Some consequences of the axiom of constructibility *

by

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Gödel, in his celebrated paper [8] of 1938, announced a proof of the consistency (with the axioms of set theory) of the following four famous undecided propositions of abstract and descriptive set theory:

(1) *the axiom of choice*;

(2) *the generalized continuum hypothesis*;

(3) *there exist linear nonmeasurable* B_2 *sets*;

(4) *there exist linear CA sets of the power of the continuum containing no nonempty perfect subsets.*

Indeed he announced the consistency of a proposition which he asserted implies (1)-(4), namely the proposition

(A) *every set is constructible*,

which has come to be called the *axiom of constructibility* (1).

A full proof of the consistency of (A), together with a proof that (A) implies (1) and (2), appeared in 1940 in Gödel's monograph [9]. For the proof that (A) implies (3) and (4), however, a closer analysis of the construction was required, and as it was inconvenient to include these details in the monograph, no mention of (3) or (4) was made there.

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* The main theorem and some of the applications of this paper were presented to the Seminar on the Foundations of Mathematics of the Mathematical Institute of the Polish Academy of Sciences in Warszawa on 27 March and 3 April 1957. The writer is indebted to the director of the seminar, Prof. Andrzej Mostowski, for his valuable advice and suggestions on an early draft of the proof of the main theorem.

Much of the material of this paper was also presented on 26 and 29 July 1957 to the Summer Institute of Symbolic Logic at Cornell University. Cf. [4].

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(1) In using this name, which seems to have been at least implicitly first suggested by Gödel himself (cf. the first sentence on p. 557 of [8]), it is not our intention to subscribe to any particular philosophic outlook.

We should perhaps note here that from his present-day absolutistic position Gödel sees proposition (A) as false.

In 1939 Kuratowski recognised that Gödel's result pertaining to (3) and (4) could be proved if one were able to show that (A) implies the proposition

(5) *there exists a PCA well-ordering of the real numbers.*

A proof of this implication was carried out by Mostowski, whose manuscript was destroyed by fire by the Nazis in the Warszawa insurrection of 1944.

Kuratowski prepared also in 1940 a manuscript presenting some interesting propositions that are implied by the proposition

(6) *there exists a projective well-ordering of the real numbers.*

This manuscript (also destroyed in 1944) was redeveloped and published in 1948 (cf. [17]). The Mostowski manuscript was never rewritten, however, and so a full proof of the consistency of (3)-(5) and of the consistency of the further propositions of [17] had never appeared.

Recognizing this need, Gödel added in 1951, in the second printing [10] of his monograph [9], a note (cf. Note 1, p. 67) calling attention to [17] and giving a brief outline of a proof that (A) implies (6), or actually, that (A) implies the stronger proposition (5).

Meanwhile at about the same time as the appearance of [10], P. S. Novikov published a long paper [20] in which he proved that (A) implies (3) and (4). He apparently did not consider (5) or (6), and the relationship of his paper to them is not clear (²).

In this same paper Novikov announced (with the statement that the proof would appear in a later paper) the consistency of the following proposition:

(7) *there exists a natural number p such that at the p-th and higher levels of the projective hierarchy the separation principles behave as they do at the 2-nd level.*

In view of the large number of interesting consequences (e. g. the propositions of [17]) of the theorem that (A) implies (5), it has seemed desirable to have a full proof of it in print. We present such a proof here, following closely the ideas given by Gödel in [10] Note 1.

Actually we prove that (A) implies a proposition (C^1) which is a little bit stronger, in three directions, than (5) (³). One of these directions, that the "PCA" of (5) can be replaced by "B_2", is trivial, and was omitted from Note 1 by Gödel only by an oversight when writing it up. Nevertheless this strengthened version already yields an immediate proof, which is somewhat simpler than that of Novikov (⁴), of the consistency of (3). It also leads by a short argument to a proof (cf. [3]) of the consistency of a strengthened version of (7) (in which $p=3$).

(²) From the comparative weakness of the results of Sodnomov [21] one gathers that Novikov was not aware of the consistency of (5).

(³) Actually we state our results and proofs, as is quite customary in descriptive set theory, for the space of irrationals (and more general spaces), rather than for the reals. But it is a routine matter to pass from our results to the results for the reals.

The second strengthening consists in going from the classical to the effective projective hierarchy, and results in strengthened versions of many of the known consequences of (A). We do not enumerate these new versions here, but simply refer the reader to [17], [21], etc. to make the obvious improvements for himself. The idea of attempting such a strengthening was suggested by the study of the analogies between classical and effective descriptive set theory (cf. [1], [2], and [3]).

The third strengthening is a generalization which has proved useful for the solution, under the assumption of (A), of a number of open problems of recursive function theory and descriptive set theory.

The principal applications of (C^1) involve an effective choice operator and we have summarized the information obtained in this direction in a table in Section 4. Some specific applications of this table are given in the following section, and the final section contains a result obtained by arguments similar to those in the main part of the paper.

1. The proposition (C^1).

We assume familiarity with [9]. We shall adopt, with minor changes, the notation of [9], p. 63-64, [3], and [12], p. 538.

We denote by "N" the set of natural numbers $0, 1, \ldots$, by "N^N" the set of functions from N into N, by "N^{N^N}" the set of functions from N^N into N, etc. As in [9] N is identified with the ordinal number ω, but we continue to use "N" to avoid an ambiguity in "ω^ω".

We use lower case Greek letters (other than "ι", "λ", "ν", "ω") with superscripts $0, 1, 2, \ldots$ as variables over $N, N^N, N^{N^N}, \ldots$, respectively. The superscript "1" is usually omitted and lower case Roman letters are usually used as variables over N in place of lower case Greek letters with superscript "0". As a variable over Ω_1 we use "ν". Capital Roman letters are used for arbitrary classes and predicates.

We denote by "$\mathfrak{N}^\tau$" the Cartesian product of $\tau(0)$ copies of N, $\tau(1)$ copies of N^N, ..., where τ has a finite positive number of values different

(⁴) This is true because any predicate well-ordering the reals is nonmeasurable (cf. [17], p. 131). And actually this fact and (5) directly imply (3) if one observes (as Kuratowski did at the Princeton Bicentennial Conference in 1946) that if a predicate well-ordering the reals belongs to a certain class of one of the hierarchies then so also does its negation.

This paper thus fills the need expressed by Mostowski in [19] for a simpler proof of this result.

from 0. Lower case German letters with superscript "τ" are used as variables over $\mathfrak{N}^\tau$. The superscripts are omitted if arbitrary or clear from context.

As indicated in [3] the categories of the C-arithmetical hierarchies are denoted by "$\Sigma_k^0[C]$", "$\Pi_k^0[C]$", the categories of the C-analytical hierarchies by "$\Sigma_k^1[C]$", "$\Pi_k^1[C]$", and the categories of the hierarchies based on quantification of higher finite types (cf. [15]) by "$\Sigma_k^t[C]$", "$\Pi_k^t[C]$" ($t > 1$). As an abbreviation "$[\emptyset]$" is omitted ([5]). Sets are classified according to their representing predicates and functions according to their graphs.

