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SOME PROPERTIES OF THE SPACE (s)

BY

C. BESSAGA, A. PEŁCZYŃSKI AND S. ROLEWICZ (WARSAW)

In the present paper we consider some isomorphic properties (i. e. the invariants of the linear homeomorphisms) of the space (s). It follows from these considerations that the isomorphic structure of the space (s) is very similar to that of the finitely dimensional spaces.

1. Notation and terminology. We denote by (s) the B_0-space [1] of all real sequences $x = (a_n)$ with the usual definition of the addition of elements and multiplication by real numbers and with the pseudonorms $\|x\|_n = \sup\limits_{i\leqslant n}|a_i|$ $(n = 1, 2, \ldots)$.

In the sequel the symbol X will be reserved for denoting an arbitrary B_0-space. We assume that convergence in X is determined by the system of homogeneous pseudonorms $(\|x\|_i)$ such that $\|x\|_i \leqslant \|x\|_{i+1}$ $(x \in X, i = 1, 2, \ldots)$. The symbol $|x|$ will always denote the norm (in general non-homogeneous) which determines the topology of the space; $|x|$, $\|x\|$ will denote homogeneous and continuous pseudonorms in the space X.

It is well known that ([5], 1.51):

($*$) *For every homogeneous and continuous pseudonorm $|x|$ there exists a constant C and an index i_0 such that for every $x \in X$ the inequality $|x| \leqslant C\|x\|_{i_0}$ is satisfied.*

The pseudonorm $|x|$ determines the quotient space $X(|.|)$ whose elements are the sets

$$\{x\} = \{y \in X\colon |x-y| = 0\},$$

where

$$\{x\}+\{y\} = \{x+y\}, \qquad t\{x\} = \{tx\}, \qquad |\{x\}| = |x|.$$

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(1) By a B_0-space we mean a linearly-metric and locally convex space. For the definition and basic properties see [5].

The above definitions of the addition, multiplication by scalars and the norm does not depend on the choice of elements from the classes $\{x\}$ and $\{y\}$.

A space X is said to be *relatively complete* [4] if it is isomorphic to the space X' such that all the spaces $X'(\|\cdot\|_i)$ $(i=1,2,\ldots)$ are complete (i. e. they are B-spaces).

A space X is called a *Montel space* [4] if in the space X for every linearly bounded sequence there is a convergent subsequence.

2. In the paper [2] the following results are obtained:

THEOREM 1. *The space X contains a subspace that is isomorphic to (s) if and only if there exists no homogeneous and continuous norm in X.*

THEOREM 2. *If all the spaces $X(\|\cdot\|_i)$ $(i=1,2,\ldots)$ are finitely dimensional, then either X is isomorphic to (s) or it has a finite dimension.*

Now we shall deduce some consequences from these theorems.

THEOREM 3. *The space (s) is the only infinitely dimensional space that is relatively complete and Montel space.*

Proof [2]. It is easily seen that (s) is a relatively complete Montel space.

Let X be a relatively complete Montel space. We may obviously assume that all the spaces $X(\|\cdot\|_i)$ are B-spaces. The spaces $X(\|\cdot\|_i)$ are Montel spaces as the linear images of the Montel space X (see [4]). Since each Montel B-space must be finitely dimensional; to complete the proof it is enough to apply Theorem 2.

THEOREM 4. *If the space X is a linear image of (s), then either X is isomorphic to (s) or it is of a finite dimension.*

Proof. Let U be a linear mapping of the space (s) onto X. Every pseudonorm $\|x\|$ defined on X determines the pseudonorm $|y| = \|U(y)\|$ on the space (s). According to property $(*)$ and Theorem 2, the space $s(|\cdot|)$ is finitely dimensional; hence so is the space $X(\|\cdot\|)$. Now it is sufficient to apply once more Theorem 2.

Let us compare Theorem 4 with the following property of finitely dimensional spaces: a linear image of a k-dimensional space X is either isomorphic to the space X or has a dimension less than k.

