Wenn $D \equiv 3 \pmod{4}$, $C \equiv 1 \pmod{2}$, folgt aus (21) $y_1 \equiv y_2 \equiv 3 \pmod{4}$, was einen Widerspruch liefert.

Wenn $D \equiv 1 \pmod{8}$, $C \equiv 6 \pmod{8}$, oder $D \equiv 7 \pmod{8}$, $C \equiv 2 \pmod{8}$, oder $D \equiv 5 \pmod{8}$, $C \equiv 6 \pmod{8}$, oder $D \equiv 3 \pmod{8}$, $C \equiv 2 \pmod{8}$, folgt aus (21) $y_1 y_2 \equiv \pm 1 \pmod{8}$, was einen Widerspruch liefert.

Wenn n die Werte $1, 2, \ldots$ durchläuft, ergibt sich die Unmöglichkeit der Fälle $q \equiv 5 \pmod{8}$, $q \equiv 9 \pmod{16}, \ldots$, d. h. von $q \equiv 1 \pmod{4}$.

Wenn $q = 3$, $D \equiv 1 \pmod{2}$, C beliebig, führt jede Gleichung (3) zu einer Gleichung der Form (7), oder

$$8 = b_2(3Ca_2^2 - Db_2^2), \tag{26}$$

$a_2 \equiv b_2 \pmod{2}$. Wenn a_2 und b_2 gerade sind, geht (26) in eine Gleichung ersten Grades in y der Form (12) über. Wenn a_2 und b_2 ungerade sind, geht (26) in $3Ca_2^2 - D - 8(D/3) = 0$ über. Aus $y = N\mathfrak{a} = \frac{1}{4}(Ca_2^2 + D)$ folgt

$$3y - D - 2(D/3) = 0. \tag{27}$$

Jede Gleichung (3) geht folglich in eine der beiden Gleichungen (12) oder (27) über. Diese haben je höchstens eine ganzzahlige ungerade Lösung.

In ähnlicher Weise kann man die Gleichungen $Cx^2 + D = 2y^q$ und $Cx^2 + D = 4y^q$ behandeln (vgl. Ljunggren [4], Stolt [10]).

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The zeta function and discriminant of a division algebra

by

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§ 1. Let D be a division algebra of finite rank $g = hf^2$ over the field Γ of rational numbers and Z its centre so that $(D: Z) = f^2$ and $(Z: \Gamma) = h$. Let $\overline{\Gamma}$ be the real number field. Siegel [10] has shown that the tensor product $\overline{D} = D \otimes \overline{\Gamma}$ has, over $\overline{\Gamma}$, an involution $x \to x^*$. Let P be the space of positive elements of $\overline{D}$, that is the set of elements $x = x^*$ all of whose characteristic roots are positive. P is a symmetric Riemannian space with the metric $ds^2 = \sigma(\xi^{-1} d\xi\, \xi^{-1} d\xi)$. Let $[d\xi]$ denote the volume element computed with this metric. We introduce the generalized gamma function

$$\Gamma_D(a, s) = \int_P (N\xi)^{s/2} e^{-\pi\sigma(a\xi)} [d\xi]$$

where $a \in P$, N and σ denote norm and trace in the regular representation of $\overline{D}$ over $\overline{\Gamma}$ and s is a complex variable whose real part is greater than $(f-1)/f$. $\Gamma_D(a, s)$ is a simple generalization of the gamma function introduced by C. L. Siegel [9] in the analytic theory of quadratic forms. Let Λ be a lattice in $\overline{D}$ and $\tilde{\Lambda}$ the complementary lattice. Let ξ be an arbitrary but fixed element of P. The function

$$\vartheta(\Lambda, \xi) = \sum_{a \in \Lambda} e^{-\pi\sigma(a^*\xi a)}$$

is called the *theta function* of the lattice. There exists a transformation formula connecting $\vartheta(\Lambda, \xi)$ and $\vartheta(\tilde{\Lambda}, \xi^{-1})$. By using this theta function and the gamma function above, we shall obtain a simple proof of the functional equation for the zeta function of D. In view of the work of Siegel on the zeta functions of indefinite forms, it seems more natural to use the representation space P of the units of a maximal order of D in the study of the zeta function of D.