The reader chiefly interested in classical descriptive set theory can assume $\mathfrak{N}^\tau$ is N^N. Then $\{\Sigma_k^1[N^N], \Pi_k^1[N^N]\}_k$ is just the projective hierarchy on the irrationals. He should keep in mind, of course, that $\Sigma_k^t[N^N] \subset \Sigma_k^t$ and $\Pi_k^t[N^N] \subset \Pi_k^t$.

We consider the proposition

(C^1) *there exists an Ω_1 well-ordering $<$ of N^N such that for any subset C of N^N and any predicate R recursive in functions in C the set $\hat{\mathfrak{a}}\hat{\beta}(E\beta_1)_{\beta_1<\beta}(E\alpha)(x)R(\mathfrak{a}, \beta_1, \alpha, x)$ is in $\Sigma_2^1[C] \cap \Pi_2^1[C]$.*

Our principal result is the following:

THEOREM 1. (A) $\to$ (C^1).

Proof. There is a natural well-ordering of the constructible elements of N^N, namely $\lambda\alpha\beta\ \mathrm{Od}`\alpha < \mathrm{Od}`\beta$. We could use this for $<$, but it will be a little more convenient to use instead $\lambda\alpha\beta\ \mathrm{Or}`\alpha < \mathrm{Or}`\beta$, where $\mathrm{Or}`\alpha$ is defined as the least ordinal $\nu$ such that $\omega\times\omega\cdot \mathrm{F}`\nu = \alpha$. Since for every constructible α, $\mathrm{Od}`\alpha < \Omega_1$, and since $\mathrm{Or}`\alpha \leqslant \mathrm{Od}`\alpha$, this gives a well-ordering of the constructible functions of type $\leqslant \Omega_1$. We hereafter denote $\lambda\alpha\beta\ \mathrm{Or}`\alpha < \mathrm{Or}`\beta$ by "$<$", and proceed to show that it has the properties required of it by the theorem. (A) clearly implies that $<$ is an Ω_1 well-ordering of N^N. Now let C be any subset of N^N and R be any predicate recursive in functions in C. We shall prove that $\hat{\mathfrak{a}}\hat{\beta}(E\beta_1)_{\beta_1<\beta}(E\alpha)(x)R(\mathfrak{a}, \beta_1, \alpha, x)$ is in $\Sigma_2^1[C] \cap \Pi_2^1[C]$. For this purpose we introduce the following definitions ([6]):

(1.0) $W(\varphi) \equiv \lambda ij\ \varphi(i, j) = 0$ well orders N,

(1.1) $\varphi_i =$ the ordinal number corresponding to i in the well-ordering $\lambda ij\ \varphi(i, j) = 0$,

(1.2) $M(\varphi, \varepsilon) \equiv W(\varphi)\ \&\ [\varepsilon(i, j) = 0 \equiv \mathrm{F}`\varphi_i \in \mathrm{F}`\varphi_j]$.

Then we have

(1.3)
$$
\begin{aligned}
&(E\beta_1)_{\beta_1<\beta}(E\alpha)(x)R(\mathfrak{a}, \beta_1, \alpha, x)\\
&\quad\equiv (E\beta_1)\big[(E\varphi)(E\varepsilon)[M(\varphi, \varepsilon)\ \&\ (Ei)[\omega\times\omega\cdot \mathrm{F}`\varphi_i = \beta_1]\\
&\qquad \&\ (\overline{Ei})[\omega\times\omega\cdot \mathrm{F}`\varphi_i = \beta]]\ \&\ (E\alpha)R(\mathfrak{a}, \beta_1, \alpha, x)\big]\\
&\quad\equiv (\varphi)(\varepsilon)\big[M(\varphi, \varepsilon)\ \&\ (Ei)[\omega\times\omega\cdot \mathrm{F}`\varphi_i = \beta] \to\\
&\qquad (E\beta_1)[\beta_1 \neq \beta\ \&\ (Ei)[\omega\times\omega\cdot \mathrm{F}`\varphi_i = \beta_1]\\
&\qquad \&\ (E\alpha)R(\mathfrak{a}, \beta_1, \alpha, x)]\big].
\end{aligned}
$$

The theorem now follows straightforwardly by the techniques of [13] from (1.3) and the following two lemmas:

LEMMA A. *There exists a $\Sigma_1^1 \cap \Pi_1^1$ predicate A such that*
$$M(\varphi, \varepsilon) \to [\omega\times\omega\cdot \mathrm{F}`\varphi_i = \beta \equiv A(\varphi, \varepsilon, \beta, i)]\ ([7]).$$

LEMMA B. $M \in \Pi_1^1$.

The next two sections are devoted to the proof of these lemmas. It is important to note for Section 6 that neither of these lemmas depends on (A).

2. Proof of Lemma A.

Assume $M(\varphi, \varepsilon)$. Then

(A.0) $\omega\times\omega\cdot \mathrm{F}`\varphi_i = \beta \equiv (k)(l)[\beta(l) = k \equiv \underbrace{\langle kl\rangle \in \mathrm{F}`\varphi_i}_{1}]$,

(A.1) $\langle kl\rangle \in \mathrm{F}`\varphi_i \equiv \big\{\{k\}\{kl\}\big\} \in \mathrm{F}`\varphi_i \equiv (Em)\big[\underbrace{\mathrm{F}`\varphi_m = \big\{\{k\}\{kl\}\big\}}_{2}\ \&\ \varepsilon(m, i) = 0\big]$,

(A.2) $\mathrm{F}`\varphi_m = \big\{\{k\}\{kl\}\big\} \equiv (n)[\varepsilon(n, m) = 0 \equiv \underbrace{\mathrm{F}`\varphi_n = \{k\}}_{3} \vee \underbrace{\mathrm{F}`\varphi_n = \{kl\}}_{4}]$,

(A.3) $\mathrm{F}`\varphi_n = \{k\} \equiv (p)[\varepsilon(p, n) = 0 \equiv \underbrace{\mathrm{F}`\varphi_p = k}_{5}]$,

(A.4) $\mathrm{F}`\varphi_n = \{kl\} \equiv (p)[\varepsilon(p, n) = 0 \equiv \underbrace{\mathrm{F}`\varphi_p = k}_{5} \vee \underbrace{\mathrm{F}`\varphi_p = l}_{5}]$,

(A.5) $\mathrm{F}`\varphi_p = k \equiv (El)(Em)[\underbrace{l = \varphi_m}_{6}\ \&\ \underbrace{\mathrm{F}`l = k}_{7}\ \&\ \underbrace{\mathrm{F}`\varphi_p = \mathrm{F}`\varphi_m}_{8}]$,

([5]) We take this opportunity to note that by changing slightly the definition of the C-arithmetical, C-analytical, ... hierarchies in [3] by letting the matrices be recursive in *functions* (instead of some *one* function) in C the notion of *linked* subset and the restrictions on certain theorems that involved it (cf. [3]) can be discarded. This change was introduced in [4] and we follow it hereafter.