The space (s) has also the following well-known property of the finitely dimensional spaces:

THEOREM 5. *All the bases of the space (s) are equivalent.*

[2] This proof is made by W. Zawadowski.

(Two bases (x_n) and (y_n) are called *equivalent* if for every real sequence (t_n) the series $\sum_{n=1}^{\infty} t_n x_n$ converges if and only if the series $\sum_{n=1}^{\infty} t_n y_n$ converges).

The proof of Theorem 5 is given in [3]; see also [1], Chap. III, Theorem 13.

An immediate consequence of Theorem 2 is the following result of S. Mazur and W. Orlicz ([6], 3.1):

THEOREM 6. *Every infinitely dimensional subspace of the space (s) is isomorphic to (s).*

We shall establish some generalizations of Theorems 5 and 6.

THEOREM 7. *In order that the space X with an infinite basis (e_n) be isomorphic to the space (s) it is necessary and sufficient that*

(α) *every subbasis $(e_n') = (e_{k_n})$ contain a subbasis $(e_n'') = (e_{r_n}')$ that is equivalent to the unit-vector-basis of the space (s)* [3].

Proof. The necessity of condition (α) is obvious.

Sufficiency. Suppose that X is not isomorphic to (s). Thus, according to Theorem 1, there is a continuous homogeneous pseudonorm $\|x\|$ such that the space $X(\|\cdot\|)$ has an infinite dimension. We can choose a subsequence $(e_n') = (e_{k_n})$ of the sequence (e_n) in such a way that $\|e_n'\| \neq 0$ $(n=1,2,\ldots)$. (This easily follows from the fact that there exists in X an infinite set whose elements are linearly independent with respect to the pseudonorm $\|x\|$). Consider a subspace $Y \subset X$ which is generated by elements (e_n'). Every element $y \in Y$ can be represented in a unique way in the form $y = \sum_{n=1}^{\infty} t_n e_n'$, because the sequence (e_n') is a basis of Y.

Let

$$|y| = \sup_n \Big\| \sum_{k=1}^{n} t_k e_k' \Big\|.$$

It is easily verified that the functional $|y|$ is a homogeneous and continuous norm. According to Theorem 1, no subspace of Y is isomorphic to (s), which contradicts condition (α), q. e. d.

THEOREM 8. *A space X is isomorphic to (s) if and only if*

(β) *every infinitely dimensional subspace Y of the space X contains a subspace Y_1 which is isomorphic to (s).*

Proof. According to Theorem 6, the necessity of condition (β) is obvious. The proof of the sufficiency is based on the following non-trivial lemma, which is proved in [3]:

[3] The unit-vector-basis is composed of elements $e_n = (0,0,\ldots,0,1,0,\ldots)$ (where 1 is on the n-th place; $n=1,2,\ldots$).

LEMMA. *If the space* $X(\|\cdot\|)$ *has an infinite dimension, then there exists a sequence* (x_n) *with* $\|x_n\| \neq 0$ $(n = 1, 2, \ldots)$ *which is a basis of a subspace* Y *of the space* X.

From this lemma we deduce in the same way as in the proof of Theorem 7 that no subspace of Y is isomorphic to (s). This contradicts condition (β).

3. In this section we shall consider F-spaces [4] which contain subspaces isomorphic to the space (s).

Theorem 1 can be generalized as follows:

THEOREM 9. *Let* E *be an* F*-space with the norm* $|x|$. E *contains a subspace which is isomorphic to the space* (s) *if and only if*

(γ) *for every* $\varepsilon > 0$ *there is in* E *an element* $x \neq 0$ *such that for every real* t *the inequality* $|tx| < \varepsilon$ *holds.*

The geometrical meaning of condition (γ) is that there exist in the space E "arbitrarily short" straight lines.

Proof. The necessity of (γ) is obvious. To prove the sufficiency suppose that the space E satisfies condition (γ). We can choose a sequence (e_n) such that $0 < \delta_1 < 1$, $\delta_{n+1} < \frac{1}{4}\delta_n$ $(n = 1, 2, \ldots)$, where $\delta_k = \sup_t |te_k|$.

