ $(i \neq j)$: and none of these vectors satisfies (5.3). Thus each of P_n, Q_m has $M \geqslant 4$, and in fact $M = 4$ since $f(x) = 4$ for $x = e_1 - e_2 - e_3 + e_4$, which satisfies (5.2), (5.3) and (5.4).

The minimal vectors of P_n are all of the form

$$x = e_a + e_b - e_c - e_d, \tag{5.7}$$

where the suffixes a, b, c, d are distinct and, by (5.3), satisfy

$$a + b \equiv c + d \pmod m. \tag{5.8}$$

A simple enumeration, as given in [3], now establishes (5.5).

For Q_m we have, in addition to these, the minimal vectors

$$x = e_a + e_b + e_c + e_d, \tag{5.9}$$

where the suffixes a, b, c, d are distinct and satisfy

$$a + b + c + d \equiv 0 \pmod m. \tag{5.10}$$

To count these, we need

LEMMA 5.1. *Let N be the number of unordered sets a, b, c, d of distinct integers chosen from $1, 2, \ldots, m$ satisfying* (5.10). *Then*

$$N = \begin{cases} \frac{1}{24}(m-1)(m-2)(m-3) & \text{if } 2 \nmid m, \\ \frac{1}{24}(m-2)(m^2-4m+6) & \text{if } 2|m,\ 4 \nmid m, \\ \frac{1}{24}(m-4)(m^2-2m+6) & \text{if } 4|m. \end{cases} \tag{5.11}$$

Proof.(1) Let

$$p_a(t) = \prod_{r=1}^m (1 + at^r) = \sum_i q_i(t) a^i.$$

Then, if $q_i(t) = \sum q_{ij} t^j$, q_{ij} is the number of sets of i distinct integers chosen from $1, 2, \ldots, m$ whose sum is j; thus

$$N = \sum_{m|j} q_{4j}.$$

If ω is a primitive mth root of unity, we therefore have

$$mN = \sum_{r=1}^m q_4(\omega^r);$$

thus mN is the coefficient of a^4 in the formal expansion of

$$p_a(\omega) + p_a(\omega^2) + \ldots + p_a(\omega^m). \tag{5.12}$$

Now ω^r is a primitive sth root of unity, where $s = m/(m, r)$, and

$$p_a(\omega^r) = \{1 + (-1)^{s-1} a^s\}^{m/s};$$

(1) I am indebted to Mr. W. B. Smith-White for the idea of this simple proof.

the formal expansion of $p_a(\omega^r)$ contains a term in a^4 only when $s = 1$, 2 or 4. Since $\varphi(1) = \varphi(2) = 1$, $\varphi(4) = 2$, we therefore have

$$mN = \begin{cases} \binom{m}{4} & \text{if } 2\nmid m, \\ \binom{m}{4}+\binom{m/2}{2} & \text{if } 2|m,\ 4\nmid m, \\ \binom{m}{4}+\binom{m/2}{2}-2\binom{m/4}{1} & \text{if } 4|m; \end{cases}$$

this gives (5.11).

Since N is the number of minimal vectors of Q_m of the type (5.9), we obtain the formulae (5.6) for s by adding (5.11) and (5.5) (with $n = m-1$).

We next deduce that Q_m is perfect for $m \geqslant 8$ from the fact that P_n is perfect for $n \geqslant 7$ (taking $n = m-1$).

The minimal vectors (5.7) are common to P_n and Q_m, and any m-dimensional quadratic form which vanishes for all of them must be of the type

$$(x_1+x_2+\ldots+x_m)(p_1x_1+p_2x_2+\ldots+p_mx_m), \tag{5.13}$$

since P_n is perfect and its lattice lies in the plane $x_1+\ldots+x_m = 0$. Since all the minimal vectors (5.9) satisfy $\sum x_i = 4$, it follows that the above quadratic form vanishes for all of these only if

$$p_a+p_b+p_c+p_d = 0 \tag{5.14}$$

whenever (5.10) is satisfied.

Let now a, b, a', b' be any four distinct suffixes $(\mathrm{mod}\, m)$ with

$$a+b \equiv a'+b'.$$

The number of distinct unordered pairs $c, d\ (\mathrm{mod}\, m)$ satisfying

$$c+d \equiv -(a+b)$$

is $\frac{1}{2}(m-1)$ if m is odd, and $\frac{1}{2}m$ or $\frac{1}{2}(m-2)$ if m is even, and so is at least 5 if $m \geqslant 11$. Hence, for $m \geqslant 11$, there exists a pair c, d with both c and d distinct from a, b, a', b'. Now (5.14) gives

$$p_a+p_b+p_c+p_d = 0, \quad p_{a'}+p_{b'}+p_c+p_d = 0,$$

whence

$$p_a+p_b = p_{a'}+p_{b'}. \tag{5.15}$$

It follows therefore that the linear form $\sum p_i x_i$ vanishes for every vector (5.7) (where (5.8) holds), i. e. for all the minimal vectors of P_n. Hence, since P_n is perfect, $p_1 = p_2 = \ldots = p_m$; for otherwise $(\sum p_i x_i)^2$ would not be of the type (5.13). From any one equation (5.14) it now follows that all $p_i = 0$ and the quadratic form (5.13) vanishes identically. Hence Q_m is perfect.