We also note that the second option of Footnote 9 of [3] should be followed for C that are not "linked" in the obvious infinite sense.

([6]) In working with partially defined functions and predicates in this paper we understand the propositional connectives in the sense of their strong truth tables (cf. [12], Section 64).

([7]) Inasmuch as the basic definitions and results of [13] were stated only for completely defined predicates and functions, we use the phraseology of this lemma, and thereby avoid having to introduce here the elementary theory of the classification of partial predicates and functions.

$$(\mathrm{A}.6)\quad l = \varphi_m \equiv (Es)\Big[(s)_l = m \,\&\, (a)_{a<l}\big[\varphi\big((s)_a, (s)_{a+1}\big) = 0\big] \,\&\, (b)[\varphi(b, m) = 0 \to (Ea)_{a<l}(s)_a = b]\Big],$$

$$(\mathrm{A}.7)\quad \lambda lk\ \mathrm{F}`l = k \quad \text{is recursive},$$

$$(\mathrm{A}.8)\quad \mathrm{F}`\varphi_p = \mathrm{F}`\varphi_m \equiv (q)[\varepsilon(q, p) = 0 \equiv \varepsilon(q, m) = 0].$$

To prove the forward implication of (A.1) and the backward implication of (A.2), (A.3), (A.4), and (A.8), one needs to use [9], 9.52, which states that every element of a constructible set is constructible and of lower order than the set. To prove the forward implication of (A.5) one needs to use the result that every natural number is a constructible set of finite order. This is easily proved by induction by applying the idea of the proof of [9], 9.66 to definition [9], 7.4. It is clear that $\lambda l \mathrm{F}`l$ is effectively calculable and hence that $\lambda lk\ \mathrm{F}`l = k$ is effectively decidable. We therefore omit the straightforward proof of (A.7).

Now by following the outline of (A.0)-(A.8) the completely defined predicate A required by the lemma is easily defined. And that $A \in \Sigma_1^1 \cap \Pi_1^1$ will follow straightforwardly from (A.0)-(A.8) by the techniques of [13].

3. Proof of Lemma B. To prove that $M \in \Pi_1^1$ we first observe that

$$
\begin{aligned}
(\mathrm{B})\quad & M(\varphi, \varepsilon) \\
& \equiv W(\varphi) \,\&\, [\varepsilon(i, j) = 0 \equiv \mathrm{F}`\varphi_i \in \mathrm{F}`\varphi_j] \\
& \equiv (i)(j)[\varphi(i, j) = 0 \equiv i \neq j \,\&\, \varphi(j, i) \neq 0] \,\& \\
& \quad (i)(j)(k)[\varphi(i, j) = 0 \,\&\, \varphi(j, k) = 0 \to \varphi(i, k) = 0] \,\& \\
& \quad \overline{(E\psi)}(i)\big[\varphi\big(\psi(i+1), \psi(i)\big) = 0\big] \,\& \\
& \quad (i)(j)\Big[\big[\varphi_j \in \mathfrak{W}(J_0) \to [\varepsilon(i, j) = 0 \equiv \mathrm{F}`\varphi_i \in \mathfrak{W}(\mathrm{F}\!\upharpoonright\!\varphi_j)]\big] \,\& \\
& \qquad \big[\varphi_j \in \mathfrak{W}(\mathrm{J}_1) \to [\varepsilon(i, j) = 0 \equiv \mathrm{F}`\varphi_i \in \{\mathrm{F}`\mathrm{K}_1`\varphi_j \mathrm{F}`\mathrm{K}_2`\varphi_j\}\ ]\big] \,\& \\
& \qquad \big[\varphi_j \in \mathfrak{W}(\mathrm{J}_2) \to [\varepsilon(i, j) = 0 \equiv \mathrm{F}`\varphi_i \in \mathrm{E} \cdot \mathrm{F}`\mathrm{K}_1`\varphi_j]\big] \,\& \\
& \qquad \big[\varphi_j \in \mathfrak{W}(\mathrm{J}_3) \to [\varepsilon(i, j) = 0 \equiv \mathrm{F}`\varphi_i \in \mathrm{F}`\mathrm{K}_1`\varphi_j - \mathrm{F}`\mathrm{K}_2`\varphi_j]\big] \,\& \\
& \qquad \big[\varphi_j \in \mathfrak{W}(\mathrm{J}_4) \to [\varepsilon(i, j) = 0 \equiv \mathrm{F}`\varphi_i \in \mathrm{F}`\mathrm{K}_1`\varphi_j \!\upharpoonright\! \mathrm{F}`\mathrm{K}_2`\varphi_j]\big] \,\& \\
& \qquad \big[\varphi_j \in \mathfrak{W}(\mathrm{J}_5) \to [\varepsilon(i, j) = 0 \equiv \mathrm{F}`\varphi_i \in \mathrm{F}`\mathrm{K}_1`\varphi_j \cdot \mathfrak{D}(\mathrm{F}`\mathrm{K}_2`\varphi_j)]\big] \,\& \\
& \qquad \big[\varphi_j \in \mathfrak{W}(\mathrm{J}_6) \to [\varepsilon(i, j) = 0 \equiv \mathrm{F}`\varphi_i \in \mathrm{F}`\mathrm{K}_1`\varphi_j \cdot (\mathrm{F}`\mathrm{K}_2`\varphi_j)^{-1}]\big] \,\& \\
& \qquad \big[\varphi_j \in \mathfrak{W}(\mathrm{J}_7) \to [\varepsilon(i, j) = 0 \equiv \mathrm{F}`\varphi_i \in \mathrm{F}`\mathrm{K}_1`\varphi_j \cdot \mathfrak{Cnv}_2(\mathrm{F}`\mathrm{K}_2`\varphi_j)]\big] \,\& \\
& \qquad \big[\varphi_j \in \mathfrak{W}(\mathrm{J}_8) \to [\varepsilon(i, j) = 0 \equiv \mathrm{F}`\varphi_i \in \mathrm{F}`\mathrm{K}_1`\varphi_j \cdot \mathfrak{Cnv}_3(\mathrm{F}`\mathrm{K}_2`\varphi_j)]\big]\Big].
\end{aligned}
$$

The proof of the lemma is now completed by showing that certain parts of the right member of equivalence (B) can be changed in such a way that it will follow by straightforward techniques of [13] from the resulting equivalence that $M \in \Pi_1^1$. That these changes can be made is certified by Lemmas C0-C8, D0-D8.