For the discriminant d of a totally real algebraic number field C. L. Siegel [6] obtained an identity which shows at once that $|d| > 1$. This identity was generalized to all fields by Müntz [5] and Calloway [1].

We will show that this and other identities of this type may be obtained quite simply from a general formula due to Siegel [8]. We shall, moreover, obtain, by the same method, an identity for the discriminant of a maximal order of D. As a simple consequence we deduce that a quaternion algebra D with Γ as centre splits over Γ if and only if it splits at all prime spots of Γ.

§ 2. Let D be a division algebra of finite rank g and $\mathfrak{o}$ a maximal order in D relative to the ring of rational integers. Let $\delta_1, \ldots, \delta_g$ be a minimal base of $\mathfrak{o}$. By means of the regular representation of D with regard to the basis $\delta_1, \ldots, \delta_g$, to every element $a \epsilon D$ is associated a matrix $\hat{a}$ with elements in Γ so that

$$a\begin{pmatrix}\delta_1\\ \vdots\\ \delta_g\end{pmatrix} = \hat{a}\begin{pmatrix}\delta_1\\ \vdots\\ \delta_g\end{pmatrix}. \tag{1}$$

Let $\bar{D} = D \otimes \bar{\Gamma}$ be the tensor product of D and $\bar{\Gamma}$ the field of real numbers. Since $\delta_1, \ldots, \delta_g$ can serve as a basis, over $\bar{\Gamma}$, of the semi-simple algebra $\bar{D}$, we obtain the algebra $\hat{\bar{D}}$ of the matrices obtained by the regular representation of $\bar{D}$ with regard to $\delta_1, \ldots, \delta_g$. Let $h = r_1 + r_3 + 2r_2$, r_2 being the number of complex infinite prime spots of Z and r_3 the number of real infinite prime spots of Z which are ramified in D. From Wedderburn's theorem one has

$$\hat{\bar{D}} \simeq \sum_{i=1}^{r_1} f M_i^f(\bar{\Gamma}) + \sum_{i=1}^{r_3} \tfrac{1}{2} f M_i^{f/2}(Q) + \sum_{i=1}^{r_2} f M_i^f(\Omega) \tag{2}$$

where Q is the division algebra of real quaternions, Ω the complex number field, $M^l(R)$ for any division ring R denotes the complete algebra of l-rowed matrices over R and the coefficients f, $f/2$ denote the number of times these algebras are repeated in the direct sum (2). For any matrix L in $M(R)$ we denote by $|L|$ the reduced norm and by $\tau(L)$ the reduced trace. Let us denote a generic element in $M^f(\bar{\Gamma})$ by X, that in $M^{f/2}(Q)$ by Y and that in $M^f(\Omega)$ by Z. If $x \epsilon \bar{D}$ and

$$\hat{x} = \sum_{i=1}^{r_1} f X_i + \sum_{i=1}^{r_3} \tfrac{1}{2} f Y_i + \sum_{i=1}^{r_2} f Z_i$$

then

$$\sigma(x) = \sum_{i=1}^{r_1} f\tau(X_i) + \sum_{i=1}^{r_3} f\tau(Y_i) + \sum_{i=1}^{r_2} f\tau(Z_i). \tag{3}$$

If we choose a basis $\varepsilon_1, \ldots, \varepsilon_g$ relative to the component algebras in (2) and put

$$\begin{pmatrix}\delta_1\\ \vdots\\ \delta_g\end{pmatrix} = \gamma\begin{pmatrix}\varepsilon_1\\ \vdots\\ \varepsilon_g\end{pmatrix} \tag{4}$$

for a real g-rowed matrix γ, then the matrix corresponding to $a \epsilon \bar{D}$ by means of the regular representation relative to $\varepsilon_1, \ldots, \varepsilon_g$ is $\gamma^{-1}\hat{a}\gamma$. Observing the fact that for every matrix A on the right side of (2), the transpose A' is also there, we see that to every $a \epsilon \bar{D}$ there exists a unique $a^* \epsilon \bar{D}$ such that

$$\gamma^{-1}\hat{a}^*\gamma = (\gamma^{-1}\hat{a}\gamma)'. \tag{5}$$

$a \to a^*$ is an involution of $\bar{D}$. We call a in $\bar{D}$ *symmetric* if $a = a^*$ and positive, written as $a > 0$, if all the characteristic roots of a are positive. The positive elements constitute a symmetric Riemannian space P of $\frac{1}{2}f(hf + r_1 - r_3)$ real dimensions and has the metric

