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On the functional equation $F(x, \varphi(x), \varphi[f(x)]) = 0$

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§ 1. The object of the present paper is the functional equation

$$F(x, \varphi(x), \varphi[f(x)]) = 0, \tag{1}$$

where $\varphi(x)$ denotes the required function, and $f(x)$ and $F(x, y, z)$ denote known functions.

Equation (1) is a natural generalization of the equation

$$\varphi(x) + \varphi[f(x)] = F(x) \tag{2}$$

solved in [2] by the second author of this paper. Equation (2), under some natural assumptions, possesses infinitely many solutions, which are continuous for every x that is not a root of the equation

$$f(x) = x. \tag{3}$$

However, if we require the solution to be continuous for $x = x_0$, fulfilling (3), then it turns out that there exists at most one such solution.

In this paper we have tried to get similar results for equation (1), with partial success only. The results regarding the solutions of equation (1) which are continuous for roots of equation (3) are not quite satisfying. Especially the hypotheses of theorem VI seem to be absolutely too strong. Likewise the problem of finding some natural criteria for the function F which would guarantee the existence of such solutions remains open.

T. Kitamura [1] discusses a similar equation. Namely, he proves that the equation

$$F(\varphi[f(x, \lambda)], \varphi(x), x, \lambda) = 0$$

possesses, under suitable conditions, a solution containing an arbitrary function. His paper, however, is less general than ours (the parameter λ is quite unnecessary, since it may be include din the definition of the functions F and f); he does not discuss the regularity of solutions and he assumes the hypotheses regarding the function F to be fulfilled in the whole space. Moreover, not all of his results are correct (see § 3 below).

A. H. Read [3] treats the functional equation

$$F(z, \varphi[f(z)]) = \varphi(z)$$

for complex z. However, he finds analytic solutions only.

§ 2. Every interval I such that $f(I) = I$ will be called a modulus-interval for the function $f(x)$. For each integer k we shall denote by $f^k(x)$ the k-th iteration of the function $f(x)$, i. e. we shall put

$$f^0(x) = x,$$

$$f^{k+1}(x) = f(f^k(x)), \quad f^{k-1}(x) = f^{-1}(f^k(x)), \quad k = 0, \pm 1, \pm 2, \ldots$$

The following lemmas may be proved:

LEMMA I. *Suppose that the function $f(x)$ is continuous and strictly increasing in an interval $\langle a, b\rangle$. In order that the interval $\langle a, b\rangle$ be a modulus-interval for the function $f(x)$, it is necessary and sufficient that a and b be roots of equation* (3).

LEMMA II. *Let $f(x)$ fulfil the hypotheses of lemma* I *and let $a < b$ be two consecutive roots of equation* (3). *Let us suppose further that $f(x) > x$ for all x in the interval (a, b). Then, for each $x \in (a, b)$, the sequences $\{f^n(x)\}$ and $\{f^{-n}(x)\}$ are monotone and*

$$\lim_{n\to\infty} f^n(x) = b, \quad \lim_{n\to\infty} f^{-n}(x) = a.$$

The proofs of these lemmas are to be found in [2].

In what follows we shall assume that:

(i) The function $f(x)$ is continuous and strictly increasing in an interval $\langle a, b\rangle$, where a and b are two consecutive roots of equation (3). The expression $f(x) - x$ is then of constant sign in (a, b). We can assume that $f(x) - x > 0$ in (a, b).

(ii) The function $F(x, y, z)$ is continuous in a convex region Ω, and possesses continuous derivatives F_y and F_z, neither of which vanishes in the region Ω.

(iii) The equation

$$F(x, y, z) = 0 \tag{4}$$

represents in Ω a simple connected piece of surface.

(iv) The interval (a, b) is contained in a set for each point of which there exist such y and z that the point (x, y, z) belongs to Ω and fulfils the equation $F(x, y, z) = 0$:

$$(a, b) \subset \mathop{E}_{x}\Big(\prod_{x} \sum_{y,z} (x, y, z) \in \Omega,\ F(x, y, z) = 0\Big).$$

On account of (ii) and (iii) surface (4) is single-foliaged in Ω with respect to the planes xy and xz (i. e. every straight line perpendicular to the plane xy or xz meets this surface in at most one point). Therefore on account of (iv) the intersection of surface (4) with the plane $x = x_0 \in (a, b)$ is a curve which is single-foliaged with respect to the axes y and z. Let

$$z = G(x_0, y), \quad y = H(x_0, z), \quad F(x_0, y, z) = 0, \quad (x_0, y, z) \in \Omega \tag{5}$$

be the equation of this curve for a fixed $x_0 \in (a, b)$ (the functions G and H are obtained by solving equation (4) with respect to z or y).