LEMMA Ch ($0 \leqslant h \leqslant 8$). *There exists a $\Sigma_1^1 \cap \Pi_1^1$ predicate C_h such that*

$$W(\varphi) \to [\varphi_j \in \mathfrak{W}(\mathrm{J}_h) \equiv C_h(\varphi, j)].$$

Proof. The predicate C_h is defined by (A.6) and the following equivalences:

$$
\begin{aligned}
&(\mathrm{C}.0)\quad Le^{\varphi}(a, b, c, d) && \equiv \varphi(b, d) = 0 \vee [b = d \,\&\, \varphi(a, c) = 0], \\
&(\mathrm{C}.1)\quad Max^{\varphi}(a, b, c) && \equiv [\varphi(a, c) = 0 \vee a = c] \,\&\, [\varphi(b, c) = 0 \vee b = c] \\
& && \quad \&\, [a = c \vee b = c], \\
&(\mathrm{C}.2)\quad R^{\varphi}(a, b, c, d) && \equiv (Ee)(Ef)\big[Max^{\varphi}(a, b, e) \,\&\, Max^{\varphi}(c, d, f) \,\& \\
& && \quad [\varphi(e, f) = 0 \vee [e = f \,\&\, Le^{\varphi}(a, b, c, d)]]\big], \\
&(\mathrm{C}.3)\quad N^{\varphi}(m) && \equiv (El)[l < 9 \,\&\, l = \varphi_m], \\
&(\mathrm{C}.4)\quad S^{\varphi}(m, a, b, n, c, d) && \equiv N^{\varphi}(m) \,\&\, N^{\varphi}(n) \,\&\, \big[R^{\varphi}(a, b, c, d) \\
& && \quad \vee [a = c \,\&\, b = d \,\&\, \varphi(m, n) = 0]\big], \\
&(\mathrm{C}.5)\quad Tr^{\varphi}(\alpha) && \equiv (v)(En)(Ec)(Ed)[N^{\varphi}(n) \,\&\, \alpha(n, c, d) = v] \\
& && \quad \&\, (n)(c)(d)(p)(e)(f)\big[S^{\varphi}(n, c, d, p, e, f) \to \\
& && \quad \varphi\big(\alpha(n, c, d), \alpha(p, e, f)\big) = 0\big], \\
&(\mathrm{C}.6)\quad J^{\varphi}(m, a, b, u) && \equiv (E\alpha)[Tr^{\varphi}(\alpha) \,\&\, \alpha(m, a, b) = u] \\
& && \equiv (\alpha)[Tr^{\varphi}(\alpha) \to \alpha(m, a, b) = u], \\
&(\mathrm{C}.7)\quad J^{\varphi}(a, b, u) && \equiv (El)[l = \varphi_n \,\&\, J^{\varphi}(l, a, b, u)], \\
&(\mathrm{C}.8)\quad C_h(\varphi, j) && \equiv (Ea)(Eb)J^{\varphi}(a, b, j).
\end{aligned}
$$

It follows straightforwardly by the techniques of [13] that the predicate C_h defined by (A.6), (C.0)-(C.8) is in $\Sigma_1^1 \cap \Pi_1^1$. From a comparison of the definition of $\lambda\varphi j\ \varphi_j \in \mathfrak{W}(J_h)$ in [9] and the definition of C_h by (A.6), (C.0)-(C.8) it should be clear that they coincide if $W(\varphi)$.

LEMMA Dh ($0 \leqslant h \leqslant 8$). *There exists a $\Sigma_1^1 \cap \Pi_1^1$ predicate D_h such that the extension of M is not changed if the expression Δ_h in* (B) *is replaced by "$D_h(\varphi, \varepsilon, i, j)$", where*

Δ_0 is $\mathrm{F}`\varphi_i \in \mathfrak{W}(\mathrm{F}\!\upharpoonright\!\varphi_j)$,
Δ_1 is $\mathrm{F}`\varphi_i \in \{\mathrm{F}`\mathrm{K}_1`\varphi_j \mathrm{F}`\mathrm{K}_2`\varphi_j\}$,
Δ_2 is $\mathrm{F}`\varphi_i \in \mathrm{E} \cdot \mathrm{F}`\mathrm{K}_1`\varphi_j$,
Δ_3 is $\mathrm{F}`\varphi_i \in \mathrm{F}`\mathrm{K}_1`\varphi_j - \mathrm{F}`\mathrm{K}_2`\varphi_j$,
Δ_4 is $\mathrm{F}`\varphi_i \in \mathrm{F}`\mathrm{K}_1`\varphi_j \!\upharpoonright\! \mathrm{F}`\mathrm{K}_2`\varphi_j$,
Δ_5 is $\mathrm{F}`\varphi_i \in \mathrm{F}`\mathrm{K}_1`\varphi_j \cdot \mathfrak{D}(\mathrm{F}`\mathrm{K}_2`\varphi_j)$,
Δ_6 is $\mathrm{F}`\varphi_i \in \mathrm{F}`\mathrm{K}_1`\varphi_j \cdot (\mathrm{F}`\mathrm{K}_2`\varphi_j)^{-1}$,
Δ_7 is $\mathrm{F}`\varphi_i \in \mathrm{F}`\mathrm{K}_1`\varphi_j \cdot \mathfrak{Cnv}_2(\mathrm{F}`\mathrm{K}_2`\varphi_j)$,
Δ_8 is $\mathrm{F}`\varphi_i \in \mathrm{F}`\mathrm{K}_1`\varphi_j \cdot \mathfrak{Cnv}_3(\mathrm{F}`\mathrm{K}_2`\varphi_j)$.

For the proof of these lemmas we need two more predicates, defined as follows:

(D.0) $K_1^\varphi(c, a) \equiv (Em)(Eb)[N^\varphi(m) \,\&\, J^\varphi(m, a, b, c)]$,

(D.1) $K_2^\varphi(c, b) \equiv (Em)(Ea)[N^\varphi(m) \,\&\, J^\varphi(m, a, b, c)]$.

Now if $W(\varphi)$, then $\hat{a}\hat{b}\hat{c}\, Max^\varphi(a, b, c)$, $\hat{c}\hat{a} K_1^\varphi(c, a)$, and $\hat{c}\hat{b} K_2^\varphi(c, b)$ are graphs of functions, which we denote by "max^φ", "$\varkappa_1^\varphi$", and "$\varkappa_2^\varphi$", respectively.

For greater clarity we assume $W(\varphi)$ in the discussions of Lemmas D0-D8 and hence freely use the functions max^φ, $\varkappa_1^\varphi$, and $\varkappa_2^\varphi$. It should be clear that these can be eliminated in obtaining the D_h (which are to be everywhere defined) by use of the following:

(D.2) $P\big(\varkappa_1^\varphi(j)\big) \equiv (Eu)[K_1^\varphi(j, u) \,\&\, P(u)]$,

(D.3) $P\big(\varkappa_2^\varphi(j)\big) \equiv (Eu)[K_2^\varphi(j, u) \,\&\, P(u)]$,

(D.4) $P\big(max^\varphi(a, b)\big) \equiv (Eu)[Max^\varphi(a, b, u) \,\&\, P(u)]$,

which hold for any predicate P if $W(\varphi)$. And of course the values of D_h do not matter if $\overline{W}(\varphi)$.