$$ds^2 = \sigma(\xi^{-1}d\xi\,\xi^{-1}d\xi)$$

in the notation of Siegel. This has the invariance property under the transformations

$$\begin{aligned} &\xi \to \beta^*\xi\beta, \\ &\xi \to \xi^{-1} \end{aligned} \tag{6}$$

where $\beta \epsilon \bar{D}$ and $N\beta \neq 0$. One easily finds that the volume element

$$[dx] = \prod_{i=1}^{r_1} |X_i|^{-(f+1)/2} \prod_{i=1}^{r_3} |Y_i|^{-(f-1)/2} \prod_{i=1}^{r_2} |Z_i|^{-f} \{dX_1\}\ldots\{dX_{r_1}\}\ldots\{dZ_1\}\ldots\{dZ_{r_2}\} \tag{7}$$

is invariant under the transformations (6). Here $\{dX_i\}$ etc. denote the Euclidean volume element.

§ 3. Let s be a complex parameter with real part $> 1 - 1/f$. If $\beta \epsilon \bar{D}$ and $N\beta \neq 0$, put

$$\Gamma_D(\beta, s) = \int_P (Nx)^{s/2} e^{-\pi\sigma(\beta^* x\beta)}[dx]. \tag{8}$$

This is the generalized gamma function associated with $\bar{D}$. That the integral exists under the conditions on s will be seen from below. Because of the invariance property (6) and the metric, it follows that

$$\Gamma_D(\beta, s) = \frac{1}{|N\beta|^s}\Gamma_D(s) \tag{9}$$

where we have written $\Gamma_D(s)$ for $\Gamma_D(1, s)$. Also

$$\Gamma_D(s) = I_1^{r_1} \cdot I_2^{r_3} \cdot I_3^{r_2} \tag{10}$$

where

$$\begin{aligned} I_1 &= \int\limits_{X>0} |X|^{sf/2-(f+1)/2} e^{-\pi f \tau(X)} \{dX\}, \\ I_2 &= \int\limits_{Y>0} |Y|^{sf/2-(f-1)/2} e^{-\pi f \tau(Y)} \{dY\}, \\ I_3 &= \int\limits_{Z>0} |Z|^{sf/2-f/2} e^{-2\pi f \tau(Z)} \{dZ\}. \end{aligned} \tag{11}$$

These integrals are similar to those considered by Siegel [9]. I_1 has been evaluated by Siegel. I_2 and I_3 can be evaluated in a similar way; for instance to evaluate I_2 we complete squares and write

$$Y = \begin{pmatrix} Y_1 & 0 \\ 0 & y \end{pmatrix} \begin{bmatrix} E & Y_1^{-1} q^* \\ 0 & 1 \end{bmatrix} = \begin{pmatrix} Y_1 & q^* \\ q & y + q^* Y_1^{-1} q \end{pmatrix}$$

where for a quaternion symmetric matrix G and a quaternion matrix H, $G[H] = H^* G H$. Since $\{dY\} = \{dY_1\}\{dq\}\{dy\}$, we get

$$I_2 = f^{-sf^2/2} \int\limits_{\substack{Y_1>0 \\ y>0 \\ q}} \{|Y_1| \cdot |y|\}^{sf/2-(f-1)/2} \cdot e^{-\pi\tau(Y_1 + y + q^* Y_1^{-1} q)} \{dY\}.$$

Since the elements of q run from $-\infty$ to ∞ and $\tau(y) = 2y$, we get

$$I_2 = f^{-sf^2/2} \frac{\Gamma(sf-f+2)}{(2\pi)^{sf-f+2}} \int\limits_{Y_1>0} |Y_1|^{sf/2-(f-3)/2} e^{-\pi\tau(Y_1)} \{dY_1\}.$$