The function $G(x, y)$ is defined and continuous and has a continuous derivative $G_y \neq 0$ for

$$a < x < b, \quad \alpha(x) < y < \beta(x),$$

and the function $H(x, z)$ is defined and continuous and has a continuous derivative $H_z \neq 0$ for

$$a < x < b, \quad \gamma(x) < z < \delta(x),$$

where the intervals $(\alpha(x), \beta(x))$ and $(\gamma(x), \delta(x))$ are, x being kept fixed, the projections of curve (5) on the axes y and z respectively. It is obvious that the image of the interval (α, β) by the function G is the interval (γ, δ) and conversely, the image of the interval (γ, δ) by the function H is the interval (α, β).

Lastly let us suppose that

$$\gamma(x) \equiv \alpha[f(x)], \quad \delta(x) \equiv \beta[f(x)] \quad \text{for} \quad x \in (a, b). \tag{v}$$

Geometrically this supposition means that the projection on the z axis of the curve obtained by the intersection of surface (4) with the plane $x = x_0$ is identical with the projection on the y axis of the curve obtained by the intersection of surface (4) with the plane $x = f(x_0)$. This supposition guarantees the possibility of inserting the value z calculated from the equation

$$z = G(x_0, y_0)$$

in the function $G(f(x_0), z)$.

THEOREM I. *Under the hypotheses* (i)-(v) *equation* (1) *possesses an infinite number of solutions which are continuous in the open interval* (a, b).

Proof. Let us take an arbitrary $x_0 \in (a, b)$ and let us put $x_n = f^n(x_0)$, $n = 0, \pm 1, \pm 2, \ldots$ Let $\bar\varphi(x)$ be an arbitrary function, defined and continuous in the interval $\langle x_0, x_1)$, such that

$$\alpha(x) < \bar\varphi(x) < \beta(x) \quad \text{for} \quad x \in \langle x_0, x_1), \tag{6}$$

$$\lim_{x\to x_1-} \bar\varphi(x) = G(x_0, \bar\varphi(x_0)). \tag{7}$$

Conditions (6) and (7) are not contradictory, for, since

$$\alpha(x_0) < \bar{\varphi}(x_0) < \beta(x_0),$$

the function G is defined at the point $(x_0, \bar{\varphi}(x_0))$ and fulfils the inequality

$$\gamma(x_0) < G(x_0, \bar{\varphi}(x_0)) < \delta(x_0),$$

i. e. according to (v):

$$\alpha(x_1) < G(x_0, \bar{\varphi}(x_0)) < \beta(x_1).$$

Let

$$(8)\qquad \varphi(x) \stackrel{\mathrm{df}}{=} \begin{cases} \bar{\varphi}(x) & \text{for} \quad x \in \langle x_0, x_1), \\ G(f^{-1}(x), \varphi[f^{-1}(x)]) & \text{for} \quad x \in \langle x_n, x_{n+1}), \\ H(x, \varphi[f(x)]) & \text{for} \quad x \in \langle x_{-n}, x_{-n+1}), \qquad n > 0. \end{cases}$$

We shall show that:

1° Formulae (8) define the function $\varphi(x)$ in the whole interval (a, b).

2° The function $\varphi(x)$ is continuous in (a, b).

3° The function $\varphi(x)$ is a solution of equation (1).

Namely, we shall show that:

(*) Formulae (8) define a function which is continuous and fulfils the inequality

$$(9)\qquad \alpha(x) < \varphi(x) < \beta(x)$$

in every interval $\langle x_0, x_n)$, and $\langle x_{-n+1}, x_1)$, $n = 1, 2, \ldots$

Hence, on account of the equality

$$(a, b) = \bigcup_{-\infty}^{\infty} \langle x_n, x_{n+1}),$$

resulting from (i) and lemma II, 1° and 2° follow. That the function $\varphi(x)$ satisfies equation (1) follows immediately from formulae (8) and equivalence (5).