The functions $\varkappa_1^\varphi$, $\varkappa_2^\varphi$ have the properties

(D.5) $\varphi_{\varkappa_1^\varphi(c)} = \mathrm{K}_1\text{‘}\varphi_c$,

(D.6) $\varphi_{\varkappa_2^\varphi(c)} = \mathrm{K}_2\text{‘}\varphi_c$,

which we use frequently in the proofs. We also use without further citation the following

(D.7) $\varphi_j \notin \mathfrak{W}(\mathrm{J}_0) \to \varphi\big(\varkappa_1^\varphi(j), j\big) = 0$,

(D.8) $\varphi_j \notin \mathfrak{W}(\mathrm{J}_0) \to \varphi\big(\varkappa_2^\varphi(j), j\big) = 0$,

which follow from [9], 9.25. We shall also use [9], 9.52, mentioned in Section 2, many times in the proofs without further citation.

We give the proof of each of the Lemmas D0-D8 by a series of equivalences, which are understood to hold under the assumptions that are guaranteed for the purposes of the lemma by the form of (B). Thus in the proof of Lemma D0, for example, $W(\varphi) \,\&\, \varphi_j \in \mathfrak{W}(\mathrm{J}_0)$ is assumed.

Whenever a constituent of one of the formulas is replaced by an expression of the form "$\varepsilon(a, b) = 0$", it is necessary to verify that in some fixed well-ordering $<^\varphi$ of N^N relative to φ $(a, b) <^\varphi (i, j)$. This is necessary, of course, because $\lambda ij\, \varepsilon(i, j) = 0$ is being defined by transfinite induction and if $(i, j) \leqslant^\varphi (a, b)$ we would be defining $\varepsilon(i, j) = 0$ in terms of itself or values coming later in the induction. The well-ordering $<^\varphi$ we use for this purpose is defined as follows:

(D.9) $(a, b) <^\varphi (i, j) \equiv R^\varphi(b, a, j, i)$.

As a convenience to the printer we abbreviate "$\varphi(a, b) = 0$" to "$a <^\varphi b$" in the following proofs. Further we omit throughout all superscripts "φ": the reader should imagine that they are there. (Since "$<$" is never used in its ordinary meaning in these proofs, no confusion should result.)

Proof of Lemma D0.

$$\begin{aligned}
(\text{D0.0})\quad \mathrm{F}\text{‘}\varphi_i \in \mathfrak{W}(\mathrm{F}\text{‘}\varphi_j) &\equiv \mathrm{F}\text{‘}\varphi_i \in \mathfrak{D}\big((\mathrm{F}\restriction\varphi_j)^{-1}\big) \\
&\equiv (EX)\langle X\,\mathrm{F}\text{‘}\varphi_i\rangle \in (\mathrm{F}\restriction\varphi_j)^{-1} \\
&\equiv (EX)\langle \mathrm{F}\text{‘}\varphi_i X\rangle \in \mathrm{F}\restriction\varphi_j \\
&\equiv (EX)\langle \mathrm{F}\text{‘}\varphi_i X\rangle \in \mathrm{F}\cdot(\mathrm{V}\times\varphi_j) \\
&\equiv (EX)[\langle \mathrm{F}\text{‘}\varphi_i X\rangle \in \mathrm{F} \,\&\, \langle \mathrm{F}\text{‘}\varphi_i X\rangle \in \mathrm{V}\times\varphi_j] \\
&\equiv (EX)[\langle \mathrm{F}\text{‘}\varphi_i X\rangle \in \mathrm{F} \,\&\, \mathrm{F}\text{‘}\varphi_i \in \mathrm{V} \,\&\, X \in \varphi_j] \\
&\equiv (Ek)_{k<j}\langle \mathrm{F}\text{‘}\varphi_i \varphi_k\rangle \in \mathrm{F} \\
&\equiv (Ek)_{k<j}\underbrace{\mathrm{F}\text{‘}\varphi_k = \mathrm{F}\text{‘}\varphi_i}_{1}.
\end{aligned}$$

$$(\text{D0.1})\quad \mathrm{F}\text{‘}\varphi_k = \mathrm{F}\text{‘}\varphi_i \equiv (l)_{l<max(i,k)}[\underbrace{\varepsilon(l, k) = 0}_{\alpha} \equiv \underbrace{\varepsilon(l, i) = 0}_{\beta}].$$

(D0.α) $(l, k) < (i, j)$. $l < max(i, k) \leqslant max(i, j)$ and $k < j \leqslant max(i, j)$, so $max(l, k) < max(i, j)$.

(D0.β) $(l, i) < (i, j)$. Case I. $i \geqslant k$: Then $l < max(i, k) = i \leqslant max(i, j)$, so $max(l, i) \leqslant max(i, j)$ and $l < i$.
Case II. $i < k$: $l < max(i, k) \leqslant max(i, j)$ and $i < k < j \leqslant max(i, j)$, so $max(l, i) < max(i, j)$.

Proof of Lemma D1.

$$\begin{aligned}
(\text{D1.0})\quad \mathrm{F}\text{‘}\varphi_i \in \{\mathrm{F}\text{‘}\mathrm{K}_1\text{‘}\varphi_j\, \mathrm{F}\text{‘}\mathrm{K}_2\text{‘}\varphi_j\} &\equiv \mathrm{F}\text{‘}\varphi_i = \mathrm{F}\text{‘}\mathrm{K}_1\text{‘}\varphi_j \vee \mathrm{F}\text{‘}\varphi_i = \mathrm{F}\text{‘}\mathrm{K}_2\text{‘}\varphi_j \\
&\equiv \underbrace{\mathrm{F}\text{‘}\varphi_{\varkappa_1(j)} = \mathrm{F}\text{‘}\varphi_i}_{1} \vee \underbrace{\mathrm{F}\text{‘}\varphi_{\varkappa_2(j)} = \mathrm{F}\text{‘}\varphi_i}_{2}.
\end{aligned}$$

(D1.1) $\mathrm{F}\text{‘}\varphi_{\varkappa_1(j)} = \mathrm{F}\text{‘}\varphi_i$. Since $\varkappa_1(j) < j$, this is just like (D0.1).

(D1.2) $\mathrm{F}\text{‘}\varphi_{\varkappa_2(j)} = \mathrm{F}\text{‘}\varphi_i$. Since $\varkappa_2(j) < j$, this is just like (D0.1).

