by x^* the coset in $\mathcal{H}/\mathcal{T}$ which contains x. For a set $\mathcal{S} \subset \mathcal{H}$ we define $\mathcal{S}^* = \{x^*: x \in \mathcal{S}\}$. In particular, we have $\mathcal{H}^* = \mathcal{H}/\mathcal{T}$. It is evident that x and x^* have parallel rotation axes (that of x^* passing through O) and equal rotation angles. Thus Φ^* and Ψ^* have the rotation angle φ and rotation axes perpendicular to each other. Such rotations are independent (cf. [2]). This means that if $\Gamma_1, \ldots, \Gamma_n \in \mathcal{H}^*$ satisfy $\Gamma_i^{\varepsilon_i} = \Phi^*$ or Ψ^*, where $\varepsilon_i = 1$ or -1 and $\Gamma_i \Gamma_{i+1} \neq e$ (e = the unity of $\mathcal{H}^*$), then $\Gamma_1 \Gamma_2 \ldots \Gamma_n \neq e$.

From assumptions 1° and 2° it follows that the face common to T_{i+1} and T_i is not parallel to any face of T_{i-1}. Hence T_{i+1} cannot be obtained from T_{i-1} by a translation. Since $T_{i+1} = \Theta_1 \ldots \Theta_{i-1} \Theta_i \Theta_{i+1} \Theta_{i-1}^{-1} \ldots \Theta_1^{-1}(T_{i-1})$, it follows that $\Theta_i \Theta_{i+1} \notin \mathcal{T}$. Thus $\Theta_i^* \Theta_{i+1}^* \neq e$ and we infer by $(\Theta_i^*)^{\varepsilon_i} = \Phi^*$ or Ψ^* and by the independence of Φ^* and Ψ^* that $\Theta_1^* \Theta_2^* \ldots \Theta_n^* \neq e$.

Let us denote by $\mathcal{S}$ the group of rotations which transform T_0 into itself. Since $\mathcal{S}^*$ is finite, we have, by the independence of Φ^* and Ψ^*, $\Theta_1^* \Theta_2^* \ldots \Theta_n^* \notin \mathcal{S}^*$. Consequently $\Theta_1 \Theta_2 \ldots \Theta_n$ is not a combination of a translation and a rotation belonging to $\mathcal{S}$. Thus $T_n = \Theta_1 \Theta_2 \ldots \Theta_n(T_0)$ and T_0 are not congruent by translation.

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ON MEASURES IN FIBRE BUNDLES

BY

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In the present paper I introduce the notion of a product measure in a fibre bundle. For the theory of fibre bundles I use the terminology and notation of Steenrod [2], and for the measure theory the terminology and notation of Halmos [1].

1. Let $\mathcal{B}(B, X, Y, G)$ be a fibre bundle with a locally compact base space X, a locally compact fibre Y, and thus a locally compact bundle space B. Consider Baire measures μ and ν given respectively in X and Y, and denote by $\mu \times \nu$ the product of those measures in the Cartesian product $X \times Y$.

A Baire measure λ in B is called the *product measure of μ and ν in the fibre bundle $\mathcal{B}$* if for every representation of $\mathcal{B}$ as a coordinate bundle $\mathcal{B}(B, X, Y, G, V_j, \varphi_j)$ and for each Baire set $Z \subset V_j \times Y$ the equality

$$\lambda(\varphi_j(Z)) = (\mu \times \nu)(Z) \tag{1}$$

holds.

2. THEOREM 1. *A product measure λ of μ and ν in a fibre bundle $\mathcal{B}(B, X, Y, G)$ exists if and only if the measure ν in Y is invariant under transformations of the group G* (1).

Proof. a) Let us suppose that there exists in $\mathcal{B}$ a product measure λ. Consider any representation of the fibre bundle as a coordinate bundle $\mathcal{B}(B, X, Y, G, V_j, \varphi_j)$ and any fixed element g of G. The coordinate bundle $\mathcal{B}'(B, X, Y, G, V_j', \varphi_j')$ with $V_j' = V_j$, $\varphi_j'(x, y) = \varphi_j(x, g^{-1}y)$ is stricly equivalent to $\mathcal{B}$. In fact, the functions $\bar{g}_{jj}(x) = \varphi_{j,x}'^{-1}\varphi_{j,x} = g$, $\bar{g}_{ji}(x) = g g_{ji}(x)$ are continuous.