By induction one gets finally

$$I_2 = f^{-sf^2/2} \prod_{i=0}^{f/2-1} \frac{\Gamma(sf-2i)}{(2\pi)^{sf-2i}}.$$

Evaluating the other integrals in a similar manner, we get

$$\Gamma_D(s) = \tag{12}$$

$$f^{-gs/2} 2^A \prod_{i=1}^{f-1} \left(\frac{\Gamma(sf-i)/2}{\pi^{sf-i/2}} \right)^{r_1} \prod_{i=0}^{f/2-1} \left(\frac{\Gamma(sf-2i)}{\pi^{sf-2i}} \right)^{r_3} \prod_{i=0}^{f-1} \left(\frac{\Gamma(sf-i)}{\pi^{sf-i}} \right)^{r_2},$$

where

$$A = -r_2 s f^2 - r_3 s \frac{f^2}{2} + \frac{r_3 f(f-2)}{4}.$$

§ 4. Let Λ be a lattice in $\bar{D}$, that is a discrete subgroup of the additive group of $\bar{D}$, not contained in a proper subspace of $\bar{D}$. Let $\omega_1, \ldots, \omega_g$ be a base of Λ. Let $\tilde{\Lambda}$ be the complementary lattice consisting of $\alpha \epsilon \bar{D}$ where $\sigma(\alpha\lambda)$ is a rational integer for every $\lambda \epsilon \Lambda$. Let $\omega_1', \ldots, \omega_g'$ be the complementary base so that

$$\sigma(\omega_i \omega_j') = \begin{cases} 0 & \text{if} \quad i \neq j, \\ 1 & \text{if} \quad i = j. \end{cases}$$

Denote by $d(\Lambda)$ the absolute value of the determinant of the matrix $(\sigma(\omega_i \omega_j))$. Then

$$d(\Lambda) \cdot d(\tilde{\Lambda}) = 1. \tag{13}$$

Let $\xi > 0$ be an arbitrary but fixed element of $\bar{D}$. The theta series of Λ is

$$\vartheta(\Lambda, \xi) = \sum_{a \epsilon \Lambda} e^{-\pi\sigma(a^* \xi a)}.$$

That it is convergent is very easy to see. We shall prove the formula

$$\vartheta(\Lambda, \xi) = \frac{1}{\sqrt{d(\Lambda)\,(N\xi)}} \vartheta(\tilde{\Lambda}, \xi^{-1}). \tag{14}$$

The proof is very simple. For let $a \epsilon \Lambda$ be a generic element of Λ so that $a = \sum_i \omega_i x_i$, x_i integers. Then

$$\sigma(a^* \xi a) = \sum_{i,j} \sigma(\omega_i^* \xi \omega_j) x_i x_j$$

is a quadratic form in $x_1, \ldots, x_g$ whose matrix is $(\sigma(\omega_i^* \xi \omega_j))$. Let

$$\xi \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix} = L \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_g \end{pmatrix};$$

L being a g-rowed real matrix. Then $|L| = N\xi$. Also

$$(\sigma(\omega_i^* \xi \omega_j)) = L(\sigma(\omega_i^* \omega_j)).$$

This proves that the determinant of the quadratic form is $d(\Lambda) N\xi$. A simple computation shows that the inverse of the matrix $(\sigma(\omega_i^* \xi \omega_j))$ is $(\sigma(\omega_i^{*\prime} \xi^{-1} \omega_j'))$. (14) is thus proved.

In particular, if $\mathfrak{o}$ is a maximal order in D and $\mathfrak{a}$ is an ideal with $\mathfrak{o}$ as left maximal order, then the complementary ideal $\tilde{\mathfrak{a}}$ (in the above sense) is an ideal with $\mathfrak{o}$ as right maximal order and

$$\vartheta(\mathfrak{a}, \xi) = \frac{1}{N\mathfrak{a}\sqrt{|d|}} \cdot \frac{1}{(N\xi)^{1/2}} \vartheta(\tilde{\mathfrak{a}}, \xi^{-1});$$

d being the discriminant (in the regular representation) of the minimal base $\delta_1, \ldots, \delta_g$ of $\mathfrak{o}$.

§ 5. Let $\Gamma(\mathfrak{o})$ be the group of units of the maximal order $\mathfrak{o}$. Let H denote the finite subgroup of units ϱ with

$$\varrho^* \varrho = 1 = \varrho \varrho^*.$$

The representation $x \to \varepsilon^* x \varepsilon$ of $\Gamma(\mathfrak{o})$ in P is discontinuous if we identify ε and $\varrho\varepsilon$. Siegel [10] has shown that there exists, in P, a fundamental domain for $\Gamma(\mathfrak{o})/H$. Let F denote the fundamental region in the subset of P consisting of x with $Nx \leqslant 1$ and F_0 the fundamental region on the determinantal surface $Nx = 1$.