If $8 \leqslant m \leqslant 10$, the result (5.15) still holds whenever $a+b \equiv a'+b'$, although the above simple counting argument breaks down; thus Q_m is perfect for $m \geqslant 8$.

Q_m is not perfect for $m \leqslant 7$, having fewer than $\frac{1}{2}m(m+1)$ minimal vectors.

6. The refinements $A_n^{t,q,r}$, $B_m^{t,q,r}$.

We define $B_m^{t,q,r}$ to be the form whose lattice is the sublattice of Γ_m given by

$$x_{iq+1} \equiv x_{iq+2} \equiv \ldots \equiv x_{(i+1)q} \pmod t \quad (i = 0, \ldots, r-1), \tag{6.1}$$

$$x_{rq+1} \equiv x_{rq+2} \equiv \ldots \equiv x_m \equiv 0 \pmod t, \tag{6.2}$$

$$x_q+x_{2q}+\ldots+x_{rq} \equiv 0 \pmod t, \tag{6.3}$$

$$\sum_1^m x_i \equiv 0 \pmod{2t}. \tag{6.4}$$

$A_n^{t,q,r}$ (with $n = m-1$) has the same definition, save that (6.4) is replaced by

$$\sum_1^m x_i = 0. \tag{6.5}$$

Here the positive integral parameters m, t, q, r are to satisfy[2]

$$t \geqslant 2, \quad r \geqslant 2, \quad q \geqslant t^2, \quad m \geqslant rq\ (\geqslant 8)$$

(and (6.2) is vacuous if $m = rq$).

Since the congruences (6.1), (6.2), (6.3) are independent and imply that

$$\sum_1^m x_i \equiv 0 \pmod t,$$

so that (6.4) is only a condition modulo 2, the determinants of $\Lambda(B_m^{t,q,r})$ and $\Lambda(A_n^{t,q,r})$ are

$$2t^{r(q-1)+(m-rq)+1} = 2t^{m-r+1},$$

$$m^{1/2}t^{r(q-1)+(m-rq)} = m^{1/2}t^{m-r}.$$

Hence

$$D(B_m^{t,q,r}) = 4t^{2m-2r+2}, \quad D(A_n^{t,q,r}) = mt^{2m-2r}.$$

[2] The conditions $t \geqslant 2$, $r \geqslant 2$ merely ensure that the forms do not reduce to a multiple of B_m or A_n.

The points of these lattices satisfying $x_1 \equiv x_2 \equiv \ldots \equiv x_m \equiv 0 \pmod t$ clearly form sublattices which are the lattices of $B_m(tx) = t^2 B_m(x)$, $A_n(tx) = t^2 A_n(x)$ respectively. Since B_m and A_n are extreme for $m \geqslant 8$ and have minimum 2, Theorem 2.1 shows that $B_m^{t,q,r}$ *and* $A_n^{t,q,r}$ *are extreme if they have minimum* $2t^2$.

We now establish that in fact $M = 2t^2$ for each form, and specify the minimal vectors. We begin with $B_m^{t,q,r}$ and set

$$x_{iq+1} \equiv \ldots \equiv x_{(i+1)q} \equiv a_{i+1} \pmod t \qquad (i = 0, \ldots, r-1)$$

where

$$0 \leqslant |a_{i+1}| \leqslant \tfrac{1}{2}t.$$

Then, since $x_j \equiv 0 \pmod t$ for $j > rq$, we clearly have

$$f(x) = \sum_1^m x_i^2 \geqslant q(a_1^2 + \ldots + a_r^2) \geqslant t^2(a_1^2 + \ldots + a_r^2).$$

Now, by (6.3), $\sum_1^r a_i \equiv 0 \pmod t$, and so $\sum_1^r a_i^2 \geqslant 2$ unless either (i) $a_1 = \ldots = a_r = 0$; or (ii) some two a_i are 1, -1 and the rest are zero; or (iii) $t = 2$, some two a_i are 1, 1 or -1, -1 and the rest are zero. In the second and third cases, $\sum_1^r a_i^2 = 2$ and $f \geqslant 2q \geqslant 2t^2$, with equality only when $q = t^2$. In the first case, all $x_i \equiv 0 \pmod t$ and the form assumes the same values as $t^2 B_m$: i. e. its least value is then $2t^2$, assumed at the points $te_i \pm te_j$ $(1 \leqslant i < j \leqslant m)$.