Let A be a Baire subset of V_j of positive finite measure μ and E a measurable set in Y. We then have

$$\lambda(\varphi_j(A \times E)) = \mu(A)\nu(E). \tag{2}$$

(1) Evidently, the measure λ is completely determined by μ and ν.

On the other hand, $\varphi_j(A\times E)=\varphi'_j(A\times gE)$, and therefore

$$\lambda(\varphi_j(A\times E))=\lambda(\varphi'_j(A\times gE))=\mu(A)\nu(gE). \tag{3}$$

From (2) and (3) follows $\nu(gE)=\nu(E)$, i. e. the measure ν is invariant.

b) Now let ν be G-invariant. We define a Baire measure ν_x in every fibre $Y_x=p^{-1}(x)$ setting for $E_x\subset Y_x$ and for any fixed representation of $\mathfrak{B}$ as a coordinate bundle

$$\nu_x(E_x)=\nu(\varphi_{j,x}^{-1}(E_x)), \tag{4}$$

where $x\in V_j$.

It follows from the G-invariance of ν that this definition is independent of the choice of V_j containing x and of the choice of the representation of $\mathfrak{B}$ as a coordinate bundle. In fact, for any two equivalent coordinate bundles $\mathfrak{B}$ and $\mathfrak{B}'$ and for $x\in V_j\cap V'_i$ we have $\varphi'^{-1}_{i,x}(E_x)=\bar{g}_{ij}(x)\varphi_{j,x}^{-1}(E_x)$, hence $\nu(\varphi'^{-1}_{i,x}(E_x))=\nu(\varphi_{j,x}^{-1}(E_x))$.

c) Let Z be any Baire set in B, and let Z_x denote the common part of Z and Y_x, i. e. $Z_x=Z\cap p^{-1}(x)$. Z_x is a Baire set in the fibre Y_x. It is easy to prove that the function $\nu_x(Z_x)$ is μ-measurable in X. In fact, the Baire set Z being σ-finite, it is contained in a denumerable family of sets $p^{-1}(V_j)$ for a fixed representation of $\mathfrak{B}$ as a coordinate bundle. Therefore it suffices to consider only sets Z which are contained in a single $p^{-1}(V_j)$, and in this case measurability follows from Fubini's theorem applied to $\varphi_j^{-1}(Z)$ in $V_j\times Y$.

The required product measure λ is now given by the formula

$$\lambda(Z)=\int_X \nu_x(Z_x)\,d\mu(x). \tag{5}$$

From (4) and from Fubini's theorem it follows that this measure is in fact a product measure of μ and ν in the fibre bundle.

COROLLARY. *For the product measure λ in a fibre bundle identity* (5) *holds.*

Generally, the following "Fubini theorem" holds: *If f is a real-valued λ-integrable function in $Z\subset B$, then f is ν_x-integrable on almost all Z_x, the function $g(x)=\int_{Z_x} f\,d\nu_x$ is μ-integrable on $p(Z)$ and*

$$\int_Z f\,d\lambda=\int_{p(Z)} g\,d\mu=\int_{p(Z)}\Big[\int_{Z_x} f\,d\nu_x\Big]d\mu. \tag{6}$$

3. Consider two fibre bundles $\mathfrak{B}$ and $\mathfrak{B}'$ with the same locally compact fibre Y and group G and with locally compact base spaces X and X' respectively. In X and X' are given respectively measures μ and μ', in Y a G-invariant measure ν. The product measures in $\mathfrak{B}$ and $\mathfrak{B}'$ are denoted respectively by λ and λ'. Let h be a bundle map $\mathfrak{B}\to\mathfrak{B}'$, and $\bar{h}$ the generated mapping $X\to X'$. From the definition of a bundle map immediately follows

$$\nu_x(Z_x)=\nu_{\bar{h}(x)}([h(Z)]_{\bar{h}(x)}). \tag{7}$$

Therefore we have for any Baire set $Z'\subset B'$

$$\lambda(h^{-1}(Z'))=\int_X \nu_x([h^{-1}(Z')]_x\,d\mu(x)=\int_{X'}\nu_{x'}(Z'_{x'})\,d\mu''(x'), \tag{8}$$

where μ'' is a Baire measure in X' generated by $\bar{h}$ from μ, i. e. $\mu''(A')=\mu(\bar{h}^{-1}(A'))$.