Let s be a complex variable with real part > 1 and $\mathfrak{k}$ a class of left ideals with $\mathfrak{o}'$ as left maximal order. The zeta function of $\mathfrak{k}$ is

$$\zeta(s, \mathfrak{k}) = \sum_{\mathfrak{b} \in \mathfrak{k}} (N\mathfrak{b})^{-s}$$

the sum extended over all integral ideals in the class $\mathfrak{k}$. The zeta function of D is defined as

$$\zeta(s, D) = \sum_{\mathfrak{k}} \zeta(s, \mathfrak{k})$$

the sum extended over all classes $\mathfrak{k}$. Let $\mathfrak{a}$ be an ideal inverse to $\mathfrak{b}$ with $\mathfrak{o}$ as right maximal order. Then

$$\zeta(s, \mathfrak{k}) = N\mathfrak{a}^s \sum_{(\lambda) \subset \mathfrak{a}} |N\lambda|^{-s}$$

where λ runs through all elements $\neq 0$ in $\mathfrak{a}$ which are not equivalent on the left by $\Gamma(\mathfrak{o})$. The type of the maximal order $\mathfrak{o}$ is determined by $\mathfrak{k}$. Using the generalized gamma integral in § 3 we get

$$|d|^{s/2} \Gamma_D(s) \zeta(s, \mathfrak{k}) = N\mathfrak{a}^s |d|^{s/2} c^{gs/2} \sum_{(a) \subset \mathfrak{a}} \int_P (Nx)^{s/2} e^{-\pi c \sigma(a^* x a)} [dx]$$

where $c > 0$ is a constant. We choose c so that

$$c^{g/2} N\mathfrak{a} |d|^{1/2} = 1. \tag{15}$$

One can interchange summation and integration and obtain

$$|d|^{s/2} \Gamma_D(s) \zeta(s, \mathfrak{k}) = \int_P \sum_{(a) \subset \mathfrak{a}} e^{-\pi c \sigma(a^* x a)} (Nx)^{s/2} [dx].$$

We split up P into $Nx \geqslant 1$ and $Nx \leqslant 1$, write the full theta series, inside the integral, sum over a fundamental region and get the integral representation

$$\begin{aligned} |d|^{s/2} \Gamma_D(s) \zeta(s, \mathfrak{k}) &= \int_{Nx \geqslant 1} \Big\{ \sum_{(a) \subset \mathfrak{a}} e^{-\pi c \sigma(a^* x a)} (Nx)^{s/2} + \sum_{(a) \subset \tilde{\mathfrak{a}}} e^{-\pi c^{-1} \sigma(a^* x a)} (Nx)^{(1-s)/2} \Big\} [dx] + \\ &\quad + \frac{1}{w} \int_F \big((Nx)^{(s-1)/2} - (Nx)^{s/2} \big) [dx]; \end{aligned} \tag{16}$$

w being the order of H. If $[dx_1]$ is the invariant volume element on the norm surface $Nx = 1$, then

$$[dx] = \lambda (Nx)^{-1} d(Nx) [dx_1]$$

where $\lambda > 0$ is a constant. We thus get the final formula

$$|d|^{s/2} \Gamma_D(s) \zeta(s, \mathfrak{k}) = \int_{Nx \geqslant 1} + 2\lambda \left(\frac{1}{s-1} - \frac{1}{s} \right) \int_{F_0} [dx_1]. \tag{17}$$

The analytic continuation of $\zeta(s, \mathfrak{k})$ to the whole plane follows from (17). Also one obtains the functional equation

$$\xi(s) = |d|^{s/2} \Gamma_D(s) \zeta(s, D) = \xi(1-s)$$

for the zeta function of D.

