COLLOQUIUM MATHEMATICUM

VOL. VII 1959 FASC. 1

A REMARK ON LINEAR FUNCTIONALS

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E. Hewitt(1) proves the following theorem:

THEOREM. *Let X be a completely regular space and $C(X)$ the set of all continuous real-valued functions on X. Then each linear p-continuous functional φ defined on $C(X)$ can be written in the form*

$$\varphi(f) = a_1 f(p_1) + \ldots + a_k f(p_k)$$

where $p_1, \ldots, p_k$ are fixed points of X and $a_1, \ldots, a_k$ are fixed real numbers.

The Hewitt proof is based on the theory of integral representation of functionals, which is developed in the paper quoted. In the present paper we give a quite elementary proof of the above theorem.

Explanations. By a *linear functional* we understand in this paper a functional φ which satisfies only the condition $\varphi(\alpha\cdot f+\beta\cdot g) = \alpha\cdot\varphi(f)+\beta\cdot\varphi(g)$, without any assumptions about continuity. Such a functional is said to be *p-continuous* if it is continuous with respect to the so-called *pointwise topology* in $C(X)$, i. e. the topology in which the neighbourhood system of a function f consists of all sets of the form $\{g: |g(p_i)-f(p_i)| < \varepsilon;\ i = 1,\ldots,s\}$, where ε is a positive number and $p_1, \ldots, p_s$ are arbitrary points of X. Of course, the pointwise topology is well-defined both in $C(X)$ and in the set $F(X)$ consisting of all real-valued functions on X. Moreover, under this topology the set $C(X)$ is dense in $F(X)$.

LEMMA. *Each linear p-continuous functional φ defined on $C(X)$ can be extended to a linear p-continuous functional φ^* defined on $F(X)$.*

Proof. Let f be any member of $F(X)$. Denote by $\mathfrak{N}$ the family of all sets of the form $\overline{\varphi(U)}$ where U is a neighbourhood of f. Since the intersection of any two neighbourhoods of f is again a neighbourhood of f (and thus contains a member of $C(X)$), the family $\mathfrak{N}$ is centred. On the other hand,

(1) E. Hewitt, *Linear functionals on spaces of continuous functions*, Fundamenta Mathematicae 37 (1950), p. 161-189.

for any $\delta > 0$ there exists a neighbourhood U of f such that the set $\overline{\varphi(U)}$ is of diameter $\leqslant \delta$. Indeed, since φ is p-continuous, we can find a number $\varepsilon > 0$ and points $p_1, \ldots, p_s$ in X such that

(1) $|\varphi(h)| < \delta$ *for each* h *in* $C(X)$ *with* $|h(p_i)| < \varepsilon$ *for* $i = 1, \ldots, s$.

Now let $U = \{g: |g(p_i) - f(p_i)| < \varepsilon/2\}$. Then U is a neighbourhood of f and if g_1, g_2 are any members of $C(X)$ lying in U, then, in virtue of (1), $|\varphi(g_1) - \varphi(g_2)| = |\varphi(g_1 - g_2)| < \delta$.

Therefore, by the well-known Riesz theorem, there exists a number a which belongs to each member of $\mathfrak{R}$. Of course, this number a satisfies the following condition:

(2) *for any* $\delta > 0$, *there exists a neighbourhood* U *of* f *such that* $|\varphi(h) - a| < \delta$ *for each member* h *of* $C(X)$ *lying in* U;

moreover, the number a is uniquely determined by property (2). Now let

(3) $\varphi^*(f) = a$, *where* a *satisfies* (2).

Of course, formula (3) defines a linear functional on $F(X)$ which is an extension of φ. One can easily check, by an argument similar to that used above, that φ^* is p-continuous. Thus the lemma is proved.

In virtue of this lemma one may assume that a functional φ is defined on the whole $F(X)$ instead of being defined only on $C(X)$. Under this assumption we give the

Proof of theorem. We divide this proof into three steps.

1st step. For any p in X we denote by f_p the characteristic function of a point p and let Z_φ be the set of all $p \in X$ such that $\varphi(f_p) \neq 0$. We shall show that the set Z_φ is finite. Indeed, assuming the contrary, let $p_1, p_2, \ldots$ be an infinite sequence of distinct members of Z_φ and let $\beta_i = 1/\varphi(f_{p_i})$. Let f be the function defined by $f(p_i) = \beta_i$ and $f(p) = 0$ for $p \neq p_i$ $(i = 1, 2, \ldots)$. If U is any neighbourhood of f and k is any integer, then there exists $n > k$ such that the function $\beta_1 f_{p_1} + \ldots + \beta_n f_{p_n}$ belongs to U. But $\varphi(\beta_1 f_{p_1} + \ldots + \beta_n f_{p_n}) = n > k$ and this leads to a contradiction.

2nd step. Suppose that f is a member of $F(X)$ and let $Z_\varphi \subset f^{-1}(0)$. We shall show that $\varphi(f) = 0$ Indeed, if U is any neighbourhood of f, then there exists a function h of the form $\beta_1 f_{p_1} + \ldots + \beta_s f_{p_s}$, where $p_1, \ldots, p_s$ lie outside the set Z_φ, which belongs to U. But $\varphi(h) = 0$ and it follows that $\varphi(f) = 0$.

3rd step. Let $Z_\varphi = \{p_1, \ldots, p_k\}$, where the points p_i are mutually distinct. Moreover, let $a_i = \varphi(f_{p_i})$. Now let f be any member of $F(X)$ and let $h = f(p_1)\cdot f_{p_1} + \ldots + f(p_k)\cdot f_{p_k} - f$. Then $Z_\varphi \subset h^{-1}(0)$, whence $\varphi(h) = 0$. But $\varphi(h) = a_1\cdot f(p_1) + \ldots + a_k\cdot f(p_k) - \varphi(f)$, and therefore

$$\varphi(f) = a_1\cdot f(p_1) + \ldots + a_k\cdot f(p_k),$$

and the theorem follows.

Of course, this theorem and its proof remain valid in "the complex case".

Reçu par la Rédaction le 10. 9. 1958