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On absolute convergence of Fourier series of some almost periodic functions

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1. In the present paper(¹) the author continues his investigations of [10]. Most of the results of [10] are special cases of the theorems presented here.

Given a function $f(x)$, almost periodic in the sense of Besicovitch with p-th power ($f\in B^p$) for a certain $1 < p \leqslant 2$(²) (for definition see [3], p. 77, 100), let the series

$$\sum_{n=1}^{\infty}(a_n\cos\lambda_n x+b_n\sin\lambda_n x) \tag{1}$$

be the Fourier series of the function $f(x)$, or a series extracted from that Fourier series, i. e. $a_n = M\{f(x)\cos\lambda_n x\}$, $b_n = M\{f(x)\sin\lambda_n x\}$, where $M\{g(x)\}$ denotes the mean value of the function $g(x)$ (see [3], p. 12). We shall always suppose $\lambda_n\uparrow\infty$(³).

Our paper is devoted to proving some sufficient convergence-conditions for the series

$$\sum_{n=1}^{\infty} n^{\beta}(|a_n|^{\gamma}+|b_n|^{\gamma}), \tag{2}$$

where $\beta\geqslant 0$, $0<\gamma\leqslant 2$. It is known that the convergence of the series (2) with $\beta = 0$, $\gamma = 1$ is equivalent to the absolute convergence of the series (1) (see [6], theorem 8.1.1). We shall suppose some generalized Hölder conditions to be satisfied or the generalized variation of the function $f(x)$ to be bounded. The theorems obtained here are of the type of Bern-

(¹) The results of this paper were presented on May the 2-nd, 1957, to the Polish Mathematical Society, Section Poznań.

(²) We will not repeat the assumption $1 < p \leqslant 2$. However, it is valid in all the theorems presented here.

(³) Since the terms of series (2) are non-negative, the assumption that λ_n is increasing is for $\beta = 0$ superfluous.

stein and Zygmund theorems (see e. g. [17], p. 135 and 136). They are generalizations of a number of known theorems (see the cited bibliography).

I wish to thank Professor Władysław Orlicz for his valuable remarks and friendly perusal of this paper.

2. First we introduce some necessary definitions and lemmas.

2.1. DEFINITIONS. (a) $\Delta^{(m)}(f; x, h)$ will denote the *difference of order* m $(m = 1, 2, \ldots)$ of the function $f(x)$, i. e.

$$\Delta^{(0)}(f; x, h) = f(x),$$

$$\Delta^{(m)}(f; x, h) = \Delta^{(m-1)}(f; x+h, h) - \Delta^{(m-1)}(f, x-h, h) \quad \text{for} \quad m = 1, 2, \ldots$$

Moreover, we write

$$a_n^{(m)} = M\{\Delta^{(m)}(f; x, h)\cos\lambda_n x\}, \qquad b_n^{(m)} = M\{\Delta^{(m)}(f; x, h)\sin\lambda_n x\}.$$

(b) The values

$$\omega_r^{(m)}(h) = \sup_{|\delta|\leqslant h}[M\{|\Delta^{(m)}(f; x, \delta/2)|^r\}]^{1/r},$$

where $r \geqslant 1$ and $m = 1, 2, \ldots$ and

$$\omega^{(m)}(h) = \lim_{T\to\infty} \operatorname*{ess\,sup}_{|x|\geqslant T, |\delta|\leqslant h} |\Delta^{(m)}(f; x, \delta/2)|,$$

where $m = 1, 2, \ldots$ will be called the *r-th integral modulus* of the function $f(x)$ of order m and the *modulus* of the function $f(x)$ of order m, respectively.

(c) Let Π be any partition $-T = x_0 < x_1 < \ldots < x_N = T$ of the interval $\langle -T, T\rangle$. Given $r \geqslant 1$, $T > 0$ and a positive integer m, we write

$$V_{r,T}^{(m)}(f) = \left[\sup_{\Pi} \sum_{n=1}^{N} \left| \Delta^{(m)}\left(f; \frac{x_{n-1}+x_n}{2}, \frac{x_n - x_{n-1}}{2m}\right)\right|^r\right]^{1/r}.$$

Then we call the value

$$V_r^{(m)}(f) = \overline{\lim_{T\to\infty}} \frac{1}{2T} V_{r,T}^{(m)}(f)$$

the *r-th variation* of the function $f(x)$ of order m.

(d) $\lambda(x)$ will denote any continuous increasing function, defined for $x \geqslant 1$ and such that $\lambda(n) = \lambda_n$, and $\mu(x)$ the function inverse to $\lambda(x)$. Similarly, given an increasing function $\varphi(x)$, we denote by $\psi(x)$ the function inverse to $\varphi(x)$.

2.2. LEMMAS. (i) *If* $a_1 \geqslant a_2 \geqslant \ldots > 0$ *and* $\varkappa$ *is a real, then the series*

$$\sum_{\nu=1}^{\infty} 2^{\varkappa\nu} a_{2^\nu} \quad \text{and} \quad \sum_{n=1}^{\infty} n^{\varkappa-1} a_n$$

are either both convergent or both divergent.