In the case $f = 1$, that is in case D coincides with Z and so is commutative, one gets the Hecke integral representation for the Dedekind zeta function of Z. In this case $\mathfrak{o} = \mathfrak{o}'$ and

$$\int_{F_0} [dx_1],$$

which is finite, is *independent* of $\mathfrak{k}$. If we put $s > 2$ real we get, since $\int_{Nx \geqslant 1}$ is positive

$$|d|^{s/2} \Gamma_D(s) \zeta(s, \mathfrak{k}) > c_1 = 2\lambda \left(\frac{1}{s-1} - \frac{1}{s} \right) \int_{F_0} [dx_1].$$

Thus since

$$\sum_{\mathfrak{k}} \zeta(s, \mathfrak{k}) = \zeta(s, D) < c_2$$

for a certain constant c_2, it follows that *there are only finitely many classes of ideals* in Z.

As has been remarked by Siegel the fact that $\int\limits_{F_0}[dx_1]$ is finite in case of Z gives at once Dirichlet's theorem on the number of generators of $\Gamma(\mathfrak{o})$.

It is easy to see that in the case of Z one obtains for the class number h_0 of Z the inequality

$$h_0 R \leqslant \frac{|d| w \zeta(2, Z)}{\pi^h 4^{r_2}}$$

where R is the regulator of Z.

§ 6. In order to study the discriminant of a division algebra, we shall state a lemma due to Siegel [8].

Let R_n be Euclidean space of n dimensions whose points we denote by $X = (x_1, \ldots, x_n)$, $x_1, \ldots, x_n$ being the coordinates of X. A point is called a *lattice point* if $x_1, \ldots, x_n$ are integers. Let S be a bounded convex set in R_n which is symmetric about the origin and has in its interior (which is non-empty) no lattice point other than the origin. Then

Lemma (Siegel).

$$2^n = V + \frac{1}{V} \sum_{l \neq 0} \Big| \int\limits_S e^{-\pi i l \cdot X} dX \Big|^2$$

where V is the volume of S, $l \cdot X$ denotes the inner product in R_n and l runs through all lattice points $\neq 0$.

The proof is very simple. Let for instance $\varphi(X)$ be a bounded function vanishing outside S. Put

$$f(X) = \sum_g \varphi(2X + 2g);$$

g running through all lattice points. $f(X)$ is a bounded periodic function of period 1 in each of $x_1, \ldots, x_n$. The Parseval formula for the Fourier series of $f(X)$ gives at once, by the use of the property of S,

$$(18) \qquad 2^n \int\limits_S |\varphi(X)|^2 dX = \Big(\int\limits_S \varphi(X) dX\Big)^2 + \sum_{g \neq 0} \Big| \int\limits_S \varphi(X) e^{-\pi i g \cdot X} dX \Big|^2.$$

If $\varphi(X)$ is, for instance, the characteristic function of S, one obtains the result stated in the lemma.

We can obtain from this Minkowski's theorem on linear forms. Let

$$L_i(x) = \sum_j a_{ij} x_j, \quad i = 1, \ldots, n,$$

be n real linear forms whose matrix has determinant $d > 0$. Let $\delta_1, \ldots, \delta_n$ be n positive real numbers and suppose that the inequalities

$$|L_i(x)| \leqslant \delta_i, \quad i = 1, \ldots, n,$$

have no integral solution except $x_i = 0$, $i = 1, \ldots, n$. The volume of the convex set S defined by these inequalities is $2^n \delta_1 \ldots \delta_n / |d|$. Using the lemma above we have

$$(19) \qquad \frac{d}{\delta_1 \ldots \delta_n} = 1 + \sum_y{}' \prod_{i=1}^n \left\{ \frac{\sin\big(\pi \sum_{l=1}^n b_{kl} \delta_l y_l\big)}{\pi \sum_{l=1}^n b_{kl} \delta_l y_l} \right\}^2$$

where $\sum'$ denotes the sum over all integral $y_1, \ldots, y_n$ not simultaneously zero. This formula is proved by Siegel [7] by quite a different method.

This formula may be used to prove the Minkowski-Hajos theorem for small values of n.