We shall prove assertion (*) for the intervals $\langle x_0, x_n)$, the proof for the intervals $\langle x_{-n+1}, x_1)$ being quite similar. The proof will be by induction.

I. For $n = 1$ the assertion is obvious.

II. $n = 2$. From (8) we have for $x \in \langle x_1, x_2)$

$$\varphi(x) = G(f^{-1}(x), \bar{\varphi}[f^{-1}(x)]).$$

For $x \in \langle x_1, x_2)$, $f^{-1}(x) \in \langle x_0, x_1)$. Consequently, according to (6), the function $G(f^{-1}(x), \bar{\varphi}[f^{-1}(x)])$ is defined and continuous (as the superposition of continuous functions) in $\langle x_1, x_2)$. Moreover, the inequalities

$$\gamma[f^{-1}(x)] < \varphi(x) < \delta[f^{-1}(x)],$$

i. e. according to (v)

$$\alpha(x) < \varphi(x) < \beta(x),$$

hold for $x \in \langle x_1, x_2)$.

Now we need to show only that $\varphi(x)$ is continuous at the point $x = x_1$. We have:

$$\varphi(x_1) = G(x_0, \bar{\varphi}(x_0));$$

$$\lim_{x\to x_1+} \varphi(x) = \varphi(x_1), \quad \text{since } \varphi(x) \text{ is continuous in } \langle x_1, x_2);$$

$$\lim_{x\to x_1-} \varphi(x) = G(x_0, \bar{\varphi}(x_0)) = \varphi(x_1) \quad \text{on account of relation (7).}$$

Hence follows the continuity of the function $\varphi(x)$ in the whole interval $\langle x_0, x_2)$.

III. Now let us suppose that the function $\varphi(x)$ is defined, continuous, and fulfils inequalities (9) in an interval $\langle x_0, x_p)$, $p \geqslant 2$.

For $x \in \langle x_p, x_{p+1})$, $f^{-1}(x) \in \langle x_{p-1}, x_p)$. Consequently, according to (9), $\varphi[f^{-1}(x)] \in (\alpha[f^{-1}(x)], \beta[f^{-1}(x)])$ for $x \in \langle x_p, x_{p+1})$, and the function

$$\varphi(x) = G(f^{-1}(x), \varphi[f^{-1}(x)])$$

is defined and continuous (as the superposition of continuous functions) in the interval $\langle x_p, x_{p+1})$. Moreover we have the inequalities

$$\gamma[f^{-1}(x)] < \varphi(x) < \delta[f^{-1}(x)], \qquad x \in \langle x_p, x_{p+1}),$$

i. e., according to (v),

$$\alpha(x) < \varphi(x) < \beta(x), \qquad x \in \langle x_p, x_{p+1}).$$

Now we need to show that $\varphi(x)$ is continuous at the point $x = x_p$. We have:

$$\varphi(x_p) = G(x_{p-1}, \varphi(x_{p-1}));$$

$$\lim_{x\to x_p+} \varphi(x) = \varphi(x_p), \quad \text{since} \quad \varphi(x) \text{ is continuous in } \langle x_p, x_{p+1});$$

$$\lim_{x\to x_p-} \varphi(x) = \lim_{x\to x_p-} G(f^{-1}(x), \varphi[f^{-1}(x)]) = G(f^{-1}(x_p), \varphi[f^{-1}(x_p)])$$

$$= G(x_{p-1}, \varphi(x_{p-1})) = \varphi(x_p)$$

on account of the continuity of the functions G and f and of the function φ for $x = x_{p-1}$. This completes the proof.

Taking as $\bar{\varphi}(x)$ all possible functions that are continuous in the interval $\langle x_0, x_1)$ and fulfil conditions (6) and (7), one can obtain all solutions of equation (1) which are continuous in (a, b) and pass through the region Ω (1).

Remark. If we take as $\bar{\varphi}(x)$ all functions defined in $\langle x_0, x_1)$ and fulfilling condition (6), then formulae (8) will define all solutions of equation (1) passing through the region Ω. From the proof of theorem I it is obvious that if we do not require the continuity of solutions, then the assumption of continuity of the function F with respect to x may be omitted.

§ 3. Theorem I, in view of the above remark, gives an extension of theorem 3 from Kitamura's paper [1] and makes it more precise. As the region Ω Kitamura admits the whole space. Kitamura omits hypotheses (iii), (iv) and (v). We shall show that these hypotheses are essential.