Let

$$Y = \Big\{x\colon E\colon x = \sum_{n=1}^{\infty} t_n e_n\Big\}.$$

Y is a linear set; it has the following properties:

(a) For every sequence (t_n) of real numbers the series $\sum_{n=1}^{\infty} t_n e_n$ converges.

Indeed, $\Big|\sum_{n=p}^{q} t_n e_n\Big| \leqslant \sum_{n=p}^{q} |t_n e_n| \leqslant \sum_{n=p}^{q} 4^{1-n}$.

(b) If $y_p = \sum_{n=1}^{\infty} t_n^p e_n \to 0$, then $\lim_{p\to\infty} t_n^p = 0$ $(n = 1, 2, \ldots)$.

Suppose that (b) is false. Then there is a sequence of indices (p_k), an index n_0 and $\eta > 0$ such that

(1) $$\lim_{k\to\infty} t_n^{p_k} = 0 \quad \text{for} \quad n < n_0,$$
$$|t_{n_0}^{p_k}| > \eta \quad (k = 1, 2, \ldots).$$

According to the definition of δ_n it is possible to find a number τ

[4] By an F-space we mean a linearly-metric and complete space. For the definition and basic properties see [1].

such that for $t \geqslant \tau\eta$ the inequality $|te_{n_0}| \geqslant \frac{3}{4}\delta_{n_0}$. holds. Hence, in particular,

$$|\tau t_{n_0}^{p_k} e_{n_0}| \geqslant \tfrac{3}{4}\delta_{n_0} \qquad (k = 1, 2, \ldots).$$

Now we consider the sequence (τy_{p_k}). It follows from (1) that there exists an index k_0 such that for every $k \geqslant k_0$

$$\Big|\sum_{n=1}^{n_0-1} t_n^{p_k} e_n\Big| \leqslant \tfrac{1}{4}\delta_{n_0}.$$

Thus for every $k \geqslant k_0$ we have

$$|\tau y_{p_k}| \leqslant |\tau t_{n_0}^{p_k} e_{n_0}| - \Big|\sum_{n=1}^{n_0-1} t_n^{p_k} e_n\Big| - \Big|\sum_{n=\eta+1}^{\infty} t_n^{p_k} e_n\Big|$$
$$\geqslant \tfrac{3}{4}\delta_{n_0} - \tfrac{1}{4}\delta_{n_0} - \sum_{n=n_0+2}^{\infty} \delta_n \geqslant \Big(\tfrac{1}{2} - \sum_{j=1}^{\infty} 4^{-j}\Big) = \tfrac{1}{6}\delta_{n_0}$$

which contradicts the condition $\tau y_{p_k} \to 0$.

(c) If $\lim_{p\to\infty} t_n^p = 0$ $(n = 1, 2, \ldots)$, then $y_p = \sum_{n=1}^{\infty} t_n^p e_n \to 0$.

Let $\varepsilon > 0$. We choose an index n_0 such that $\sum_{n=n_0}^{\infty} \delta_n < \varepsilon/2$. Now we choose an index P such that for $p > P$ the inequality $\Big|\sum_{n=1}^{n_0-1} t_n^p e_n\Big| < \varepsilon/2$ holds. For $p > P$ we have

$$|y_p| \leqslant \Big|\sum_{n=1}^{\infty} t_n^p e_n\Big| + \Big|\sum_{n=n_0+1}^{\infty} t_n^p e_n\Big| < \varepsilon.$$

Conditions (a), (b) and (c) imply that the correspondence $U(t_n) = \sum_{n=1}^{\infty} t_n e_n$ is an isomorphic mapping of the space (s) onto X, q. e. d.

COROLLARY. E *contains a subspace that is isomorphic to* (s) *if and only if there is in* E *a sequence* (x_n) *such that for every real sequence* (t_n) *the series* $\sum_{n=1}^{\infty} t_n x_n$ *converges.*

THEOREM 8a. *Let* E *be an* F*-space with a basis* (e_n). *If every subspace of the space* E *contains a subspace isomorphic to* (s), *then* E *is isomorphic to the space* (s).