This shows that $M = 2t^2$, as asserted. To calculate s, the number of pairs of minimal vectors, we note first the $m(m-1)$ vectors $te_i \pm te_j$ which exist in all cases; and these are all if $q > t^2$. If $q = t^2$ and $t > 2$, we obtain $\frac{1}{2}r(r-1)$ further minimal vectors by choosing, in the above notation, $a_i = 1$, $a_j = -1$ for $1 \leqslant i < j \leqslant r$. Finally, if $t = 2$, $q = t^2 = 4$, all additional minimal vectors are obtained by choosing any two sets $x_{4i+1}, \ldots, x_{4i+4}$; $x_{4j+1}, \ldots, x_{4j+4}$ $(1 \leqslant i < j \leqslant r)$ and taking the values of these variables to be any permutation of

$$(1_8), (1_6, -1_2), (1_4, -1_4)$$

(with all other variables zero). This yields in all

$$\binom{r}{2}\left(1 + \binom{8}{2} + \frac{1}{2}\binom{8}{4}\right) = 32r(r-1)$$

pairs of vectors. Thus for $B_m^{t,q,r}$ we have

$$s = \begin{cases} m(m-1) & \text{if} \quad q > t^2, \\ m(m-1) + \frac{1}{2}r(r-1) & \text{if} \quad q = t^2, \quad t > 2, \\ m(m-1) + 32r(r-1) & \text{if} \quad q = t^2 = 4. \end{cases}$$

It now follows at once that $A_n^{t,q,r}$ has also $M = 2t^2$, its minimal vectors being those of $B_m^{t,q,r}$ which satisfy (6.5). We thus find, for $A_n^{t,q,r}$,

$$s = \begin{cases} \frac{1}{2}n(n+1) & \text{if} \quad q > t^2, \\ \frac{1}{2}n(n+1) + \frac{1}{2}r(r-1) & \text{if} \quad q = t^2, \quad t > 2, \\ \frac{1}{2}n(n+1) + \frac{35}{2}r(r-1) & \text{if} \quad q = t^2 = 4. \end{cases}$$

It is noteworthy that no two of the above forms are equivalent, despite the fact that different forms (either A or B) with $q > t^2$ may agree in minimum, determinant, number of minimal vectors and even in the geometrical configuration of their minimal vectors. To see this, we need only consider forms with the same m, t, r and different $q > t^2$, the inequivalence being otherwise trivial. Such forms take the values $2t^2, 4t^2, \ldots$ at precisely the same points, when all variables are congruent to zero $(\bmod t)$. But the above analysis shows that the least value assumed at any other point is $2q$, so that forms with different values of q cannot be equivalent.

Finally we note that the absolutely extreme forms E_7, E_8 are equivalent to $A_7^{2,4,2}$, $B_8^{2,4,2}$ respectively; T_9 of [7] is equivalent to $B_9^{2,4,2}$; and J_{12} (found by Chaundy and given in [6]) is equivalent to $B_{12}^{2,4,3}$.

7. The refinement A_t^n. For $t \geqslant 2$, we define A_n^t to be the form

$$f(x) = A_n(x) = \sum_1^n x_i^2 + \Big(\sum_1^n x_i\Big)^2 \tag{7.1}$$

with lattice the sublattice of Γ_n given by

$$x_1 \equiv x_2 \equiv \ldots \equiv x_n \pmod t. \tag{7.2}$$

Coxeter [5] has defined a form A_n^t under the condition $t|n+1$; it is easily verified that this agrees with the above definition when $t|n+1$.

Since $D(A_n) = n+1$ and the lattice (7.2) has determinant t^{n-1},

$$D(A_n^t) = (n+1)t^{2n-2}. \tag{7.3}$$

Now the sublattice of (7.2) given by $x_i \equiv 0 \pmod t$ $(i = 1, \ldots, n)$ is $t\Gamma_n$, and so corresponds to the form $A_n(tx) = t^2 A_n(x)$, which is extreme with minimum $2t^2$. Hence A_n^t is a refinement of $t^2 A_n$ and, by Theorem 2.1, A_n^t *is extreme if its minimum is* $2t^2$; in this case, we have

$$\Delta(A_n^t) = \frac{n+1}{t^2}.$$

It is easy to show that $M(A_n^t) < 2t^2$ if $n \leqslant 2t-1$. For if we choose

$$x_1 = \ldots = x_{n-1} = 1, \quad x_n = 1-t$$

we obtain

$$f(x) = (n-1)+(1-t)^2+(n-t)^2 \leqslant 2t-2+(1-t)^2+(t-1)^2 = 2t(t-1).$$

For the application of Theorem 2.1, it suffices then to assume that

$$m = n+1 > 2t. \tag{7.4}$$

THEOREM 7.1. (i) *We have*

$$M(A_n^t) = 2t^2, \tag{7.5}$$

and A_n^t is extreme, if and only if either

$$m-\sqrt{m} \geqslant 2t, \tag{7.6}$$

or

$$m \geqslant 3t; \tag{7.7}$$

except for the (imperfect) forms A_5^2, A_6^2.

(ii) *The corresponding number of minimal vectors s is given by*

$$s = n^2 \quad \text{if} \quad m-\sqrt{m} = 2t,$$

$$s = \tfrac{1}{2}n(n+1) \quad \text{otherwise},$$

with the following four exceptions:

$$A_7^2\,(s=63), \quad A_8^2\,(s=71), \quad A_8^3\,(s=120), \quad A_9^3\,(s=129).$$

For the proof of these results, we denote by $\varphi(a)$ the minimum of $A_n(x)$ for

$$x_i \equiv a \pmod t \quad (i = 1, \ldots, n), \quad \boldsymbol{x} \neq \boldsymbol{O};$$

thus $M(A_n^t)$ is the minimum of $\varphi(a)$ for $0 \leqslant a \leqslant \frac{1}{2}t$.