(ii) *Suppose that* $r \geqslant 1$, $h > 0$, $m = 1, 2, \ldots$ *Then*

$$M\{|\Delta^{(m)}(f; x, h)|^r\} \leqslant 2mh[V_r^{(m)}(f)]^r.$$

(iii) *Given* $q > 0$, $h > 0$ *and a positive integer* m, *we have*

$$|a_n^{(m)}|^q + |b_n^{(m)}|^q = 2^{mq}(|a_n|^q + |b_n|^q)|\sin\lambda_n h|^{mq}.$$

(iv) *If* $1 < p \leqslant 2$, $1/p + 1/q = 1$ *and* $f \in B^p$, *then*

$$\left[\sum_{n=1}^{\infty}(|a_n|^q + |b_n|^q)\right]^{1/q} \leqslant [M\{|f(x)|^p\}]^{1/p}$$

(the Young-Hausdorff inequality).

(v) *If* $1 < p \leqslant 2$, $1/p + 1/q = 1$, $f \in B^p$ *and* $A_\nu = \{n: 2^{\nu-1}\pi \leqslant \lambda_n < 2^\nu\pi\}$, *then for arbitrary positive integer* m,

$$\sum_{n\in A_\nu}(|a_n|^q + |b_n|^q) \leqslant 2^{-mq/2}[M\{|\Delta^{(m)}(f; x, 2^{-\nu-1})|^p\}]^{1/(p-1)}.$$

(vi) *By the assumptions of* (v) *we have*

$$\sum_{n\in A_\nu} n^\beta(|a_n|^q + |b_n|^q)^{\gamma/q}$$
$$\leqslant n_\nu^\beta[\mu(2^\nu\pi) - \mu(2^{\nu-1}\pi) + 1]^{1-\gamma(p-1)/p}\left[\sum_{n\in A_\nu}(|a_n|^q + |b_n|^q)\right]^{\gamma(p-1)/p},$$

where $n_\nu = \max_{n\in A_\nu} n$.

(vii) *Given* $0 < \gamma \leqslant q$, *the series*

$$\sum_{n=1}^{\infty} n^\beta(|a_n|^\gamma + |b_n|^\gamma) \quad \text{and} \quad \sum_{n=1}^{\infty} n^\beta(|a_n|^q + |b_n|^q)^{\gamma/q}$$

are either both convergent or both divergent.

2.3. Proofs of the lemmas. Lemma (i) is known (see e. g. [10], p. 13).

To prove lemma (ii)(4), suppose that $V_r^{(m)}(f) < \infty$. Given $\varepsilon > 0$, let us choose a number $T_0 > 0$ such that for $T > T_0$,

$$[V_{r,T+3mh}^{(m)}(f)]^r \leqslant 2(T+3mh)\{[V_r^{(m)}(f)]^r + \varepsilon\}.$$

(4) This lemma has been proved for $m = 1$ for periodic functions by L. C. Young ([16], p. 259) and for almost periodic functions by the author ([10], p. 12, lemma 1) with Φ-th variation instead of r-th variation. Lemma (ii) remains true, of course, also for Φ-th variations.

For fixed $T > T_0$, let us consider the partition $-T = x_0 < x_1 < \ldots < x_N = T$, defined by formulas $x_n - x_{n-1} = 2mh$ for $n = 1, 2, \ldots, N-1$, $x_N - x_{N-1} \leqslant 2mh$. Then

$$\int_{-T}^{T} |\Delta^{(m)}(f; x, h)|^r dx = \int_{-mh}^{mh} \sum_{n=1}^{N-1} \left| \Delta^{(m)}\left(f; \frac{x_{n-1}+x_n}{2} + t, \frac{x_n - x_{n-1}}{2m}\right)\right|^r dt +$$

$$+ \int_{-(x_N - x_{N-1})/2}^{(x_N - x_{N-1})/2} \left| \Delta^{(m)}\left(f; \frac{x_{N-1}+x_N}{2} + t, h\right)\right|^r dt$$

$$\leqslant 2mh [V_{r, T+3mh}^{(m)}(f)]^r < 4mh(T+3mh)\{[V_r^{(m)}(f)]^r + \varepsilon\}.$$

Thus

$$M\{|\Delta^{(m)}(f; x, h)|^r\} \leqslant 2mh\{[V_r^{(m)}(f)]^r + \varepsilon\}.$$

Equality (iii) follows from formulas

$$a_n^{(m+1)} = 2b_n^{(m)} \sin\lambda_n h, \qquad b_n^{(m+1)} = -2a_n^{(m)} \sin\lambda_n h \qquad (m = 0, 1, 2, \ldots).$$

The proof of the Young-Hausdorff inequality (iv) may be obtained as in the periodic case without essential changes (compare the proof for periodic functions in [8]).