EXAMPLE I. Consider the equation

$$e^x + e^{\varphi(x)} + e^{\varphi[f(x)]} = 0. \tag{10}$$

In this case the function $F(x, y, z) = e^x + e^y + e^z$ is defined and continuous in the whole space. Also the derivatives $F_y = e^y$ and $F_z = e^z$ are continuous and do not vanish in the whole space. Meanwhile equation (10) evidently has no solution. Hypothesis (iii) is not fulfilled here, because the equation

$$e^x + e^y + e^z = 0$$

represents the empty set.

EXAMPLE II. Consider the equation

$$e^{\varphi(x+1)+\varphi(x)} - x = 0. \tag{11}$$

In this case the function $F(x, y, z) = e^{z+y} - x$ is defined and continuous in the whole space. Also the derivatives $F_y = F_z = e^{z+y}$ are continuous and do not vanish in the whole space. Hypothesis (iii) is fulfilled, but hypothesis (iv) is not, because the interval $(a, b) = (-\infty, \infty)$ is not contained in the set $\underset{x}{E}\big(\prod\limits_{x}\sum\limits_{y,z} e^{z+y} - x = 0\big) = (0, \infty)$. It is obvious that equation (11) has no solution in the whole interval $(a, b) = (-\infty, \infty)$, because it is not fulfilled for any $x \leqslant 0$.

EXAMPLE III. Consider the equation

$$\varphi(x) - e^{\varphi[f(x)]} = 0. \tag{12}$$

(1) I. e. such that $(x, \varphi(x), \varphi[f(x)]) \in \Omega$.

In this case the function $F(x, y, z) = y - e^z$ is defined and continuous in the whole space. Also the derivatives $F_y \equiv 1$ and $F_z = -e^z$ are continuous and do not vanish in the whole space.

Hypotheses (iii) and (iv) are evidently fulfilled. But hypothesis (v) is not fulfilled, because $\big(\alpha(x), \beta(x)\big) = (0, \infty)$ and $\big(\gamma(x), \delta(x)\big) = (-\infty, \infty)$, and consequently the equality $\gamma(x) \equiv \alpha[f(x)]$ does not hold.

We shall show that equation (12) has no solution defined in the whole interval (a, b), which is the modulus-interval for the function $f(x)$. The proof follows by *reductio ad absurdum*.

Let us suppose that the function $\varphi(x)$ satisfies equation (12) in (a, b). Let us take an arbitrary $x_0 \in (a, b)$. We shall define the sequence $\{\varepsilon_n\}$:

$$\varepsilon_0 = 0, \qquad \varepsilon_{n+1} = e^{\varepsilon_n}.$$

Since $\varepsilon_n \xrightarrow[n\to\infty]{} \infty$, there exists an index N such that $\varphi(x_0) < \varepsilon_N$. According to (12) there must be

$$\varphi[f(x_0)] = \ln\varphi(x_0) < \varepsilon_{N-1};$$

next

$$\varphi[f^2(x_0)] = \ln\varphi[f(x_0)] < \varepsilon_{N-2};$$

continuing this procedure, we shall get

$$\varphi[f^N(x_0)] = \ln\varphi[f^{N-1}(x_0)] < \varepsilon_0 = 0, \qquad \varphi[f^{N+1}(x_0)] = \ln\varphi[f^N(x_0)].$$

But $\varphi[f^N(x_0)]$ is negative, and the function $\varphi(x)$ is not defined at the point $x = f^{N+1}(x_0) \in (a, b)$.

Hypothesis (i) differs from the analogous hypotheses in Kitamura's paper in so far as Kitamura assumes the function $f(x)$ strictly monotone (not necessarily increasing). Kitamura commits here an error. One can discuss equation (1) with the function $f(x)$ decreasing, but it is quite a different problem, and the methods used by Kitamura and by us in the present paper in general are not applicable here. In Kitamura's paper theorem 2 is wrong even for $f(x) = 1/x$.

§ 4. We shall discuss now the continuity of solutions of equation (1) at the ends of the interval $\langle a, b\rangle$, where a and b are two consecutive roots of equation (3). In what follows we shall assume that hypotheses (i)-(v) are fulfilled, but in hypotheses (iv) and (v) the open interval (a, b) must be replaced by the closed interval $\langle a, b\rangle$.