Now let D be the division algebra of the previous section. Consider the convex set S given by the $a \in \bar{D}$ with

$$(20) \qquad \sigma(a^* a) \leqslant g,$$

where $a = \delta_1 x_1 + \ldots + \delta_g x_g$. Now

$$|Na|^2 \leqslant \left\{\frac{\sigma(a^* a)}{g}\right\}^g \leqslant 1,$$

so that since for $a \in \mathfrak{o}$, $a \neq 0$, $|Na| \geqslant 1$, it follows that S has in its interior no lattice points other than the origin. S is clearly convex. Its volume V is given by

$$V = \frac{\pi^{g/2} \cdot g^{g/2}}{\sqrt{|d|} \cdot \Gamma(g/2 + 1)}.$$

An elementary calculation gives

$$\int\limits_S e^{-\pi i x \cdot l} dx = (2g)^{g/2} \frac{J_{g/2}\big(\pi \sqrt{g T^{-1}[l]}\big)}{\big(T^{-1}[l]\big)^{g/4}}$$

where $T = \big(\sigma(\delta_i \delta_j)\big)$. Using Siegel's lemma one has the identity

$$(21) \qquad \sqrt{|d|} = \left(\frac{\pi}{4}\right)^{g/2} \frac{g^{g/2}}{\Gamma(g/2+1)} + \frac{\Gamma(g/2+1)}{(2\pi)^{g/2}} \sum_{\substack{\lambda \in \vartheta^{-1} \\ \lambda \neq 0}} \left(\frac{J_{g/2}\big(\pi \sqrt{g \sigma(\lambda^* \lambda)}\big)}{\big(\sigma(\lambda^* \lambda)\big)^{g/4}} \right)^2;$$

J_μ being the ordinary Bessel function, ϑ^{-1} is the complementary ideal to $\mathfrak{o}$ by means of the regular representation.

Unfortunately (21) cannot, in general, be used to prove the well-known theorem of Hasse-Brauer-Noether. This is because d is not the discriminant of D. If $h=1$ and $f=2$ and Δ is the discriminant of D, then $d=16\Delta$, and so (21) gives

$$|\Delta| \geqslant \tfrac{1}{4}\left(\frac{\pi}{4}\right)^2 \cdot \frac{16}{\Gamma(3)} = \frac{\pi^2}{8} > 1.$$

This proves that a quaternion algebra with Γ as centre splits if and only if it splits over every completion of Γ.

Let us take the special case $f=1$ so that d is the discriminant of the algebraic number field Z. (21) then gives

$$|d|^{1/2} \geqslant \left(\frac{\pi}{4}\right)^{h/2} \cdot \frac{h^{h/2}}{\Gamma(h/2+1)} > 1. \tag{22}$$

Using Stirling's formula for Γ-function, one sees that $|d| \to \infty$ as $h \to \infty$.

Let Z have r_1 real infinite and r_2 complex infinite prime spots so that $r_1+2r_2=h$. Consider the convex set S in R_n defined by

$$\begin{aligned} &|\xi^{(i)}| \leqslant 1, \quad 1 \leqslant i \leqslant r_1,\\ &\xi^{(i)2}+\eta^{(i)2} \leqslant 1, \quad r_1 < i \leqslant r_1+r_2, \end{aligned} \tag{23}$$

where

$$\xi^{(i)} = x_1\delta_1^{(i)}+\ldots+x_n\delta_n^{(i)}, \quad i \leqslant r_1,$$

$$\left.\begin{aligned} \xi^{(j)}+i\eta^{(j)} &= x_1\delta_1^{(j)}+\ldots+x_n\delta_n^{(j)}\\ \xi^{(j)}-i\eta^{(j)} &= x_1\delta_1^{(j+r_2)}+\ldots+x_n\delta_n^{(j+r_2)} \end{aligned}\right\} \quad r_1 < j \leqslant r_1+r_2.$$

Clearly S is convex and satisfies the conditions of the lemma. The volume V is given by

$$V = \frac{\pi^{r_2}\cdot 2^{r_1+r_2}}{\sqrt{|d|}}.$$

Evaluating the appropriate integrals, one obtains the formula

$$\sqrt{|d|} = \left(\frac{\pi}{2}\right)^{r_2} + \sum_{\substack{\lambda\epsilon\vartheta^{-1}\\ \lambda\neq 0}} \prod_{i=1}^{r_1}\left(\frac{\sin\pi\lambda^{(i)}}{\pi\lambda^{(i)}}\right)^2 \prod_{i=r_1+1}^{r_1+r_2}\left(\frac{J_1(2\pi|\lambda^{(i)}|)}{\sqrt{2}|\lambda^{(i)}|}\right)^2. \tag{24}$$

The special case $r_2=0$ is due to Siegel [6]. Using the fact that π is transcendental, one gets from (24) Minkowski's inequality [4]

$$|d| > \left(\frac{\pi}{2}\right)^{2r_2} \tag{25}$$

if $r_2>0$. But if $r_2=0$ one uses the fact that for $r_1>0$, ϑ^{-1} contains properly the ring of rational integers and so there are infinitely many terms on the right of (24) which do not vanish. Thus (25) is true even if $r_2=0$.