Clearly $\varphi(0) = 2t^2 = \min A_n(t\boldsymbol{x})$, and is attained at the $s_0 = \frac{1}{2}n(n+1)$ pairs of points $\pm t\boldsymbol{e}_i$, $t\boldsymbol{e}_i - t\boldsymbol{e}_j$ $(i \neq j)$.

If $0 < a \leqslant \frac{1}{2}t$, it is easily seen that $\varphi(a)$ is attained when each x_i is a or $a-t$; and if k coordinates are $a-t$ and $n-k$ are a we obtain

$$f(\boldsymbol{x}) = a(n+1)(na-2kt)+(k^2+k)t^2.$$

Writing for convenience

$$q = \frac{m}{t} > 2,$$

we may write this as

$$f(\boldsymbol{x}) = t^2(k+\tfrac{1}{2}-aq)^2+t^2(aq-\tfrac{1}{4})-ta^2q. \tag{7.8}$$

$\varphi(a)$ is thus the least value of this expression for integral k with $0 \leqslant k \leqslant n$, and so is attained when $k = [aq]$:

$$\varphi(a) = t^2([aq]+\tfrac{1}{2}-aq)^2+t^2(aq-\tfrac{1}{4})-ta^2q \quad (0 < a \leqslant \tfrac{1}{2}t). \tag{7.9}$$

We can now prove

LEMMA 7.1. *We have*

$$M = M(A_n^t) = \min\{2t^2, \varphi(1)\},$$

and $\varphi(a) > M$ for $2 \leqslant a \leqslant \frac{1}{2}t$.

Proof. If $a \geqslant 3$, then $t \geqslant 2a \geqslant 6$ and (7.9) gives

$$\varphi(a) \geqslant t^2(aq-\tfrac{1}{4})-ta^2q = ma(t-a)-\tfrac{1}{4}t^2$$
$$> 6t(t-3)-\tfrac{1}{4}t^2 > 2t^2.$$

If $a = 2$, then $t \geqslant 4$, $m > 2t \geqslant 8$, and

$$\varphi(2) \geqslant t^2(2q-\tfrac{1}{4})-4tq = 2m(t-2)-\tfrac{1}{4}t^2 > 4t(t-2)-\tfrac{1}{4}t^2.$$

Hence certainly $\varphi(2) > 2t^2$ if $t \geqslant 5$. If however $t = 4$, then

$$\varphi(2) \geqslant 2m(t-2)-\tfrac{1}{4}t^2 = 4m-4,$$

whereas, by (7.9),

$$\varphi(1) \leqslant \tfrac{1}{4}t^2+t^2(q-\tfrac{1}{4})-tq = 3m;$$

thus $\varphi(2) > \varphi(1)$, since $m > 8$.

The lemma follows at once, since $M = \min\varphi(a)$ $(0 \leqslant a \leqslant \frac{1}{2}t)$.

LEMMA 7.2. *Let s_1 denote the number of pairs $\pm\boldsymbol{x}$ satisfying*

$$f(\boldsymbol{x}) = \varphi(1), \quad x_i \equiv 1 \pmod t \quad (i = 1, \ldots, n).$$

Then

$$s_1 = \begin{cases} \binom{n}{[q]} & \text{if } q \text{ is not integral}, \\ \binom{n+1}{q} & \text{if } q \text{ is integral and } t > 2, \\ \binom{n}{q} & \text{if } q \text{ is integral and } t = 2. \end{cases} \tag{7.10}$$

Proof. $\varphi(1)$ is the least value of (7.8), with $a = 1$, and is attained only when $k = [q]$ if q is not integral; this gives the first result of (7.10).

If q is integral, $\varphi(1)$ is attained when $k = q$ or $q-1$, and we obtain

$$\binom{n}{q} + \binom{n}{q-1} = \binom{n+1}{q}$$

representations if $t \neq 2$. If however $t = 2$, the vectors with $k = q-1$ are simply the negatives of those with $k = q$, since now

$$1-t \equiv -1 \pmod{t}.$$

The proof of Theorem 7.1 is now easily completed. By (7.4), $q > 2$; if now (7.7) is not satisfied we have $2 < q < 3$ and so, by (7.9),

$$\varphi(1) = t^2\left(\tfrac{5}{2} - q\right)^2 + \left(q - \tfrac{1}{4}\right)t^2 - tq = m^2 - 4mt - m + 6t^2.$$

By Lemma 7.1, (7.5) holds if and only if $\varphi(1) \geqslant 2t^2$, i. e.

$$(m-2t)^2 - m \geqslant 0, \quad \text{or} \quad m - \sqrt{m} \geqslant 2t,$$

which is (7.6).

If now (7.6) holds with inequality, we have $\varphi(1) > 2t^2$, whence $s = s_0 = \frac{1}{2}n(n+1)$; if however (7.6) holds with equality, then $\varphi(1) = 2t^2 = M$ and, using (5.9),

$$s = s_0 + s_1 = \tfrac{1}{2}n(n+1) + \binom{n}{2} = n^2.$$

This establishes Theorem 7.1 when $q < 3$, i. e. when (7.7) does not hold.