From (8) it follows that if $\bar{h}$ transforms the measure μ into μ', i. e. if $\mu''=\mu'$, then $\lambda(h^{-1}(Z'))=\lambda'(Z')$, i. e. h transforms the product measure λ into λ'. In particular the following lemma holds:

LEMMA. *If h is a bundle map of a fibre bundle $\mathfrak{B}$ into itself and $\bar{h}$ preserves the measure μ, then h preserves the measure λ.*

4. For a given Baire measure λ in the bundle space B and a given G-invariant measure ν in Y we consider the following condition:

(C) *For any two Baire sets Z and Z' contained in B if $\nu_x(Z_x)=k\nu_x(Z'_x)$ (k is a real positive constant) for every $x\in X$, then $\lambda(Z)=k\lambda(Z')$.*

THEOREM 2. *If the measures λ and ν satisfy the condition* (C), *then there exists in X a Baire measure μ such that λ is the product measure of μ and ν* ([2]).

Proof. It is sufficient to define the mesaure μ for sets contained in single V_j-s of an arbitrary representation of the fibre bundle $\mathfrak{B}$ as a coordinate bundle.

We fix arbitrarily a Baire set E in Y of finite positive measure $\nu(E)$, and set for $A\subset V_j$

$$\mu(A)=\frac{\lambda(\varphi_j(A\times E))}{\nu(E)}. \tag{9}$$

From condition (C) it follows that the above definition does not depend on the choice of the representation of $\mathfrak{B}$ as coordinate bundle and on the choice of the set E.

Indeed, if $A\subset V_j\cap V'_i$ for two representations, then $\nu_x(\varphi_j(A\times E))=\nu_x(\varphi'_j(A\times E))$ for every x, and consequently $\lambda(\varphi_j(A\times E))=\lambda(\varphi'_i(A\times E))$.

([2]) Evidently, condition (C) is also a necessary one.

If we take another set E' and write $k = \nu(E')/\nu(E)$, then we have

$$\nu_x([\varphi_j(A\times E')]_x) = \nu(E') = k\nu(E) = k\nu_x([\varphi_j(A\times E)]_x)$$

and hence

$$\lambda(\varphi_j(A\times E')) = k\lambda(\varphi_j(A\times E)).$$

Consequently

$$\frac{\lambda(\varphi_j(A\times E'))}{\nu(E')} = \frac{k\lambda(\varphi_j(A\times E))}{k\nu(E)} = \frac{\lambda(\varphi_j(A\times E))}{\nu(E)}.$$

It is clear that the product measure λ^* of μ and ν coincides with λ for sets of the form $\varphi_j(A\times E)$, where the A-s are Baire subsets of the V_j-s and the E-s are Baire sets in Y. Consequently λ^* coincides with λ for all Baire sets in B.

5. If the G-invariant measure ν in Y is unique up to a constant factor, condition (C) in theorem 2 may be replaced by a weaker one (the principle of Cavalieri):

(C′) *If* $\nu_x(Z_x) = \nu_x(Z'_x)$ *for every* x, *then* $\lambda(Z) = \lambda(Z')$.

(This is the case for instance if Y is a factor space G/H of the group G and its closed subgroup H).

From (C′) follows the independence of the above definition of μ of the choice of the representation, but not of the choice of the set E.

Let μ_E be the measure in X defined for a fixed $E\subset Y$. Denote by λ^* the product measure of μ_E and ν. The measure λ^* then coincides with λ for sets of the form $\varphi_j(A\times E)$, $A\subset V_j$. Fixing the set $A\subset V_j\subset X$, we take under consideration two measures in Y: $\nu_1(E') = \lambda^*(\varphi_j(A\times E')) = \mu_E(A)\nu(E')$ and $\nu_2(E') = \lambda(\varphi_j(A\times E'))$.

The measure ν_1 is G-invariant, for it is proportional to ν. The G-invariance of ν_2 follows from the invariance of ν and from condition (C′). The measure ν_1 coincides with ν_2 for the fixed set E; consequently, it follows from the uniqueness of the G-invariant measure in Y that $\nu_1 = \nu_2$, and thus $\lambda^* = \lambda$, and the measure μ_E is independent of the choice of E.

However, the uniqueness of the G-invariant measure in Y is not necessary for the possibility of replacing condition (C) by (C′) in the above theorem, as we see in the following example:

B is a Cartesian plane (with x-axis X and y-axis Y) regarded as a fibre bundle with trivial group G, consisting of the identical transformation of Y only. Every measure in Y is of course G-invariant. However, for arbitrary Baire measure λ in B and ν in Y condition (C′) is a sufficient one for the existence of a measure μ in X such that $\lambda = \mu\times\nu$.