The values $c = \varphi(a)$ and $d = \varphi(b)$, assumed by a solution of equation (1) at the ends of the interval $\langle a, b\rangle$, must fulfil the equations

$$F(a, c, c) = 0, \tag{13}$$

$$F(b, d, d) = 0. \tag{14}$$

A priori equations (13) and (14) may have no solutions, a finite number of solutions, or infinitely many solutions. We shall prove, however, the following two theorems:

THEOREM II. *If $F_y F_z > 0$ in the region Ω, then there exist exactly one value c and exactly one value d such that the points (a, c, c) and (b, d, d) belong to the closure $\bar{\Omega}$ of the region Ω and equations* (13) *and* (14) *are fulfilled.*

Proof. Relation (13) is equivalent to the relation

$$c = G(c, c). \tag{15}$$

The function $z = G(a, y)$, $y \in \langle \alpha(a), \beta(a) \rangle$, is decreasing, because $z' = -F_y/F_z < 0$ in Ω. Consequently there exists at most one value c such that $c = G(a, c)$.

On the other hand, let us suppose that (15) does not hold for any c from the interval $\langle \alpha(a), \beta(a) \rangle$, i. e. that $G(a, y) \neq y$ for $y \in \langle \alpha(a), \beta(a) \rangle$. For example let $G(a, y) > y$ for $y \in \langle \alpha(a), \beta(a) \rangle$.

The values which the function $G(a, y)$ assumes in the interval $\langle \alpha(a), \beta(a) \rangle$ fill the interval $\langle \gamma(a), \delta(a) \rangle$. Since $G(a, y)$ is decreasing, we have

$$\gamma(a) = G[a, \beta(a)] > \beta(a) > \alpha(a),$$

which contradicts hypothesis (v).

The proof for the point (b, d, d) is quite analogous.

THEOREM III. *If $F_y F_z < 0$ in the region Ω, then there exist at least two values c and two values d such that the points (a, c, c) and (b, d, d) belong to the closure $\bar{\Omega}$ of the region Ω, and equations* (13) *and* (14) *are fulfilled.*

Proof. The function $z = G(a, y)$ is increasing in the interval $\langle \alpha(a), \beta(a) \rangle$. Then, according to (v),

$$G[a, \alpha(a)] = \gamma(a) = \alpha(a), \qquad G[a, \beta(a)] = \delta(a) = \beta(a),$$

which proves that $c_1 = \alpha(a)$ and $c_2 = \beta(a)$ both fulfil (15), whence also (13) (these values may be infinite [2]).

The proof for the point (b, d, d) is quite analogous.

COROLLARY. *Equations* (13) *and* (14) *always possess at least one solution (finite or not) in $\bar{\Omega}$.*

Now we shall prove the following theorem:

[2] We shall call $c = \infty$ the root of equation (13) if

$$\lim_{y \to \infty} G(a, y) = \lim_{z \to \infty} H(a, z) = \infty.$$

THEOREM IV. *Let c and d be fixed finite roots of equations* (13) *and* (14). *If $|F_y(x, y, z)/F_z(x, y, z)| \geqslant 1$ ($|F_z(x, y, z)/F_y(x, y, z)| \geqslant 1$) in a neighbourhood of the point (b, d, d) ((a, c, c)), then there exists at most one function $\varphi(x)$, continuous in the interval $(a, b\rangle$ ($\langle a, b)$), which satisfies equation* (1) *and the condition $\varphi(b) = d$ ($\varphi(a) = c$).*

Proof. Let us assume that

$$\left| \frac{F_y(x, y, z)}{F_z(x, y, z)} \right| \geqslant 1 \tag{16}$$

in a neighbourhood of the point (b, d, d). Further, let us suppose that there exist functions $\varphi(x)$ and $\psi(x) \not\equiv \varphi(x)$, which are continuous in the interval $(a, b\rangle$ and satisfy equation (1) and the condition $\varphi(b) = \psi(b) = d$. Let us put

$$\varrho(x) \overset{\text{df}}{=} \psi(x) - \varphi(x).$$

The function $\varrho(x)$ is evidently continuous in $(a, b\rangle$, $\varrho(b) = 0$. We have