Suppose now that (7.7) holds, so that $q \geqslant 3$. By (5.8),

$$\varphi(1) \geqslant t^2\left(q - \tfrac{1}{4}\right) - tq \geqslant \tfrac{11}{4}t^2 - 3t,$$

so that certainly $\varphi(1) > 2t^2$ if $t \geqslant 5$. Thus

$$M = 2t^2, \quad s = s_0 = \tfrac{1}{2}n(n+1) \quad \text{if} \quad t \geqslant 5.$$

We now consider separately the values 2, 3, 4 of t.

(a) If $t = 2$, (7.9) gives

$$\varphi(1) = 4\left([q] - q + \tfrac{1}{2}\right)^2 + 2q - 1,$$

whence $\varphi(1) > 8 = 2t^2$ if $q \geqslant 5$. The remaining possible values 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$ of $q = \frac{1}{2}m$ give respectively $\varphi(1) = 6, 6, 8, 8$. Thus

$$M = \varphi(1) < 2t^2 \quad \text{for} \quad A_5^2, A_6^2;$$
$$M = \varphi(1) = 2t^2 \quad \text{for} \quad A_7^2, A_8^2;$$
$$M = 2t^2 < \varphi(1) \quad \text{for} \quad A_n^2,\ n \geqslant 9.$$

Here the forms A_5^2, A_6^2 are easily seen to be imperfect; for A_7^2, A_8^2 we have $s = s_0 + s_1 = 63, 71$ respectively; for A_n^2 $(n \geqslant 9)$, $s = s_0 = \frac{1}{2}n(n+1)$.

(b) If $t = 3$, (7.9) gives

$$\varphi(1) = 9\left([q] - q + \tfrac{1}{2}\right)^2 + 6q - \tfrac{9}{4},$$

whence $\varphi(1) > 18 = 2t^2$ if $q \geqslant 4$. The remaining possible values $3, \frac{10}{3}, \frac{11}{3}$ of $q = \frac{1}{3}m$ give respectively $\varphi(1) = 18, 18, 20$. Thus

$$M = \varphi(1) = 2t^2 \quad \text{for} \quad A_8^3, A_9^3;$$
$$M = 2t^2 < \varphi(1) \quad \text{for} \quad A_n^3,\ n \geqslant 10.$$

For A_n^3 $(n \geqslant 10)$ we therefore have $s = s_0 = \frac{1}{2}n(n+1)$; for A_8^3, A_9^3, $s = s_0 + s_1 = 120, 129$ respectively.

(c) If $t = 4$, (7.9) gives

$$\varphi(1) = 16\left([q] - q + \tfrac{1}{2}\right)^2 + 12q - 4,$$

whence $\varphi(1) > 2t^2 = 32$ for all $q \geqslant 3$. Thus here $M = 2t^2$, $s = s_0 = \frac{1}{2}n(n+1)$.

This completes the proof of Theorem 7.1.

To settle the possible equivalence of a form A_n^t with one of the forms $A_n^{t,a,r}$ of § 6, we prove:

THEOREM 7.2. *The only equivalences among the extreme forms* A_n^t, $A_n^{t,a,r}$ *are*

$$(7.11) \qquad A_{2q-1}^2 \sim A_{2q-1}^{2,q,2}, \quad A_{2q}^2 \sim A_{2q}^{2,q,2} \quad (q \geqslant 4).$$

Proof. The relations (7.11) are easily verified, the forms being in fact identical. Comparison of the values of s and Δ, viz.

$$\Delta(A_n^t) = \frac{n+1}{t^2}, \quad \Delta(A_n^{t,a,r}) = \frac{n+1}{t^{2r-2}},$$

shows that the only other possible equivalence is

$$(7.12) \qquad A_n^t \sim A_n^{t,q,2}, \quad q > t^2$$

(each form having $\Delta = (n+1)/t^2$, $s = \frac{1}{2}n(n+1)$).

Now both A_n^t, $A_n^{t,q,2}$ take the values $2t^2, 4t^2, \ldots$ of $A_n(tx)$ at precisely the same points (when all variables are zero modulo t). As was shown in § 6, the least value assumed by $A_n^{t,q,2}$ at any other point is $2q$. For A_n^t, the above analysis shows that the least value assumed at any other point is

$$v = \min_a \varphi(a) \quad (1 \leqslant a \leqslant \tfrac{1}{2}t).$$

Now (7.9) gives

$$v \geqslant t^2\left(a\frac{m}{t}-\frac{1}{4}\right)-a^2m = ma(t-a)-\tfrac{1}{4}t^2$$
$$\geqslant m(t-1)-\tfrac{1}{4}t^2.$$

Since $m > 2q > 2t^2$, we obtain

$$v > m(t-\tfrac{9}{8}) > 2q(t-\tfrac{9}{8}),$$

whence $v > 2q$ if $t \geqslant 3$.