Now we prove (v). Applying (iv) to the function $\Delta^{(m)}(f; x, 2^{-\nu-1})$ and then (iii), we obtain

$$\sum_{n=1}^{\infty} (|a_n|^q + |b_n|^q) |\sin\lambda_n 2^{-\nu-1}|^{mq} \leqslant 2^{-mq}\left[M\{|\Delta^{(m)}(f; x, 2^{-\nu-1})|^p\}\right]^{1/(p-1)}.$$

Since for $n \in A_\nu$, $|\sin\lambda_n 2^{-\nu-1}| \geqslant 2^{-1/2}$, we have

$$\sum_{n\in A_\nu} (|a_n|^q + |b_n|^q) \leqslant 2^{mq/2} \sum_{n=1}^{\infty} (|a_n|^q + |b_n|^q)|\sin\lambda_n 2^{-\nu-1}|^{mq}$$

$$\leqslant 2^{-mq/2}\left[M\{|\Delta^{(m)}(f; x, 2^{-\nu-1})|^p\}\right]^{1/(p-1)}.$$

To prove (vi) let us denote by $|A_\nu|$ the number of elements of the set A_ν. It follows from the Hölder inequality that

$$\sum_{n\in A_\nu} n^\beta (|a_n|^q + |b_n|^q)^{\gamma/q} \leqslant \Big[\sum_{n\in A_\nu} n^{\beta q/(q-\gamma)}\Big]^{1-\gamma/q} \Big[\sum_{n\in A_\nu} (|a_n|^q + |b_n|^q)\Big]^{\gamma/q}$$

$$\leqslant n_\nu^\beta |A_\nu|^{1-\gamma/q} \Big[\sum_{n\in A_\nu} (|a_n|^q + |b_n|^q)\Big]^{\gamma/q}.$$

Since $|A_\nu| \leqslant \mu(2^\nu\pi) - \mu(2^{\nu-1}\pi) + 1$, we obtain (vi).

Lemma (vii) follows from inequality $\frac{1}{2}(1+x^q)^{\gamma/q} \leqslant 1 + x^\gamma \leqslant 2(1+x)^{\gamma/q}$, valid for $0 \leqslant x \leqslant 1$, $0 < \gamma \leqslant q$.

3. Now we prove two theorems concerning the convergence of series (2), analogous to theorems 1 and 2 of [10]. The first generalizes the known theorem of Bernstein ([1] and [2]), Szász ([14] and [15], p. 376, theorem 3.1) and Hardy ([7], p. 631, theorem 8 and [17], p. 143, example 6), the second one the theorems of Zygmund ([17], pp. 136 and 138), Haršiladze ([4], p. 203) and others. We shall apply indications introduced in 2.

3.1. *Given $\beta \geqslant 0$ and $0 < \gamma < 2$, let us suppose that $f \in B^p$ and*

$$\sum_{\nu=1}^{\infty} n_\nu^\beta [\mu(2^\nu\pi) - \mu(2^{\nu-1}\pi) + 1]^{1-\gamma(p-1)/p} [\omega_p^{(m)}(2^{-\nu})]^\gamma < \infty$$

for a positive integer m. Then series (2) is convergent.

3.2. *If $\beta \geqslant 0$, $0 < \gamma < 2$, $f \in B^p$, $f(x)$ is for a certain $1 \leqslant r \leqslant p$ of finite r-th variation of order m and*

$$\sum_{\nu=1}^{\infty} n_\nu^\beta [\mu(2^\nu\pi) - \mu(2^{\nu-1}\pi) + 1]^{1-\gamma(p-1)/p} 2^{-\gamma r/p} [\omega^{(m)}(2^{-\nu})]^{\gamma(p-r)/p} < \infty,$$

then series (2) is convergent(5).

3.3. Theorem 3.1 follows from (vii), (vi) and (v).

To prove 3.2 let us remark that, given $T > 0$, we have

$$M\{|\Delta^{(m)}(f; x, 2^{-\nu-1})|^p\} \leqslant \operatorname*{ess\,sup}_{|x|\geqslant T,\ |\delta|\leqslant 2^{-\nu}} |\Delta^{(m)}(f; x, \delta/2)|^{p-r} M\{|\Delta^{(m)}(f; x, 2^{-\nu-1})|^r\}.$$

Supposing $T \to \infty$ we obtain

$$M\{|\Delta^{(m)}(f; x, 2^{-\nu-1})|^p\} \leqslant [\omega^{(m)}(2^{-\nu})]^{p-r} M\{|\Delta^{(m)}(f; x, 2^{-\nu-1})|^r\}.$$

To the above inequality we apply inequality (ii) and then the inequality obtained to (v). Then according to (vii) and (vi) we obtain 3.2 in the same way as 3.1.

Now we shall consider some applications of theorems 3.1 and 3.2 assuming that λ_n satisfies further conditions.