$$\begin{aligned} 0 &= F\big(x, \psi(x), \psi[f(x)]\big) - F\big(x, \varphi(x), \varphi[f(x)]\big) \\ &= F\big(x, \varphi(x) + \varrho(x), \varphi[f(x)] + \varrho[f(x)]\big) - F\big(x, \varphi(x), \varphi[f(x)]\big) \\ &= F_y\big(x, \varphi(x) + \vartheta\varrho(x), \varphi[f(x)] + \vartheta\varrho[f(x)]\big)\varrho(x) + \\ &\quad + F_z\big(x, \varphi(x) + \vartheta\varrho(x), \varphi[f(x)] + \vartheta\varrho[f(x)]\big)\varrho[f(x)]. \end{aligned}$$

Hence

$$\varrho[f(x)] = -\frac{F_y\big(x, \varphi(x) + \vartheta\varrho(x), \varphi[f(x)] + \vartheta\varrho[f(x)]\big)}{F_z\big(x, \varphi(x) + \vartheta\varrho(x), \varphi[f(x)] + \vartheta\varrho[f(x)]\big)}\varrho(x). \tag{17}$$

Since $\varrho(x) \not\equiv 0$, there exists a point x_0 such that $\varrho(x_0) \neq 0$. Let us put $x_n = f^n(x_0)$, $n = 0, 1, 2, \ldots$ By (17) $\varrho(x_n) \neq 0$ for $n = 0, 1, 2, \ldots$ and

$$\varrho(x_{n+1}) = -\frac{F_y\big(x_n, \varphi(x_n) + \vartheta_n\varrho(x_n), \varphi(x_{n+1}) + \vartheta_n\varrho(x_{n+1})\big)}{F_z\big(x_n, \varphi(x_n) + \vartheta_n\varrho(x_n), \varphi(x_{n+1}) + \vartheta_n\varrho(x_{n+1})\big)}\varrho(x_n). \tag{18}$$

$x_n \xrightarrow[n \to \infty]{} b$, and consequently, since $\varrho(b) = 0$ and the functions $\varphi(x)$ and $\varrho(x)$ are continuous at the point $x = b$, we have

$$\varrho(x_n) \xrightarrow[n \to \infty]{} 0, \quad \varphi(x_n) + \vartheta_n\varrho(x_n) \xrightarrow[n \to \infty]{} d, \quad \varphi(x_{n+1}) + \vartheta_n\varrho(x_{n+1}) \xrightarrow[n \to \infty]{} d. \tag{19}$$

Hence, according to (16), there exists an index N such that for $n \geqslant N$

$$\left| \frac{F_y\big(x_n, \varphi(x_n) + \vartheta_n\varrho(x_n), \varphi(x_{n+1}) + \vartheta_n\varrho(x_{n+1})\big)}{F_z\big(x_n, \varphi(x_n) + \vartheta_n\varrho(x_n), \varphi(x_{n+1}) + \vartheta_n\varrho(x_{n+1})\big)} \right| \geqslant 1$$

and by (18) for $n \geqslant N$

$$|\varrho(x_{n+1})| \geqslant |\varrho(x_n)| \geqslant \ldots \geqslant |\varrho(x_N)| > 0,$$

which contradicts relation (19).

The proof for the interval $\langle a, b)$ is quite analogous.

Remark. If $|F_y/F_z| < 1$ in a neighbourhood of the point (b, d, d), then equation (1) may possess infinitely many solutions which are continuous in $(a, b\rangle$ and admit the common value at the point $x = b$. For example, every solution of the equation

$$\varphi(x)+2\varphi(\sqrt{x})-x+1 = 0$$

which is continuous in the interval $(0, 1)$ fulfils the condition

$$\lim_{x\to 1-}\varphi(x) = 0.$$

§ 5. The question now arises how to find that unique solution of equation (1) which is continuous in the interval $(a, b\rangle$ or $\langle a, b)$. In this section we shall try to give a partial answer to this question.