In the case of an atomic measure ν with unequal measures on the atoms condition (C) cannot be replaced by (C′). For example, let $Y = \{a, b\}$ be a two-point set and $X = \langle 0, 1\rangle$ a unit interval, $\nu(\{a\}) = 1/3$, $\nu(\{b\}) = 2/3$. Condition (C′) is satisfied for every measure λ in $X\times Y$, because there exists no pair of different sets Z and Z' in $X\times Y$ with $\nu(Z_x) = \nu(Z'_x)$ for every x. The measure λ may be taken in such a way that it would not be a product measure in $X\times Y$, e. g. $\lambda(Z) = \frac{1}{2}(|\{x\colon (x, a)\in Z\}| + |\{x\colon (x, b)\in Z\}|)$, where $|\ |$ denotes the Lebesgue measure in X.

6. Now let B be a locally compact topological group and G its closed subgroup admitting a local cross section. Then B may be regarded as a fibre bundle with the factor space $X = B/G$ as a base space, with the fibre $Y = G$, and with group G acting on itself by left translations (cf. [2], § 7).

THEOREM 3. *If there exists in* $X = B/G$ *a* B*-invariant measure* μ, *then the product measure* λ *of* μ *and the left invariant Haar measure* ν *in* $Y = G$ *is a left invariant Haar measure in* B.

Conversely, if the Haar measure λ *in* B *is the product measure of a certain Baire measure* μ *in* $X = B/G$ *and the Haar measure* ν *in* G, *then* μ *is* B*-invariant.*

Proof. The left translation of B by an element b being a bundle map h of B onto itself generating the b-translation $\bar{h}$ in X, the first part of the theorem follows from lemma of section 3. The second part is a simple consequence of the fact that for $Z = \varphi_j(A\times E)$, $A\subset V_j$, we have $\lambda(Z) = \mu(A)\nu(E)$ and $\lambda(gZ) = \int_X \nu_x([gZ]_x) = \nu(E)\mu(gA)$, for $p(gZ) = gA$.

7. The well-known necessary and sufficient condition of the existence of an invariant measure in $X = B/G$ (cf. [3], § 9)

$$\Delta(g) = \delta(g) \quad (g\in G), \tag{10}$$

where $\Delta(g) = \lambda(Zg)/\lambda(Z)$, $\delta(g) = \nu(Eg)/\nu(E)$ (λ denotes the left invariant Haar measure in B, ν the Haar measure in G, $Z\subset B$, $E\subset G$) has a very simple intuitive interpretation when the group is regarded as a fibre bundle.

a) A right translation of B by an element $g\in G$ is a homeomorphism $h_g\colon B\to B$ preserving fibres (i. e. $h_g(Y_x) = Y_x$), but in general not a bundle map. For any Baire subset Z_x of Y_x we have $\nu_x(Z_x g) = \delta(g)\nu_x(Z_x)$.

If there exists an invariant measure μ in $X = B/G$, then λ is the product measure of μ and ν and consequently

$$\lambda(Z) = \int_X \nu_x(Z_x)\,d\mu(x).$$

Hence

$$\lambda(Zg) = \int_X \nu_x([Zg]_x)\,d\mu(x) = \int_X \nu_x(Z_x g)\,d\mu(x)$$
$$= \delta(g)\int_X \nu_x(Z_x)\,d\mu(x) = \delta(g)\lambda(Z).$$

On the other hand, $\lambda(Zg) = \Delta(g)\lambda(Z)$, and thus $\Delta(g) = \delta(g)$.

b) Now suppose that (10) holds. In order to prove the existence of the invariant measure μ in B/G it suffices to show that λ and ν satisfy condition (C) (or (C′) on account of the uniqueness of the Haar measure). We shall use an argument closely related to Weil's proof.

Let Z and Z' be two sets in B such that

(11) $$\nu_x(Z'_x) = k\nu_x(Z_x) \qquad (k = \text{const})$$

for every x. Let $\chi(b)$ and $\chi'(b)$ denote the characteristic functions of Z and Z' respectively, i. e.

$$\chi(b) = \begin{cases} 0 \text{ if } b\notin Z, \\ 1 \text{ if } b\in Z, \end{cases} \qquad \chi'(b) = \begin{cases} 0 \text{ if } b\notin Z', \\ 1 \text{ if } b\in Z'. \end{cases}$$

Then

(12) $$\lambda(Z) = \int_B \chi(b)\,d\lambda(b), \qquad \lambda(Z') = \int_B \chi'(b)\,d\lambda(b).$$