4. We consider an increasing sequence $\varphi_n > 0$ such that for every $\varepsilon > 0$ there exists a positive integer k not depending on n such that $\varphi_n \leqslant \varepsilon\varphi_{kn}$ for $n = 1, 2, \ldots$ Let us suppose that $\varphi_n = O(\varphi_{n-1})$ and that the sequence $\lambda_n \uparrow \infty$ satisfies the condition $\varphi_n = O(\lambda_n)$. We choose increasing functions $\varphi(x)$ and $\lambda(x)$ such that $\varphi(n) = \varphi_n$ and $\lambda(n) = \lambda_n$. Then for every $\varepsilon > 0$ there exists a number $k' > 0$ such that $\varphi(x) \leqslant \varepsilon\varphi(k'x)$ for $x \geqslant 1$. Moreover, $\varphi(x) = O[\varphi(x-1)]$ and $\varphi(x) = O[\lambda(x)]$. Let us denote by $\psi(x)$ and $\mu(x)$ the functions inverse to $\varphi(x)$ and $\lambda(x)$, respectively. Then, given any $k > 0$, there exists a k' such that $\psi(kx) \leqslant k'\psi(x)$,

(5) In 3.1 and 3.2 and in all the subsequent theorems in this paper as well one must pay attention to (3).

whence $\varphi(x) = O[\lambda(x)]$ implies $\mu(x) = O[\psi(x)]$. Especially, we obtain

$$\mu(2^{\nu}\pi) - \mu(2^{\nu-1}\pi) + 1 \leqslant \mu(2^{\nu}\pi) + 1 = O(\psi_{2^{\nu}}),$$

where $\psi_n = \psi(n)$. Similarly, $n_{\nu} = \max_{n\in\Delta_\nu} n = O(\psi_{2^{\nu}})$. Hence theorems 3.1 and 3.2 can be formulated as follows.

4.01. *Let us consider an increasing sequence $\varphi_n > 0$ satisfying the following condition: given an arbitrary $\varepsilon > 0$, there exists a positive integer k such that $\varphi_n \leqslant \varepsilon\varphi_{kn}$ for $n = 1, 2, \ldots$ Let us suppose that $\varphi_n = O(\varphi_{n-1})$, $f \in B^p$, $\varphi_n = O(\lambda_n)$ and*

$$\sum_{\nu=1}^{\infty} \psi_{2^{\nu}}^{\beta+1-\gamma(p-1)/p} [\omega_p^{(m)}(2^{-\nu})]^{\gamma} < \infty \tag{3}$$

for certain $\beta \geqslant 0$, $0 < \gamma < 2$ and a positive integer m. Then series (2) *is convergent.*

4.02. *Let us suppose the sequence $\varphi_n > 0$ to satisfy the same condition as in* 4.01 *except* (3). *Further let us assume that $f(x)$ is of finite r-th variation of order m for a certain $1 \leqslant r \leqslant p$. If we have*

$$\sum_{\nu=1}^{\infty} \psi_{2^{\nu}}^{\beta+1-\gamma(p-1)/p} 2^{-\gamma\nu/p} [\omega^{(m)}(2^{-\nu})]^{\gamma(p-r)/p} < \infty \tag{4}$$

for certain $\beta \geqslant 0$ and $0 < \gamma < 2$, then series (2) *is convergent.*

4.1 If we put in 4, $\varphi_n = n^{\varrho}\log_2^{\delta} n$, where $\varrho > 0$ and δ is a real, and take $\varphi(x) = x^{\varrho}\log_2^{\delta} x$, then $\psi_{2^{\nu}} = O(2^{\nu/\varrho}\nu^{-\delta/\varrho})$. Thus, if we have $\varphi_n = O(\lambda_n)$ then conditions (3) and (4) follow from the conditions

$$\sum_{\nu=1}^{\infty} 2^{[\beta+1-\gamma(p-1)/p]\nu/\varrho} \nu^{-\delta[\beta+1-\gamma(p-1)/p]/\varrho} [\omega_p^{(m)}(2^{-\nu})]^{\gamma} < \infty \tag{5}$$

and

$$\sum_{\nu=1}^{\infty} 2^{[\beta p + p(1-\gamma) + \gamma(1-\varrho)]\nu/\varrho p} \nu^{-\delta[\beta+1-\gamma(p-1)/p]/\varrho} [\omega^{(m)}(2^{-\nu})]^{\gamma(p-r)/p} < \infty, \tag{6}$$

respectively.

To simplify the notation we first consider the case $\delta = 0$ in detail.

4.2. Suppose that $n^{\varrho} = O(\lambda_n)$ for a certain $\varrho > 0$. Then lemma (i) implies that conditions (5) and (6) are equivalent to the conditions

$$\sum_{n=1}^{\infty} n^{[\beta+1-\varrho-\gamma(p-1)/p]/\varrho} [\omega_p^{(m)}(n^{-1})]^{\gamma} < \infty \tag{7}$$

and

$$\sum_{n=1}^{\infty} n^{[(\beta+1-\varrho-\gamma)p+\gamma(1-\varrho)]/\varrho p} [\omega^{(m)}(n^{-1})]^{\gamma(p-r)/p} < \infty, \tag{8}$$

respectively. Hence we have the following theorems.