Proof of Lemma D2.

$$\begin{aligned}
(\text{D2.0})\quad \mathrm{F}\text{‘}\varphi_i \in \mathrm{E}\cdot\mathrm{F}\text{‘}\mathrm{K}_1\text{‘}\varphi_j &\equiv \mathrm{F}\text{‘}\varphi_i \in \mathrm{E} \,\&\, \mathrm{F}\text{‘}\varphi_i \in \mathrm{F}\text{‘}\mathrm{K}_1\text{‘}\varphi_j \\
&\equiv \underbrace{\mathrm{F}\text{‘}\varphi_i \in \mathrm{E}}_{1} \,\&\, \underbrace{\varepsilon\big(i, \varkappa_1(j)\big)}_{\alpha} = 0.
\end{aligned}$$

$$\begin{aligned}
(\text{D2.1})\quad \mathrm{F}\text{‘}\varphi_i \in \mathrm{E} &\equiv (EX)(EY)[X \in Y \,\&\, \mathrm{F}\text{‘}\varphi_i = \langle XY\rangle] \\
&\equiv (Ek)(El)_{k,l<i}[\underbrace{\varepsilon(k, l) = 0}_{\beta} \,\&\, \underbrace{\mathrm{F}\text{‘}\varphi_i = \langle \mathrm{F}\text{‘}\varphi_k \mathrm{F}\text{‘}\varphi_l\rangle}_{2}].
\end{aligned}$$

$$\begin{aligned}\text{(D2.2)}\quad \mathrm{F}`\varphi_i = \langle \mathrm{F}`\varphi_k \mathrm{F}`\varphi_l\rangle &\equiv \mathrm{F}`\varphi_i = \{\{\mathrm{F}`\varphi_k\}\{\mathrm{F}`\varphi_k \mathrm{F}`\varphi_l\}\}\\ &\equiv (Em)_{m<i}[\underset{3}{\underline{\mathrm{F}`\varphi_m = \{\mathrm{F}`\varphi_k\}}} \,\&\, \underset{\gamma}{\underline{\varepsilon(m,i)=0}}]\\ &\quad \& (Em)_{m<i}[\underset{4}{\underline{\mathrm{F}`\varphi_m = \{\mathrm{F}`\varphi_k \mathrm{F}`\varphi_l\}}} \,\&\, \underset{\gamma}{\underline{\varepsilon(m,i)=0}}]\\ &\quad \& (m)_{m<i}[\underset{\gamma}{\underline{\varepsilon(m,i)=0}} \equiv \underset{3}{\underline{\mathrm{F}`\varphi_m = \{\mathrm{F}`\varphi_k\}}}\\ &\quad \vee \underset{4}{\underline{\mathrm{F}`\varphi_m = \{\mathrm{F}`\varphi_k \mathrm{F}`\varphi_l\}}}].\end{aligned}$$

$$\text{(D2.3)}\quad \mathrm{F}`\varphi_m = \{\mathrm{F}`\varphi_k\} \equiv \underset{\beta}{\underline{\varepsilon(k,m)=0}} \,\&\, (n)_{n<m}[\underset{\beta}{\underline{\varepsilon(n,m)=0}} \equiv \underset{5}{\underline{\mathrm{F}`\varphi_n = \mathrm{F}`\varphi_k}}].$$

$$\begin{aligned}\text{(D2.4)}\quad \mathrm{F}`\varphi_m = \{\mathrm{F}`\varphi_k \mathrm{F}`\varphi_l\} &\equiv \underset{\beta}{\underline{\varepsilon(k,m)=0}} \,\&\, \underset{\beta}{\underline{\varepsilon(l,m)=0}}\\ &\quad \& (n)_{n<m}[\underset{\beta}{\underline{\varepsilon(n,m)=0}} \equiv \underset{5}{\underline{\mathrm{F}`\varphi_n = \mathrm{F}`\varphi_k}}\\ &\quad \vee \underset{5}{\underline{\mathrm{F}`\varphi_n = \mathrm{F}`\varphi_l}}].\end{aligned}$$

$$\text{(D2.5)}\quad \mathrm{F}`\varphi_n = \mathrm{F}`\varphi_k \equiv (p)_{p<max(n,k)}[\underset{\beta}{\underline{\varepsilon(p,n)=0}} \equiv \underset{\beta}{\underline{\varepsilon(p,k)=0}}].$$

(D2.α) $(i,\varkappa_1(j)) < (i,j)$. $max(i,\varkappa_1(j)) \leqslant max(i,j)$, $i=i$, and $\varkappa_1(j) < j$.

(D2.β) $(k,l) < (i,j)$. $max(k,l) < i \leqslant max(i,j)$, so $max(k,l) < max(i,j)$.

(D2.γ) $(m,i) < (i,j)$. $max(m,i) = i \leqslant max(i,j)$ and $m < i$.

Proof of Lemma D3.

$$\begin{aligned}\text{(D3.0)}\quad \mathrm{F}`\varphi_i \in \mathrm{F}`\mathrm{K}_1`\varphi_j - \mathrm{F}`\mathrm{K}_2`\varphi_j &\equiv \mathrm{F}`\varphi_i \in \mathrm{F}`\mathrm{K}_1`\varphi_j \,\&\, \mathrm{F}`\varphi_i \notin \mathrm{F}`\mathrm{K}_2`\varphi_j\\ &\equiv \underset{(D2.\alpha)}{\underline{\varepsilon(i,\varkappa_1(j))=0}} \,\&\, \underset{(D2.\alpha)}{\underline{\varepsilon(i,\varkappa_2(j))\neq 0}}.\end{aligned}$$

Proof of Lemma D4.

$$\begin{aligned}\text{(D4.0)}\quad \mathrm{F}`\varphi_i \in \mathrm{F}`\mathrm{K}_1`\varphi_j \upharpoonright \mathrm{F}`\mathrm{K}_2`\varphi_j &\equiv \mathrm{F}`\varphi_i \in \mathrm{F}`\mathrm{K}_1`\varphi_j \cdot (\mathrm{V}\times \mathrm{F}`\mathrm{K}_2`\varphi_j)\\ &\equiv \underset{(D2.\alpha)}{\underline{\varepsilon(i,\varkappa_1(j))=0}} \,\&\, \underset{1}{\underline{\mathrm{F}`\varphi_i \in \mathrm{V}\times \mathrm{F}`\mathrm{K}_2`\varphi_j}}.\end{aligned}$$

$$\begin{aligned}\text{(D4.1)}\quad \mathrm{F}`\varphi_i \in \mathrm{V}\times \mathrm{F}`\mathrm{K}_2`\varphi_j &\equiv (EX)(EY)[\mathrm{F}`\varphi_i = \langle XY\rangle \,\&\, Y \in \mathrm{F}`\mathrm{K}_2`\varphi_j]\\ &\equiv (Ek)(El)_{k,l<i}[\underset{(D2.2)}{\underline{\mathrm{F}`\varphi_i = \langle \mathrm{F}`\varphi_k \mathrm{F}`\varphi_l\rangle}}\\ &\quad \& \underset{\alpha}{\underline{\varepsilon(l,\varkappa_2(j))=0}}].\end{aligned}$$

(D4.α) $(l,\varkappa_2(j)) < (i,j)$. $max(l,\varkappa_2(j)) \leqslant max(i,j)$ and $l < i$.