Thus the equivalence (7.12) cannot hold if $t \geqslant 3$. For $t = 2$, (7.12) reduces to (7.11) (with $q > 4$). This proves the theorem.

We note finally the possibility that A_n^t may be perfect or extreme with $M < 2t^2$ (for which Theorem 2.1 would not apply). A slight extension of the above analysis shows that this occurs only when $m = 2t$. This case has been dealt with by Coxeter [5] and so we merely state:

THEOREM 7.3. *If $m = n+1 = 2t$, then A_n^t is extreme, with*

$$M = 2t(t-1), \qquad s = \tfrac{1}{2}n(n+1),$$

and so

$$\Delta = (n+1)t^{n-2}/(t-1)^n.$$

8. The refinement B_m^t. Continuing the analogy between forms A and B, we define B_m^t as

$$f(x) = \sum_1^m x_i^2 \tag{8.1}$$

with lattice the sublattice of Γ_m given by

$$x_1 \equiv x_2 \equiv \ldots \equiv x_n \pmod{t}, \tag{8.2}$$

$$\sum_1^m x_i \equiv 0 \pmod{2t}. \tag{8.3}$$

Here $t \geqslant 2$ and, as always, $m = n+1$. Clearly

$$D(B_m^t) = 4t^{2m-2},$$

and B_m^t is a refinement of $B_m(tx) = t^2B_m(x)$. Hence, by Theorem 2.1, *B_m^t is extreme provided that $M(B_m^t) = 2t^2$*; in this case, $\Delta(B_m^t) = 4/t^2$.

It is easily verified that

$$B_{2q}^2 = B_{2q}^{2,q,2}, \qquad B_{2q+1}^2 = B_{2q+1}^{2,q,2}$$

so that these forms (which are extreme for $q \geqslant 4$) have been dealt with in § 6.

We therefore suppose henceforward that $t \geqslant 3$, and define a unique integer k by

$$n \equiv \pm k \pmod{2t}, \qquad 0 \leqslant k \leqslant t. \tag{8.4}$$

THEOREM 8.1. *For $t \geqslant 3$, B_m^t is extreme with $M = 2t^2$, if and only if*

$$n \geqslant 2t^2-k^2 \quad \text{when} \quad 0 \leqslant k \leqslant t-1, \tag{8.5}$$

$$n \geqslant t(t+2) \quad \text{when} \quad k = t. \tag{8.6}$$

The number s of minimal vectors is then given by

$$s = m(m-1),$$

unless either

(i) $n = t(t+2)$ and t is odd, when $s = m^2-1$; or

(ii) $n = t(t+2)-1$ and t is odd, when $s = m^2$; or

(iii) $n = 2t^2-k^2$, $0 \leqslant k \leqslant t-2$ and $k^2 \pm k \equiv 0 \pmod{2t}$, when $s = m(m-1)+1$.

As in § 7, we define $\varphi(a)$ to be the minimum of $f(x)$ subject to

$$x_1 \equiv \ldots \equiv x_n \equiv a \pmod{t}, \qquad 0 \leqslant a \leqslant \tfrac{1}{2}t \qquad (x \neq O);$$

then $M(B_m^t) = \min\varphi(a)$.

We have at once $\varphi(0) = M(t^2B_m) = 2t^2$, attained at the $s_0 = m(m-1)$ point $x = te_i \pm te_j$ $(i < j)$.

If $a = 1$, (8.3) and (8.4) give

$$x_m = -\sum_1^n x_i \equiv -n \equiv \pm k \pmod{t},$$

and the least possible value of $|x_m|$ is therefore either k or $t-k$. From this, and (8.3), it follows easily that $\varphi(1)$ is the lesser of

$$v_1 = n+k^2,$$

attained when $x_1 = x_2 = \ldots = x_n = 1$, $x_m = \mp k$, and

$$v_2 = n-1+(t-1)^2+(t-k)^2,$$

attained when $x_1 = 1-t$, $x_2 = \ldots = x_n = 1$, $x_m = \pm(t-k)$. Since

$$v_2 > v_1 \text{ for } 0 \leqslant k \leqslant t-2, \qquad v_2 = v_1 \text{ for } k = t-1, \qquad v_2 < v_1 \text{ for } t = k,$$

we have

$$\varphi(1) = \begin{cases} n+k^2 & \text{if} \quad 0 \leqslant k \leqslant t-1, \\ n-1+(t-1)^2 & \text{if} \quad k = t. \end{cases}$$

The conditions (8.5), (8.6) are therefore necessary for $M(B_m^t) = 2t^2$. Also, they imply that always
$$n \geqslant 2t^2-(t-1)^2 = t^2+2t-1;$$
for $2 \leqslant a \leqslant \frac{1}{2}t$ we therefore have crudely
$$\varphi(a) \geqslant na^2 \geqslant 4(t^2+2t-1) > 2t^2.$$
It follows that the inequalities (8.5), (8.6) are necessary and sufficient to ensure that $M(B_m^t) = 2t^2$.

To determine s, the number of (pairs of) minimal vectors, we observe first that $s = s_0 = m(m-1)$, unless equality holds in (8.5) or (8.6), when $s = s_0+s_1$.