Let c and d be fixed finite roots of equations (13) and (14). We shall define two functional sequences $\{h_n(x)\}$ and $\{g_n(x)\}$:

$$(20)\qquad h_0(x)\equiv d,\quad h_{n+1}(x) = H\big(x, h_n[f(x)]\big),\quad x\in(a, b\rangle,$$

$$(21)\qquad g_0(x)\equiv c,\quad g_{n+1}(x) = G\big(f^{-1}(x), g_n[f^{-1}(x)]\big),\quad x\in\langle a, b).$$

THEOREM V. *If the functions F, G, H are continuous in the closure $\bar{\Omega}$ of the region Ω and if the sequence $h_n(x)$ $(g_n(x))$ converges for $x = x_0$, then it converges also for $x = f(x_0)$ and $x = f^{-1}(x_0)$, and moreover the function $h(x) = \lim_{n\to\infty} h_n(x)$ $(g(x) = \lim_{n\to\infty} g_n(x))$ satisfies equation* (1) *for $x = x_0$.*

Proof. Let us suppose that $h_n(x_0)\to h(x_0)$. We have by (20)

$$h_n[f^{-1}(x_0)] = H\big(f^{-1}(x_0), h_{n-1}(x_0)\big).$$

Passing to the limit, we obtain on account of the continuity of the function H

$$h[f^{-1}(x_0)] = \lim_{n\to\infty} h_n[f^{-1}(x_0)] = H\big(f^{-1}(x_0), h(x_0)\big).$$

The second relation of (20) is equivalent to the relation

$$h_n[f(x)] = G\big(x, h_{n+1}(x)\big).$$

Hence, by the continuity of the function G,

$$h[f(x_0)] = \lim_{n\to\infty} h_n[f(x_0)] = G\big(x_0, h(x_0)\big).$$

The second relation of (20) is also equivalent to the relation

$$F\big(x, h_n(x), h_{n-1}[f(x)]\big) = 0.$$

Passing to the limit, we obtain on account of the continuity of the function F

$$F\big(x_0, h(x_0), h[f(x_0)]\big) = 0,$$

which proves that the function $h(x)$ satisfies equation (1) for $x = x_0$.

The proof for $g_n(x)$ is quite analogous.

We shall prove the following theorem, which partially answers the question stated at the beginning of this section:

THEOREM VI. *If $|F_y(x, y, z)/F_z(x, y, z)|\geqslant 1$ in the whole region Ω and if $\varphi(x)$ is the solution of equation* (1) *which is continuous in the interval $(a, b\rangle$ and fulfils the condition $\varphi(b) = d$, then*

$$h_n(x) \underset{\langle a+\eta, b\rangle}{\rightrightarrows} \varphi(x)$$

for every $\eta > 0$.

Similarly, if $|F_z(x, y, z)/F_y(x, y, z)|\geqslant 1$ in the whole region Ω and if $\psi(x)$ is the solution of equation (1) *which is continuous in the interval $\langle a, b)$ and fulfils the condition $\psi(a) = c$, then*

$$g_n(x) \underset{\langle a, b-\eta\rangle}{\rightrightarrows} \psi(x)$$

for every $\eta > 0$.

Proof. Let us assume that $|F_y(x, y, z)/F_z(x, y, z)|\geqslant 1$ in Ω. We shall define the functional sequence $\{\bar{h}_n(x)\}$:

$$\bar{h}_0(x) = \varphi(x),\qquad \bar{h}_{n+1}(x) = H\big(x, \bar{h}_n[f(x)]\big).$$

It is easy to show that $\bar{h}_n(x) = \varphi(x)$ for $n = 0, 1, 2, \ldots$ This follows immediately from the relation

$$\varphi(x) = H\big(x, \varphi[f(x)]\big),$$

which is equivalent to the relation

$$F\big(x, \varphi(x), \varphi[f(x)]\big) = 0.$$

We have

$$\begin{aligned}|\bar{h}_n(x)-h_n(x)| &= \big|H\big(x, \bar{h}_{n-1}[f(x)]\big)-H\big(x, h_{n-1}[f(x)]\big)\big|\\ &= \Big|H_z\Big(x, h_{n-1}[f(x)]+\vartheta_1\big(\bar{h}_{n-1}[f(x)]-h_{n-1}[f(x)]\big)\Big)\Big|\times\\ &\quad\times\big|\bar{h}_{n-1}[f(x)]-h_{n-1}[f(x)]\big|\qquad (0<\vartheta_1<1).\end{aligned}$$