In this way one can obtain many other identities taking for S suitable convex bodies. If one takes for S the convex set

$$|\xi^{(i)}| \leqslant 1, \quad 1 \leqslant i \leqslant r_1,$$

$$|\xi^{(i)}| \leqslant \frac{1}{\sqrt{2}}, \quad |\eta^{(i)}| \leqslant \frac{1}{\sqrt{2}}, \quad r_1 < i \leqslant r_1+r_2,$$

then the lemma is applicable and gives the identity given by Müntz [5].

If in (18) we take $\varphi(X)$ to be some other function, not the characteristic function, we do get identities for $|d|$ but they do not give any interesting estimates for $|d|$. For instance, if $\varphi(X)$ is given by (if for simplicity we put $r_2=0$ and take S as defined by (23))

$$\varphi(X) = \begin{cases} \prod_{i=1}^{h}(1-|\xi^{(i)}|) & \text{if} \quad X\epsilon S,\\ 0 & \text{otherwise}, \end{cases}$$

$\xi = \delta_1x_1+\ldots+\delta_hx_h$, one then gets the identity

$$\sqrt{|d|} = (\tfrac{3}{4})^h\left(1+\sum_{\substack{\lambda\neq 0\\ \lambda\epsilon\vartheta^{-1}}}\prod_{i=1}^{h}\left(\frac{\sin\frac{1}{2}(\pi\lambda^{(i)})}{\frac{1}{2}(\pi\lambda^{(i)})}\right)^4\right). \tag{26}$$

By using summation processes of multiple Fourier series one can obtain from the Fourier series of the function $f(X)$ defined in the proof of the lemma of Siegel, other identities for $|d|$. But they do not give the sharp estimates for $|d|$ that we have obtained.

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Added in proof (5 August 1959). Professor Carl Siegel informs me (23 July 1959) that the theorem on splitting of rational quaternion algebras, which is deduced in § 6 as a consequence of formula (21), has been proved already by E. Witt (*Über ein Gegenbeispiel zum Normensatz*, Math. Zeit. 39 (1935), p. 467) by using number geometric methods.

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Note on a theorem of S. Uchiyama

by

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Let $Z(m, n)$ denote the number of systems of complex numbers $(z_1, z_2, \ldots, z_n)$ satisfying the system of equations

$$s_{m+1} = s_{m+2} = \ldots = s_{m+n-1} = 0,$$

where

$$s_k = z_1^k + z_2^k + \ldots + z_n^k$$

and m is an integer $\geqslant 0$. Two systems $(z_1, z_2, \ldots, z_n)$ and $z_1', z_2', \ldots, z_n'$ are *equivalent* in $Z(m, n)$ if there exists a complex number $\lambda \neq 0$ such that

$$f(x; z_1, z_2, \ldots, z_n) = f(x; \lambda z_1', \lambda z_2', \ldots, \lambda z_n'),$$

where

$$f(x; z_1, z_2, \ldots, z_n) = \prod_{j=1}^{n} (x - z_j).$$

Let $B(m, n)$ denote the number of classes of non-trivial sets relative to this equivalence relation. In a recent paper [1], Uchiyama has proved that

$$\sum_{d|(m,n)} a(d) B\left(\frac{m}{d}, \frac{n}{d}\right) = \frac{(m+n-1)!}{m!\, n!}, \tag{1}$$

where

$$a(1) = 1 \quad \text{and} \quad a(n) = n^{-1} \prod_{p|n} (1-p),$$

the product extending over all distinct prime divisors of n. A consequence of (1) is the elegant reciprocity relation

$$B(m, n) = B(n, m), \tag{2}$$

as noted by Uchiyama.