It is easy to show that

(13) $$\int_G \chi(bg)\,d\nu(g) = \nu_{p(b)}(Z_{p(b)}),$$

and similarly

(14) $$\int_G \chi'(bg)\,d\nu(g) = \nu_{p(b)}(Z'_{p(b)}).$$

In fact, let $\chi_{b^{-1}}(c)$ denote the characteristic function of the set $b^{-1}Z$. Then $\chi(bg) = \chi_{b^{-1}}(g)$ and $Z_{p(b)} = b^{-1}Z_{p(e)}$ (e = the identity element of the set B). Therefore

$$\int_G \chi(bg)\,d\nu(g) = \int_G \chi_{b^{-1}}(g)\,d\nu(g) = \nu_{p(e)}([b^{-1}Z]_{p(e)}),$$

as the fibre containing e is the subgroup G, and $\nu_{p(e)}$ is identical with ν. On the other hand the left translation by b is a bundle map and $\nu_{p(e)}(b^{-1}Z)_{p(e)} = \nu_{p(b)}(Z)$ which proves (13).

From (13) follows equality

(15) $$\int_G \frac{\chi(bg)\,d\nu(g)}{\nu_{p(b)}(Z_{p(b)})} = 1.$$

Multiplying the integrand in the second equality of (12) by the left term of (15), we obtain

$$\lambda(Z') = \int_B \left[\int_G \frac{\chi(bg)\,d\nu(g)}{\nu_{p(b)}(Z_{p(b)})}\,\chi'(b)\right] d\lambda(b).$$

We change the order of integration:

$$\lambda(Z') = \int_G \left[\int_B \frac{\chi(bg)\chi'(b)}{\nu_{p(b)}(Z_{p(b)})}\,d\lambda(b)\right] d\nu(g)$$

and introduce a new variable $c = bg$, whence $p(c) = p(b)$. We now have (cf. [3], § 8)

$$\lambda(Z') = \int_G \left[\int_B \frac{\chi(c)\chi'(cg^{-1})}{\nu_{p(c)}(Z_{p(c)})}\,\Delta(g^{-1})\,d\lambda(c)\right] d\nu(g).$$

Changing the order of integration again, we obtain

$$\lambda(Z') = \int_B \frac{\chi(c)}{\nu_{p(c)}(Z_{p(c)})}\left[\int_G \chi'(cg^{-1})\,\Delta(g^{-1})\,d\nu(g)\right] d\lambda(c).$$

We now apply (10) to the integral over G, introduce a new variable $h = g^{-1}$, and then apply (14). We obtain

$$\int_G \chi'(cg^{-1})\,\Delta(g^{-1})\,d\nu(g) = \int_G \chi'(cg^{-1})\,\delta(g^{-1})\,d\nu(g)$$
$$= \int_G \chi'(ch)\,d\nu(h) = \nu_{p(c)}(Z'_{p(c)}),$$

and, consequently,

$$\lambda(Z') = \int_B \frac{\nu_{p(c)}(Z'_{p(c)})}{\nu_{p(c)}(Z_{p(c)})}\,\chi(c)\,d\lambda(c).$$

On account of (11) the ratio $\nu_{p(c)}(Z'_{p(c)})/\nu_{p(c)}(Z_{p(c)}) = k = \text{const}$, whence $\lambda(Z') = k\int\limits_B \chi(c)\,d\lambda(c) = k\lambda(Z)$, q. e. d.

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COMPACTNESS AND PRODUCT SPACES

BY

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In this paper we are concerned with the preserving of different sorts of compactness under the Cartesian multiplication. We shall use the following terminology:

countably compact = each countable open covering contains a finite subcovering;

compact = each open covering contains a finite subcovering;

Lindelöf space = each open covering contains a countable subcovering;

pseudo-compact = each real-valued continuous function is bounded (see [2]).

I. M. Katětov has proved the following theorem (see [3]):

The Cartesian product of two countably compact spaces, one of which is compact, is also countably compact.

In [6] C. Ryll-Nardzewski has proved a similar theorem:

The Cartesian product of two countably compact spaces, one of which satisfies the first axiom of countability, is also countably compact.

Using the theory of Moore-Smits nets (for the definition, properties, notation and terminology see [4], p. 65) we may obtain, by a uniform method, the following theorem:

(i) *The Cartesian product of two countably compact spaces, X and Y, one of which is either compact or sequentially compact, is also countably compact.*

We recall that a space is said to be *sequentially compact* if each sequence of elements of the space contains a convergent subsequence. Of course, each countably compact space satisfying the first axiom of countability is sequentially compact, but not conversely.