Proceeding thus, we shall obtain after n steps

$$|\bar{h}_n(x)-h_n(x)| = \prod_{\nu=1}^{n}\left|H_z\Big(f^{\nu-1}(x),\, h_{n-\nu}[f^{\nu}(x)]+\vartheta_\nu\big(\bar{h}_{n-\nu}[f^{\nu}(x)]-h_{n-\nu}[f^{\nu}(x)]\big)\Big)\right|\times$$
$$\times\, |\bar{h}_0[f^n(x)]-h_0[f^n(x)]|$$
$$=\prod_{\nu=1}^{n}\left|H_z\Big(f^{\nu-1}(x),\, h_{n-\nu}[f^{\nu}(x)]+\vartheta_\nu\big(\bar{h}_{n-\nu}[f^{\nu}(x)]-h_{n-\nu}[f^{\nu}(x)]\big)\Big)\right|\times$$
$$\times\, |\varphi[f^n(x)]-d|\,.$$

By the assumption $|H_z| = |F_z/F_y| \leqslant 1$ we have $\prod |H_z| \leqslant 1$ and hence

$$|\bar{h}_n(x)-h_n(x)| < |\varphi[f^n(x)]-d|\,.$$

Let us take an arbitrary $\varepsilon>0$. Since $\varphi(x)\xrightarrow[x\to b]{} d$, there exists $\delta>0$ such that

$$|\varphi(x)-d|<\varepsilon \quad \text{ for } \quad x\,\epsilon\,(b-\delta,\, b\rangle.$$

$f^n(a+\eta)\xrightarrow[n\to\infty]{} b$, and therefore there exists an index N such that for $n>N$

$$f^n(a+\eta)\,\epsilon\,(b-\delta,\, b\rangle.$$

Now let us take an arbitrary $x\,\epsilon\,\langle a+\eta,\, b\rangle$. $f^n(x)\geqslant f^n(a+\eta)$, for $f^n(x)$ is increasing with $f(x)$. Consequently, for $n>N$, $f^n(x)\,\epsilon\,(b-\delta,\, b\rangle$ and $|\varphi[f^n(x)]-d|<\varepsilon$, whence, for $n>N$ and $x\,\epsilon\,\langle a+\eta,\, b\rangle$

$$|\bar{h}_n(x)-h_n(x)|<\varepsilon,$$

i. e.

$$|\varphi(x)-h_n(x)|<\varepsilon,$$

which proves that $h_n(x) \underset{\langle a+\eta,\, b\rangle}{\Rightarrow} \varphi(x)$.

The second part of this theorem may be proved in a quite similar manner.

References

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ANNALES
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On a problem of S. L. Cheng concerning sequences of functions with convergent k-th differences

by K. Urbanik (Wrocław)

In the present note we use the notation

$$\Delta_h^{(k)} f(x) = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} f(x+jh) \qquad (k = 1, 2, \ldots).$$

The aim of this note is to prove the following theorem, which is the solution of a problem raised by S. L. Cheng.

THEOREM. *Let $f_n(x)$ $(n = 1, 2, \ldots)$ be a sequence of Lebesgue measurable real-valued functions on the line. The convergence*

$$(*) \qquad \lim_{n\to\infty} \Delta_h^{(k)} f_n(x) = 0$$

for each h uniform with respect to x in every finite interval is equivalent to the equalities

$$(**) \qquad f_n(x) = \sum_{j=0}^{k-1} a_{jn}x^j + g_n(x) \qquad (n = 1, 2, \ldots),$$

where a_{jn} $(j = 0, 1, \ldots, k-1;\ n = 1, 2, \ldots)$ are constants and the sequence $g_n(x)$ $(n = 1, 2, \ldots)$ converges to 0 uniformly in every finite interval.

Remarks. (a) H. Whitney ([2], p. 67-68) has proved the following fundamental theorem:

For each integer $k\geqslant 1$ there is a number C_k with the following property. Let I be any closed finite interval. Then for any continuous function $f(x)$ in I there is a polynomial $P(x)$ of degree at most $k-1$ such that

$$\max_{x\,\epsilon\, I}|f(x)-P(x)| \leqslant C_k \max_{x+jh\,\epsilon\, I;\ j=0,1,\ldots,k} |\Delta_h^{(k)} f(x)|\,.$$

If $f_n(x)$ $(n = 1, 2, \ldots)$ are continuous functions and if the convergence $(*)$ is uniform with respect to h and x in every finite square, then $(**)$ is a direct consequence of the theorem of Whitney.