Proof. Suppose that E is not isomorphic to (s). Hence the basis (e_n) is not equivalent to the unit-vector-basis of (s) (see [3], Wniosek 2), therefore there is a real sequence (t_n) such that the series $\sum_{n=1}^{\infty} t_n e_n$ is not convergent. We may suppose (Theorem 4 of [7]) that the norm of the space E satisfies the condition

$$\Big|\sum_{n=1}^{p} t_n e_n\Big| \leqslant \Big|\sum_{n=1}^{q} t_n e_n\Big| \quad \text{for} \quad p \leqslant q. \tag{2}$$

Since the series $\sum_{n=1}^{\infty} t_n e_n$ does not converge, there exist an $\varepsilon > 0$ and a subspace (k_n) of indices such that $\Big|\sum_{i=k_n+1}^{k_{n+1}} t_i e_i\Big| > \varepsilon$.

Write

$$e_n' = \sum_{i=k_n+1}^{k_{n+1}} t_i e_i \qquad (n = 1, 2, \ldots).$$

Let E' be a subspace generated by the elements $e_1', e_2', \ldots$ According to (2) and Theorem 5 of [7], the sequence (e_n') is a basis of the space E'.

Let $y = \sum_{n=1}^{\infty} \tau_n e_n$ be an arbitrary element of E'. We denote by u_0 the first index such that $\tau_{n_0} \neq 0$. Let $\lambda = 1/\tau_{n_0}$. We have $|\lambda \tau_{n_0} e_{n_0}'| > \varepsilon$. Hence, by (2), $\varepsilon < |\lambda \tau_{n_0} e_{n_0}'| < |y|$. According to Theorem 9, no subspace of E' is isomorphic to the space (s), q. e. d.

We do not know whether Theorem 8 holds true for arbitrary F-spaces, i. e. whether for an arbitrary F-space E condition (β) (without any additional assumption) implies that E is isomorphic to the space (s).

Let us note that Theorem 7 is false in the case of F-spaces. This is a consequence of the following

Example. Let H be the set of all real sequences for which

$$|x| = \sum_{n=1}^{\infty} \frac{1}{n} \frac{|t_n|}{1+|t_n|} < +\infty. \tag{3}$$

If we define the addition of elements and multiplication by scalars, then H is an F-spaces under norm (3). It is easy to see that the sequence of unit vectors (x_n) is a basis of H. Since the basis (x_n) is not equivalent to the unit-vector-basis of the space (s), using Theorem 5 we see that H is not isomorphic to (s).

Let (x_{k_n}) be an arbitrary subsequence of the basis (x_n). Let us choose an increasing sequence of indices (r_n) in such a way that

$$\sum_{n=1}^{\infty} \frac{1}{k_{r_n}} < +\infty.$$

The subbasis $(x_{k_{r_n}})$ is equivalent to the unit-vector-basis of (s).

REFERENCES

[1] S. Banach, *Théorie des opérations linéaires*, Warszawa 1932.

[2] C. Bessaga and A. Pełczyński, *On a class of B_0-spaces*, Bulletin de l'Académie Polonaise des Sciences, Classe III, 5 (1957), p. 375-377.

[3] — *Własności baz w przestrzeniach typu B_0 (Properties of bases in B_0-spaces)*, Prace Matematyczne (in print).

[4] J. A. Dieudonné, *Recent developments in the theory of locally convex spaces*, Bulletin of the American Mathematical Society 59 (1953), p. 495-514.

[5] S. Mazur and W. Orlicz, *Sur les espaces métriques linéaires I*, Studia Mathematica 10 (1948), p. 184-208.

[6] — *On linear methods of summability*, ibidem 14 (1954), p. 129-160.

[7] В. Н. Никольский, *Наилучшее приближение и базис в пространстве Фреше*, Доклады Академии Наук СССР 59 (1948), p. 639-642.

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