If equality holds in (8.6) we have, with (8.4),
$$n = t(t+2) \equiv t \pmod{2t},$$
whence t is odd; and the representations of v_2 above give $s_1 = n$.

If equality holds in (8.5) and $k = t-1$ we have, with (8.4),
$$n = t(t+2)-1 \equiv t-1 \pmod{2t},$$
whence t is odd; and the representations of v_1 and v_2 above give $s_1 = n+1 = m$.

If equality holds in (8.5) and $0 \leqslant k \leqslant t-2$ we obtain (iii) of Theorem 8.1; and $s_1 = 1$ from the single representation of v_1 above.

This completes the proof of Theorem 8.1.

By an analysis very similar to that of § 7 we may establish:

THEOREM 8.2. *The only equivalences among the extreme forms B_m^t, $B_m^{t,q,r}$ are given by*

(8.7) $$B_{2q}^2 \sim B_{2q}^{2,q,2}, \quad B_{2q+1}^2 \sim B_{2q+1}^{2,q,2};$$

(8.8) $$B_{2q}^t \sim B_{2q}^{t,q,2} \quad \textit{if} \quad t \geqslant 3 \textit{ and } t|q \textit{ or } t|(q-1);$$

(8.9) $$B_{2q+1}^t \sim B_{2q+1}^{t,q,2} \quad \textit{if} \quad t \geqslant 3 \textit{ and } t|q.$$

We note that (8.7) has been noted above, while the equivalences in (8.8) and (8.9) arise simply from a change of sign of the variables $x_{q+1}, \ldots, x_{2q}$ or $x_{q+1}, \ldots, x_{2q+1}$.

The simplest new forms B_m^t are:

B_{15}^3 with $s = 15^2 = 225$,

B_{16}^3 with $s = 16^2-1 = 255$,

B_{17}^3 with $s = 17\cdot 16 = 272$,

B_{18}^3 with $s = 18\cdot 17+1 = 307$ $(\sim B_{18}^{3,9,2})$,

B_{19}^3 with $s = 19\cdot 18+1 = 343$;

all of these having $\Delta = \frac{4}{9}$.

9. Refinements of L_m^r, M_n^r. We may obtain several types of refinement of L_m^r, M_n^r by the same devices as were used above to refine B_m and A_n, beginning with the basic form

(9.1) $$f(\boldsymbol{x}) = \sum_{i=1}^{r}(x_i^2-x_ix_{i+r}+x_{i+1}^2)+\sum_{k=2r+1}^{m}x_k^2 \quad (m \geqslant 2r)$$

and taking sublattices of $f(t\boldsymbol{x})$. The analysis becomes rather complicated if done with complete generality, and most of the resulting forms require large values of n.

We shall therefore consider here only one type of refinement, which yields new perfect or extreme forms for all $n \geqslant 11$.

We define $M_n^{r,2}$ to be the form (9.1) with integral $\boldsymbol{x}$ subject to

(9.2) $$\sum_1^m x_i = 0,$$

(9.3) $$x_1 \equiv x_2 \equiv \ldots \equiv x_n \pmod 2.$$

Since the sublattice of $M_n^{r,2}$ with all $x_j \equiv 0 \pmod 2$ clearly gives the form $M_n^r(2\boldsymbol{x})$, $M_n^{r,2}$ is a refinement of $2M_n^r$. By Theorem 2.1 and § 4, $M_n^{r,2}$ *is perfect if $m \geqslant 2r \geqslant 6$ and extreme if also $m \leqslant 4r-1$ provided that it has minimum* 8. In this case, we have

(9.4) $$\Delta(M_n^{r,2}) = 3^r(n+2r+1)/2^{2r+2}.$$

Let $\varphi(a)$ be the minimum of $f(\boldsymbol{x})$ for $\boldsymbol{x} \neq O$ subject to (9.2) and $x_j \equiv a \pmod 2$ $(1 \leqslant j \leqslant n)$; then $M = \min\{\varphi(0), \varphi(1)\}$.

If $a = 0$, (9.2) shows that also $x_m \equiv 0 \pmod 2$; hence, by § 4, $\varphi(0) = 8$, with $s_0 = \frac{1}{2}n(n+1)+\frac{1}{2}r(r-3)$ representations.

If $a = 1$, (9.2) gives

(9.5) $$x_m \equiv -nx_1 \equiv n \pmod 2.$$

We set for convenience $\mu = m$ or $m-1$ according as m is even or odd; thus μ is even and $\mu \geqslant 2r$. Now, by (9.5), x_m is odd when m is even, and even when m is odd, and x_m^2 is least for $x_m = \pm 1$ or $x_m = 0$. Since each $\psi(x_i, x_{i+r}) \geqslant 1$, with equality only when $(x_i, x_{i+1}) = \pm(1,1)$, we therefore have
$$\varphi(1) \geqslant r+(\mu-2r) = \mu-r.$$
It is easy to see that the equality sign holds unless $\mu = 2r$ and r is odd, since (9.2) may be satisfied by suitably choosing $(x_i, x_{i+r}) = \pm(1,1)$, $x_k = \pm 1$ $(k = 2r+1, \ldots, \mu)$. If however $\mu = 2r$ and r is odd, we have $\varphi(1) = \mu-r+2$, and is attained, for example, when $(x_i, x_{i+r}) = (1,1)$ for $1 \leqslant i \leqslant \frac{1}{2}(r-1)$, $(x_i, x_{i+r}) = (-1,-1)$ for $\frac{1}{2}(r+1) \leqslant i \leqslant r-1$, and $(x_r, x_{2r}) = (1,-1)$.