Proof of Lemma D5.

$$\text{(D5.0)}\quad \mathrm{F}`\varphi_i \in \mathrm{F}`\mathrm{K}_1`\varphi_j \cdot \mathfrak{D}(\mathrm{F}`\mathrm{K}_2`\varphi_j) \equiv \underset{(D2.\alpha)}{\underline{\varepsilon(i,\varkappa_1(j))=0}} \,\&\, \underset{1}{\underline{\mathrm{F}`\varphi_i \in \mathfrak{D}(\mathrm{F}`\mathrm{K}_2`\varphi_j)}}.$$

$$\begin{aligned}\text{(D5.1)}\quad \mathrm{F}`\varphi_i \in \mathfrak{D}(\mathrm{F}`\mathrm{K}_2`\varphi_j) &\equiv (EX)[\langle X\mathrm{F}`\varphi_i\rangle \in \mathrm{F}`\mathrm{K}_2`\varphi_j]\\ &\equiv (Ek)_{k<\varkappa_2(j)}\langle \mathrm{F}`\varphi_k \mathrm{F}`\varphi_i\rangle \in \mathrm{F}`\mathrm{K}_2`\varphi_j\\ &\equiv (Ek)(El)_{k,l<\varkappa_2(j)}[\underset{2}{\underline{\mathrm{F}`\varphi_l = \langle \mathrm{F}`\varphi_k \mathrm{F}`\varphi_i\rangle}}\\ &\quad \& \underset{\alpha}{\underline{\varepsilon(l,\varkappa_2(j))=0}}].\end{aligned}$$

$$\begin{aligned}\text{(D5.2)}\quad \mathrm{F}`\varphi_l = \langle \mathrm{F}`\varphi_k \mathrm{F}`\varphi_i\rangle &\equiv \mathrm{F}`\varphi_l = \{\{\mathrm{F}`\varphi_k\}\{\mathrm{F}`\varphi_k \mathrm{F}`\varphi_i\}\}\\ &\equiv (Em)_{m<l}[\underset{3}{\underline{\mathrm{F}`\varphi_m = \{\mathrm{F}`\varphi_k\}}} \,\&\, \underset{\alpha}{\underline{\varepsilon(m,l)=0}}]\\ &\quad \& (Em)_{m<l}[\underset{4}{\underline{\mathrm{F}`\varphi_m = \{\mathrm{F}`\varphi_k \mathrm{F}`\varphi_i\}}} \,\&\, \underset{\alpha}{\underline{\varepsilon(m,l)=0}}]\\ &\quad \& (m)_{m<l}[\underset{\alpha}{\underline{\varepsilon(m,l)=0}} \equiv \underset{3}{\underline{\mathrm{F}`\varphi_m = \{\mathrm{F}`\varphi_k\}}}\\ &\quad \vee \underset{4}{\underline{\mathrm{F}`\varphi_m = \{\mathrm{F}`\varphi_k \mathrm{F}`\varphi_i\}}}].\end{aligned}$$

$$\text{(D5.3)}\quad \mathrm{F}`\varphi_m = \{\mathrm{F}`\varphi_k\} \equiv \underset{\alpha}{\underline{\varepsilon(k,m)=0}} \,\&\, (n)_{n<m}[\underset{\alpha}{\underline{\varepsilon(n,m)=0}} \equiv \underset{5}{\underline{\mathrm{F}`\varphi_n = \mathrm{F}`\varphi_k}}].$$

$$\begin{aligned}\text{(D5.4)}\quad \mathrm{F}`\varphi_m = \{\mathrm{F}`\varphi_k \mathrm{F}`\varphi_i\} &\equiv \underset{\alpha}{\underline{\varepsilon(k,m)=0}} \,\&\, \underset{(D2.\alpha)}{\underline{\varepsilon(i,m)=0}} \,\&\, (n)_{n<m}[\underset{\alpha}{\underline{\varepsilon(n,m)=0}}\\ &\equiv \underset{5}{\underline{\mathrm{F}`\varphi_n = \mathrm{F}`\varphi_k}} \vee \underset{(D1.2)}{\underline{\mathrm{F}`\varphi_n = \mathrm{F}`\varphi_i}}].\end{aligned}$$

$$\text{(D5.5)}\quad \mathrm{F}`\varphi_n = \mathrm{F}`\varphi_k \equiv (p)_{p<max(n,k)}[\underset{\alpha}{\underline{\varepsilon(p,n)=0}} \equiv \underset{\alpha}{\underline{\varepsilon(p,k)=0}}].$$

(D5.α) $(l,\varkappa_2(j)) < (i,j)$. $max(l,\varkappa_2(j)) = \varkappa_2(j) < j \leqslant max(i,j)$.

Proof of Lemma D6.

$$\text{(D6.0)}\quad \mathrm{F}`\varphi_i \in \mathrm{F}`\mathrm{K}_1`\varphi_j \cdot (\mathrm{F}`\mathrm{K}_2`\varphi_j)^{-1} \equiv \underset{(D2.\alpha)}{\underline{\varepsilon(i,\varkappa_1(j))=0}} \,\&\, \underset{1}{\underline{\mathrm{F}`\varphi_i \in (\mathrm{F}`\mathrm{K}_2`\varphi_j)^{-1}}}.$$

$$\begin{aligned}\text{(D6.1)}\quad \mathrm{F}`\varphi_i \in (\mathrm{F}`\mathrm{K}_2`\varphi_j)^{-1} &\equiv (EX)(EY)[\langle XY\rangle = \mathrm{F}`\varphi_i \,\&\, \langle YX\rangle \in \mathrm{F}`\mathrm{K}_2`\varphi_j]\\ &\equiv (Ek)(El)_{k,l<i,\varkappa_2(j)}[\underset{(D2.2)}{\underline{\mathrm{F}`\varphi_i = \langle \mathrm{F}`\varphi_k \mathrm{F}`\varphi_l\rangle}}\\ &\quad \& \underset{2}{\underline{\langle \mathrm{F}`\varphi_l \mathrm{F}`\varphi_k\rangle \in \mathrm{F}`\mathrm{K}_2`\varphi_j}}].\end{aligned}$$

$$\begin{aligned}\text{(D6.2)}\quad \langle \mathrm{F}`\varphi_l \mathrm{F}`\varphi_k\rangle \in \mathrm{F}`\mathrm{K}_2`\varphi_j &\equiv (Em)_{m<\varkappa_2(j)}[\underset{(D5.\alpha)}{\underline{\varepsilon(m,\varkappa_2(j))=0}}\\ &\quad \& \underset{(D5.2)}{\underline{\mathrm{F}`\varphi_m = \langle \mathrm{F}`\varphi_l \mathrm{F}`\varphi_k\rangle}}].\end{aligned}$$

Proof of Lemma D7.