4.21. *If $n^{\varrho} = O(\lambda_n)$, $f \in B^p$, $\beta \geqslant 0$, $0 < \gamma < 2$ and condition* (7) *holds, then series* (2) *is convergent.*

4.22. *If $n^{\varrho} = O(\lambda_n)$, $f \in B^p$, $V_r^{(m)}(f) < \infty$ for a certain $1 \leqslant r \leqslant p$, $\beta \geqslant 0$, $0 < \gamma < 2$ and forapositive integer m and if condition* (8) *holds, then the series* (2) *is convergent.*

Theorems 4.21 and 4.22 contain theorems 3 and 4 of [10] if we put $\beta = 0$, $p = 2$ and $m = 1$. Moreover, theorems 4.21 and 4.22 imply for almost periodic functions a number of known theorems, e. g. some theorems of Bernstein [2] and Stečkin [13] for $\beta = 0$, $m = \varrho = \gamma = 1$, $p = 2$, a theorem of Zygmund (see e. g. [13], p. 231) for $\beta = 0$, $m = \varrho = \gamma = r = 1$, $p = 2$, a theorem of Szász ([15], p. 376, theorem 3.1) and others(6).

4.3. Now let us suppose that $n^{\varrho} = O(\lambda_n)$ and $\omega_p^{(m)}(h) = O(h^{\alpha})$ for small h for certain $\varrho > 0$, $\alpha > 0$. Then the following corollary results from 4.21.

4.31. *If $n^{\varrho} = O(\lambda_n)$, $f \in B^p$, $\omega_p^{(m)}(h) = O(h^{\alpha})$ for small h and*

$$\gamma > \frac{p(\beta+1)}{p+\alpha\varrho p-1}$$

for certain $\varrho > 0$, $\alpha > 0$, $\beta \geqslant 0$ and $0 < \gamma < 2$, then series (2) *is convergent.*

Remark. Since, for $\beta \geqslant 0$, $p(\beta+1)/(p+\alpha\varrho p-1)$ decreases by increasing p, the minimal value of γ also decreases by increasing p.

If we put in 4.31, $\beta = 0$ and $m = \varrho = 1$, then we obtain the Szász-Hardy theorem (see [14], [7], p. 631, theorem 8 and [17], p. 143, example 5) for almost periodic functions and $1 < p \leqslant 2$.

Theorem 4.31 yields for $\beta = 0$ and $\gamma = 1$ the following condition of absolute convergence of series (1).

4.32. *If $n^{\varrho} = O(\lambda_n)$, $f \in B^p$ and $\omega_p^{(m)}(h) = O(h^{\alpha})$ for small h for certain $\varrho > 0$ and $\alpha > 1/\varrho p$, then series* (1) *is absolutely convergent.*

4.4. We obtain from 4.22 the following corollary.

4.41. *Let us suppose that the function $f \in B^p$ satisfies for certain $\varrho > 0$, $1 \leqslant r \leqslant p$, $\alpha > 0$, $\beta \geqslant 0$, $0 < \gamma < 2$ and a positive integer m the conditions $n^{\varrho} = O(\lambda_n)$, $V_r^{(m)}(f) < \infty$, $\omega^{(m)}(h) = O(h^{\alpha})$ for small h and*

$$\gamma > \frac{p(\beta+1)}{(p+\alpha\varrho p-1)+\varrho(1-\alpha r)}. \tag{9}$$

Then series (2) *is convergent.*

(6) Theorem 4.21 has been proved in the case $\beta = 0$, $\varrho = \gamma = 1$, $p = 2$ independently by the author ([10], p. 14, where it is called the Bernstein condition and follows from theorem 3 on p. 13) and by Kupcov (see [9], p. 169, theorem 3).

If $p = 2$ then inequality (9) may be replaced by the inequality

$$\gamma > r(\beta+1)/(r+\varrho-1) \quad \text{for} \quad 1-ar \geqslant \varrho^{-1} \text{(7)}.$$

This theorem constitutes a generalization of known results of Zygmund-Waraszkiewicz (see [17], p. 138) and Haršiladze (see [4], p. 203, theorem 4) for almost periodic functions. The first may be obtained from 4.41 by $\beta = 0$, $m = \varrho = r = 1$, $p = 2$, the second by $\beta = 0$, $\varrho = r = \gamma = 1$ and $m = p = 2$.

Theorem 4.41 yields the following condition of absolute convergence of series (1).

4.42. *If $f \in B^p$, $n^\varrho = O(\lambda_n)$, $V_r^{(m)}(f) < \infty$ and $\omega^{(m)}(h) = O(h^a)$ for small h for certain constants $\varrho > 0$, $1 \leqslant r < p$, $a > 0$ and $a > (1-\varrho)/\varrho(p-r)$, then series (1) is absolutely convergent.*

4.5. Now we shall investigate conditions which appear if we replace the assumptions $\omega_p^{(m)}(h) = O(h^a)$ and $\omega^{(m)}(h) = O(h^a)$ by $\omega_p^{(m)}(h) = O(h^a \log_2^{-a'} h)$ and $\omega^{(m)}(h) = O(h^a \log_2^{-a'} h)$ $(h > 0)$, respectively(8). We formulate only a theorem, analogous to 4.32.