Summing up, we have

$$(9.6)\qquad \begin{aligned} &\varphi(1) = \mu - r + 2 \quad \text{if} \quad \mu = 2r \text{ and } r \text{ is odd,} \\ &\varphi(1) = \mu - r \quad \text{otherwise;} \end{aligned}$$

and $M_n^{r,2}$ has minimum 8 if and only if $\varphi(1) \geqslant 8$, i. e. if

$$(9.7)\qquad \begin{aligned} &\mu \geqslant r + 6 \quad \text{if} \quad \mu = 2r \text{ and } r \text{ is odd,} \\ &\mu \geqslant r + 8 \quad \text{otherwise.} \end{aligned}$$

To determine the value of s, we note that $s = s_0$ if inequality holds in (9.7); also the equality sign cannot hold in $(9.7)_1$, since it would give $r = 6$ and 6 is not odd. Equality in $(9.7)_2$ implies that r is even and $r+8 = \mu \geqslant 2r$, $r \leqslant 8$; thus equality holds in (9.7) only for the extreme forms $M_{11}^{4,2}$, $M_{12}^{4,2}$ $M_{13}^{6,2}$, $M_{14}^{6,2}$, $M_{15}^{8,2}$, $M_{16}^{8,2}$. For these we find respectively $s_1 = 22, 22, 20, 20, 35, 35$ representations of $\varphi(1) = 8$; and $s = s_0 + s_1$, where $s_0 = s(M_n^r) = \frac{1}{2}n(n+1) + \frac{1}{2}r(r-3)$.

We may inquire, as with A_n^t, whether $M_n^{r,2}$ can be perfect or extreme even when M_n^r is not, i. e. when $r \leqslant 2$. This can happen only when $M_n^{r,2}$ has more minimal vectors, i. e. when equality holds in $(9.7)_2$. Excluding $r = 0$ (which reduces us to A_n^2), we have as the only possibility $r = 2$, $\mu = 10$, i. e. $m = 10$ or 11. The corresponding forms $M_9^{2,2}$, $M_{10}^{2,2}$ are however not perfect (in spite of their relatively large number of minimal vectors, viz. 70 and 80 respectively); all minimal vectors in fact satisfy the relation $x_1x_3 - x_2x_4 = 0$.

10. Conclusion. The method of "refinement" described in this article is clearly capable of much wider application, though it is unlikely that any new forms would appear for small values of n. A list of the distinct forms discussed here for $n = 7$, 8 and 9 follows; it includes all previously known perfect or extreme forms for these values of n, and will serve as a basis of reference for part II. The table gives the name of the form, its number s of minimal vectors, its value of $\Delta = (2/M)^n D$, and indicates whether the form is extreme (E) or perfect and non-extreme (P).

Form	s	Δ	Type	Form	s	Δ	Type
A_7	28	8	E	A_8	36	9	E
B_7	42	4	E	B_8	56	4	E
L_7^2	30	$3^4/2^4$	P	L_8^2	39	$3^4/2^4$	P
L_7^3	36	$3^5/2^6$	E	L_8^3	46	$3^5/2^6$	P
M_7^3	28	$3^3\cdot 7/2^5$	E	L_8^4	54	$3^6/2^8$	E
P_7	36	4	E	M_8^3	36	$3^4\cdot 5/2^6$	E
A_7^2	63	2	E	M_8^4	38	$3^4\cdot 17/2^8$	E
A_7^4	28	$2^{13}/3^7$	E	P_8	54	$3^6/2^8$	E

Form	s	Δ	Type	Form	s	Δ	Type
Q_8	45	4	E	M_9^4	47	$3^6/2^7$	E
A_8^2	71	$3^2/2^2$	E	M_9^5	50	$3^5\cdot 5/2^8$	E
$A_8^3(\sim B_8^2)$	120	1	E	P_9	80	$5^3/2^6$	E
A_9	45	10	E	Q_9	68	$3^4/2^5$	E
B_9	72	4	E	$A_9^{2,4,2}$	80	$5/2$	E
L_9^2	49	$3^4/2^4$	P	A_9^2	45	$5/2$	E
L_9^3	57	$3^5/2^6$	P	A_9^3	129	$2\cdot 5/3^2$	E
L_9^4	66	$3^6/2^8$	E	A_9^5	45	$5^8/2^{17}$	E
M_9^3	45	$3^3/2^2$	E	B_9^2	136	1	E

(The extreme forms L_8^4 and P_8, though agreeing in their values of s and Δ, are not equivalent. For example, it is easily proved that the group of automorphisms of L_8^4 is transitive on its minimal vectors, while that of P_8 is not.)

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Reçu par la Rédaction le 20. 3. 1958