4.51. *Let us suppose that the function $f \in B^p$ satisfies the conditions $n^\varrho = O(\lambda_n)$ and $\omega_p^{(m)}(h) = O(h^a \log_2^{-a'} h)$ for small h, where $\varrho > 0$, $a > 0$ and a' real are certain constants. Further let us assume that either $a > 1/\varrho p$ or $a = 1/\varrho p$ and $a' > 1$. Then series (1) is absolutely convergent.*

4.6. Now we return to the general case $\varphi_n = n^\varrho \log_2^\delta n$ (see 4.1) for $\varrho > 0$ and $\delta \geqslant 0$. We formulate theorems 4.61 and 4.62, analogous to 4.32 and 4.42, respectively.

4.61. *Let us suppose that $f \in B^p$, $n^\varrho \log_2^\delta n = O(\lambda_n)$ and $\omega_p^{(m)}(h) = O(h^a)$ for small h for certain $\varrho > 0$, $\delta \geqslant 0$ and $a > 0$. Moreover, assume either $a > 1/\varrho p$ or $a = 1/\varrho p > 1/\delta$. Then series (1) is absolutely convergent.*

4.62. *Let us suppose that the function $f \in B^p$ satisfies the conditions $n^\varrho \log_2^\delta n = O(\lambda_n)$, $V_r^{(m)}(f) < \infty$ and $\omega^{(m)}(h) = O(h^a)$ for small h for certain $\varrho > 0$, $\delta \geqslant 0$, $1 \leqslant r < p$ and $a > 0$. Further let us suppose that either*

(7) If $p = 2$ then $f \in B^p$ for $1 \leqslant p \leqslant 2$. Let us put $\gamma(p)$ in place of the right hand side of inequality (9). If $\gamma > \gamma(p)$ for a certain $1 < p \leqslant 2$, then one may apply 4.41. Evidently, the best choice of p is such that $\gamma(p)$ is minimal. Since for $1-ar < \varrho^{-1}$, $\gamma(p)$ decreases, $\gamma(2) \leqslant \gamma(p)$ and the best value is $p = 2$. However, for $1-ar > \varrho^{-1}$, $\gamma(p)$ increases, $\gamma(1) \leqslant \gamma(p)$ and the best value is $p = 1$. Thus it suffices to take $\gamma > \gamma(1)$. For $1-ar = \varrho^{-1}$, $\gamma(p) = \text{const}$.

(8) Compare [5]. The first part of theorem 2 in [5], p. 804, is a special case of theorem 4.51 formulated below. Theorem 3 in [5] can also be generalized by applying the method used here.

$a > (1-\varrho)/\varrho(p-r)$ or $a = (1-\varrho)/\varrho(p-r) > (1-\varrho)/(\delta-\varrho r)$. Then series (1) is absolutely convergent.

4.7. Theorems 4.31-4.42 remain true if we replace the hypothesis $n^\varrho = O(\lambda_n)$ by the hypothesis $n^\varrho \log_2^\delta n = O(\lambda_n)$, where $\delta \leqslant 0$.

4.8. Let us also remark that the terms of the series (2) may be replaced by more general ones of form $n^\beta \log_2^{\beta'} n$, where $\beta \geqslant 0$ and $\beta' \geqslant 0$. Also, instead of the sequence n^β, a sequence $\sigma_n \geqslant 0$, $\sigma_n \uparrow \infty$ can be considered if the following condition is satisfied: given any positive integer k, there exists a number $k' > 0$ such that $\sigma_{kn} \leqslant k' \sigma_n$ for $n = 1, 2, \ldots$ The method of proof is analogous to that used above. The same applies to the results presented below.

5. Now we consider an increasing, differentiable and weakly concave(9) function $\varphi(x) > 0$, defined for $x \geqslant 1$. We suppose that the sequence λ_n satisfies for a certain $q > 1$ the inequality $\lambda_{n+1}/\lambda_n \geqslant q^{\varphi(n+1)-\varphi(n)}$(10). Then the sequence $\lambda_n/q^{\varphi(n)}$ is non-decreasing. We choose a function $\lambda(x)$ such that $\lambda(n) = \lambda_n$ and that $\lambda(x)/q^{\varphi(x)}$ is a non-decreasing function. It is easily seen that the function $\lambda(x)$ is increasing. Denote by $\psi(x)$ and $\mu(x)$ the functions inverse to $\varphi(x)$ and $\lambda(x)$, respectively. We have $\varphi\big(\mu(x)\big) \leqslant \log_q(\lambda_1^{-1} q^{\varphi(1)} x)$. Moreover, $\varphi(y) = \log_q x - \log_q\big(\lambda(y)/q^{\varphi(y)}\big)$ for $y = \mu(x)$. Hence, if we indicate $y_1 = \varphi\big(\mu(x)\big)$ and $y_2 = \varphi\big(\mu(2x)\big)$, then we obtain $y_2 - y_1 = O(1)$ and $\mu(2x) - \mu(x) \leqslant (y_2 - y_1)\psi'(y_2) = O\{\psi'[\log_q 2\lambda_1^{-1} q^{\varphi(1)} x]\}$, whence $\mu(2^\nu \pi) - \mu(2^{\nu-1}\pi) + 1 = O[\psi'(\log_q C2^\nu)]$, where $C = \lambda_1^{-1} q^{\varphi(1)} \pi$. Moreover, $n_\nu = \max_{n \in A_\nu} n \leqslant \mu(2^\nu \pi) \leqslant \psi(k\nu)$, where $k = \log_q 2C$. Hence theorems 3.1 and 3.2 imply the following theorems.

5.01. *Let us suppose that the function $f \in B^p$ satisfies the condition $\lambda_{n+1}/\lambda_n \geqslant q^{\varphi(n+1)-\varphi(n)}$ for a certain $q > 1$ and an increasing, weakly concave and differentiable function $\varphi(x) > 0$. Further let us suppose that for certain $\beta \geqslant 0$ and $0 < \gamma < 2$,*

$$\sum_{\nu=1}^{\infty} \psi^\beta(l\nu)[\psi'(\log_q C2^\nu)]^{1-\gamma(p-1)/p}[\omega_p^{(m)}(2^{-\nu})]^\gamma < \infty \tag{10}$$

for every $l \geqslant 1$(11). Then series (2) is convergent.

(9) I. e. $\varphi[(x+y)/2] \geqslant [\varphi(x)+\varphi(y)]/2$.

(10) For $\varphi(n) = n$ this means that series (1) is lacunary.

(11) If the function $\varphi(x)$ satisfies the following condition: given any $\varepsilon > 0$, there exists a number $k > 0$ such that $\varphi(x) \leqslant \varepsilon\varphi(kx)$ for every $x \geqslant 1$ (compare the assumptions of theorems 4.01 and 4.02) then it suffices to take $l = 1$ in theorems 5.01 and 5.02.

5.02. *Let us consider a function* $f \in B^p$ *such that* $V_r^{(m)}(f) < \infty$ *and* $\lambda_{n+1}/\lambda_n \geqslant q^{\varphi(n+1)-\varphi(n)}$ *for certain* $1 \leqslant r \leqslant p$, $q > 1$ *and an increasing, weakly concave and differentiable function* $\varphi(x) > 0$. *If*

$$(11) \qquad \sum_{\nu=1}^{\infty} \psi^{\beta}(l\nu)[\psi'(\log_q C2^{\nu})]^{1-\gamma(p-1)/p} 2^{-\gamma\nu/p}[\omega^{(m)}(2^{-\nu})]^{\gamma(p-r)/p} < \infty$$

for certain $\beta \geqslant 0$, $0 < \gamma < 2$ *and every* $l \geqslant 1$, *then series* (2) *is convergent.*

5.1. Both the above theorems imply the following interesting corollary.

5.11. *Given an increasing, weakly concave and differentiable function* $\varphi(x) > 0$, *defined for* $x \geqslant 1$ *and satisfying the condition* $2^{-\varepsilon\varphi(x)} = O[\varphi'(x)]$ *for every* $\varepsilon > 0$, *we consider a function* $f \in B^p$ *such that* $\lambda_{n+1}/\lambda_n \geqslant q^{\varphi(n+1)-\varphi(n)}$ *for a certain* $q > 1$. *Let us suppose that either of the following two conditions is satisfied:*

(α) $\omega_p^{(m)}(h) = O(h^{\alpha})$ *for small* h *and for a certain* $\alpha > 0$,

(β) $V_r^{(m)}(f) < \infty$ *and* $\omega^{(m)}(h) < \infty$ *for certain* $1 \leqslant r \leqslant p$ *and* $h > 0$.

Then series (2) *is convergent for every* $\beta \geqslant 0$ *and* $0 < \gamma < 2$.

To prove this corollary let us first remark that the condition $2^{-\varepsilon\varphi(x)} = O[\varphi'(x)]$ implies $\psi'(x) = O(2^{\varepsilon x})$ for large x. Indeed, for a fixed $\varepsilon > 0$ we can choose a number K_ε such that for $x \geqslant 1$, $2^{-\varepsilon\varphi(x)} \leqslant K_\varepsilon \varphi'(x)$. Hence $1/\varphi'(x) \leqslant K_\varepsilon 2^{\varepsilon\varphi(x)}$ and we obtain $\psi'(x) = 1/\varphi'(\psi(x)) \leqslant K_\varepsilon 2^{\varepsilon x}$.

Thus we have $\psi'(\log_q C2^{\nu}) \leqslant 2^{\varepsilon\nu} K_\varepsilon \log_q 2C$ (supposing $C \geqslant 1$). Moreover, $\psi(x) \leqslant \psi(1) + x\psi'(x)$. Hence, given $l \geqslant 1$, there exists a constant K'_ε such that

$$\psi^{\beta}(l\nu)[\psi'(\log_q C2^{\nu})]^{1-\gamma(p-1)/p} \leqslant K'_\varepsilon \nu^{\beta} 2^{\varepsilon\nu(l\beta+\log_q 2C)}.$$

Now let us suppose (α) to be satisfied and choose

$$\varepsilon < \frac{\alpha\gamma}{l\beta+\log_q 2C}.$$

Then series (10) is convergent and we can apply theorem 5.01.

If the condition (β) is satisfied it suffices to choose

$$\varepsilon < \frac{\gamma}{p(l\beta+\log_q 2C)}$$

to obtain the convergence of series (11) and thus also of series (2). This completes the proof of 5.11.

As an example of function $\varphi(x)$ satisfying the assumption of theorem 5.11 we can choose for instance $\varphi(x) = x^{\varrho}$, where $0 < \varrho \leqslant 1$. Thus, 5.11

is applicable to lacunary series (1) ($\varrho = 1$)([12]) and to series (1) such that the sequence λ_n increases more slowly than for lacunary series but yet sufficiently rapidly ($0 < \varrho < 1$).

5.2. Applying (i), 5.01 and 5.02 one can obtain results analogous to theorems 4.21, 4.22 and others and constituting generalizations of theorems 6, 6′ and 7 of [10].

5.3. We remark that theorems 5.01 and 5.02 contain for $\lambda_n = Cn^{\varrho}$ ($\varrho > 0$) and $\beta = 0$ theorems 4.21 and 4.22, respectively. Indeed, the sequence $\lambda_n = Cn^{\varrho}$ satisfies the condition $\lambda_{n+1}/\lambda_n = q^{\varphi(n+1)-\varphi(n)}$ with $q = 2$ for $\varphi(x) = \varrho\log_2 x$. Since $\psi(x) = 2^{x/\varrho}$, we have $\psi'(\log_2 C2^{\nu}) = O(2^{\nu/\varrho})$.

6. We also remark that the results obtained here for almost periodic functions of one variable can be generalized to functions of several variables. One can apply here the method used by the author in [11] and [12] for periodic functions.

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([12]) Compare also [10], p. 14-15. This result is partly connected with Sidon's theorem on lacunary trigonometric series (see [17], p. 139), generalized to almost periodic functions by T. Kawata and M. Udagawa in the paper *Some gap theorems*, Kōdai Math. Sem. Rep. (1949), p. 99-102, for S^2 a. p. functions and by *Е. А. Бредихина* in the paper *Об абсолютной сходимости рядов Фурье почти-периодических функций*, ДАН СССР 111 (1956), p. 1163-1166, for u. a. p. functions.

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Remarque sur un travail de J. Schauder

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Le but de cette note est de combler une lacune dans le travail de J. Schauder sur l'équation hyperbolique quasilinéaire aux dérivées partielles du second ordre [2]. La lacune en question pourrait faire croire au lecteur que la méthode, appliquée par J. Schauder dans son travail fondamental pour la théorie des équations hyperboliques, n'est pas rigoureuse. C'est pourquoi nous avons cru utile de montrer que cette lacune ne tient qu'à l'application d'un théorème peu général et qu'elle peut être comblée grâce à une généralisation de ce théorème. Or, dans le travail cité, J. Schauder fait intervenir un théorème sur les limitations a priori, dû à K. Friedrichs et H. Lewy [1], qui peut être énoncé de la façon suivante.

Soit

$$(1)\quad \sum_{i,k=1}^{n} A_{ik}(x_1,\ldots,x_n)\frac{\partial^2 u}{\partial x_i x_k}+\sum_{j=1}^{n} B_j(x_1,\ldots,x_n)\frac{\partial u}{\partial x_j}+C(x_1,\ldots,x_n)u = F(x_1,\ldots,x_n)\qquad (A_{ik}=A_{ki})$$

une équation hyperbolique normale. Supposons que dans une pyramide P_n, dont la base b_{n-1} est située dans le plan $x_n=\text{const}$ et les faces latérales possèdent l'orientation d'espace par rapport à l'équation (1), les coefficients satisfassent aux conditions suivantes:

1° A_{ik}, B_j, C, F sont de classe C^1,

2° la forme quadratique $\sum_{i,k=1}^{n-1} A_{ik}\lambda_i\lambda_k$ est positive définie et $A_{nn}<0$,

3° $|A_{ik}|$, $|B_j|$, $|C|$, $\left|\dfrac{\partial A_{ik}}{\partial x_j}\right| \leqslant M_1$.

Ceci admis, il existe deux nombres positifs $h(M_1)$ et $C_1(M_1)$ dépendant de M_1 tels que, pour toute solution u de l'équation (1) définie et