$$\begin{aligned}\text{(D7.0)}\quad \mathrm{F}`\varphi_i \in \mathrm{F}`\mathrm{K}_1`\varphi_j \cdot \mathfrak{Cnv}_2(\mathrm{F}`\mathrm{K}_2`\varphi_j) &\equiv \underset{(D2.\alpha)}{\underline{\varepsilon(i,\varkappa_1(j))=0}}\\ &\quad \& \underset{1}{\underline{\mathrm{F}`\varphi_i \in \mathfrak{Cnv}_2(\mathrm{F}`\mathrm{K}_2`\varphi_j)}}.\end{aligned}$$

$$
\begin{aligned}
\text{(D7.1)}\ \ \text{F‘}\varphi_i \in \mathfrak{Cnv}_2(\text{F‘K}_2\text{‘}\varphi_j) &\equiv (EX)(EY)(EZ)[\text{F‘}\varphi_i = \langle XYZ\rangle \\
&\quad \& \langle YZX\rangle \in \text{F‘K}_2\text{‘}\varphi_j] \\
&\equiv (Ek)(El)(Em)_{k,l,m<i,\varkappa_2(j)}[\underbrace{\text{F‘}\varphi_i = \langle \text{F‘}\varphi_k\text{F‘}\varphi_l\text{F‘}\varphi_m\rangle}_{2} \\
&\quad \& \underbrace{\langle \text{F‘}\varphi_l\text{F‘}\varphi_m\text{F‘}\varphi_k\rangle \in \text{F‘K}_2\text{‘}\varphi_j}_{3}].
\end{aligned}
$$

$$
\begin{aligned}
\text{(D7.2)}\ \ \text{F‘}\varphi_i = \langle \text{F‘}\varphi_k\text{F‘}\varphi_l\text{F‘}\varphi_m\rangle &\equiv \text{F‘}\varphi_i = \langle \text{F‘}\varphi_k\langle \text{F‘}\varphi_l\text{F‘}\varphi_m\rangle\rangle \\
&\equiv \text{F‘}\varphi_i = \big\{\{\text{F‘}\varphi_k\}\{\text{F‘}\varphi_k\langle \text{F‘}\varphi_l\text{F‘}\varphi_m\rangle\}\big\} \\
&\equiv (En)_{n<i}[\underbrace{\text{F‘}\varphi_n = \{\text{F‘}\varphi_k\}}_{\text{(D2.3)}} \& \underbrace{\varepsilon(n, i) = 0}_{\text{(D2.}\gamma)}] \\
&\quad \& (En)_{n<i}[\underbrace{\text{F‘}\varphi_n = \{\text{F‘}\varphi_k\langle \text{F‘}\varphi_l\text{F‘}\varphi_m\rangle\}}_{4} \\
&\quad \& \underbrace{\varepsilon(n, i)\ 0}_{\text{(D2.}\gamma)}] \\
&\quad \& (n)_{n<i}[\underbrace{\varepsilon(n, i) = 0}_{\text{(D2.}\gamma)} \equiv \underbrace{\text{F‘}\varphi_n = \{\text{F‘}\varphi_k\}}_{\text{(D2.3)}} \\
&\quad \vee \underbrace{\text{F‘}\varphi_n = \{\text{F‘}\varphi_k\langle \text{F‘}\varphi_l\text{F‘}\varphi_m\rangle\}}_{4}].
\end{aligned}
$$

$$
\begin{aligned}
\text{(D7.3)}\ \ \langle \text{F‘}\varphi_l\text{F‘}\varphi_m\text{F‘}\varphi_k\rangle \in \text{F‘K}_2\text{‘}\varphi_j &\equiv (Ep)_{p<\varkappa_2(j)}[\underbrace{\text{F‘}\varphi_p = \langle \text{F‘}\varphi_l\text{F‘}\varphi_m\text{F‘}\varphi_k\rangle}_{5} \\
&\quad \& \underbrace{\varepsilon\big(p, \varkappa_2(j)\big) = 0}_{\text{(D5.}\alpha)}].
\end{aligned}
$$

$$
\begin{aligned}
\text{(D7.4)}\ \ \text{F‘}\varphi_n = \{\text{F‘}\varphi_k\langle \text{F‘}\varphi_l\text{F‘}\varphi_m\rangle\} &\equiv \underbrace{\varepsilon(k, n) = 0}_{\text{(D2.}\beta)} \& \underbrace{\langle \text{F‘}\varphi_l\text{F‘}\varphi_m\rangle \in \text{F‘}\varphi_n}_{6} \\
&\quad \& (q)_{q<n}[\underbrace{\varepsilon(q, n) = 0}_{\text{(D2.}\beta)} \equiv \underbrace{\text{F‘}\varphi_q = \text{F‘}\varphi_k}_{\text{(D2.5)}} \\
&\quad \vee \underbrace{\text{F‘}\varphi_q = \langle \text{F‘}\varphi_l\text{F‘}\varphi_m\rangle}_{\text{(D2.2)}}].
\end{aligned}
$$

$$
\begin{aligned}
\text{(D7.5)}\ \ \text{F‘}\varphi_p = \langle \text{F‘}\varphi_l\text{F‘}\varphi_m\text{F‘}\varphi_k\rangle &\equiv (Er)_{r<p}[\underbrace{\text{F‘}\varphi_r = \text{F‘}\varphi_l}_{\text{(D5.3)}} \& \underbrace{\varepsilon(r, p) = 0}_{\text{(D5.}\alpha)}] \\
&\quad \& (Er)_{r<p}[\underbrace{\text{F‘}\varphi_r = \{\text{F‘}\varphi_l\langle \text{F‘}\varphi_m\text{F‘}\varphi_k\rangle\}}_{7} \\
&\quad \& \underbrace{\varepsilon(r, p) = 0}_{\text{(D5.}\alpha)}] \\
&\quad \& (r)_{r<p}[\underbrace{\varepsilon(r, p) = 0}_{\text{(D5.}\alpha)} \equiv \underbrace{\text{F‘}\varphi_r = \{\text{F‘}\varphi_l\}}_{\text{(D5.3)}} \\
&\quad \vee \underbrace{\text{F‘}\varphi_r = \{\text{F‘}\varphi_l\langle \text{F‘}\varphi_m\text{F‘}\varphi_k\rangle\}}_{7}].
\end{aligned}
$$

$$
\text{(D7.6)}\ \ \langle \text{F‘}\varphi_l\text{F‘}\varphi_m\rangle \in \text{F‘}\varphi_n \equiv (Es)_{s<n}[\underbrace{\varepsilon(s, n) = 0}_{\text{(D2.}\beta)} \& \underbrace{\text{F‘}\varphi_s = \langle \text{F‘}\varphi_l\text{F‘}\varphi_m\rangle}_{\text{(D2.2)}}].
$$

$$
\begin{aligned}
\text{(D7.7)}\ \ \text{F‘}\varphi_r = \{\text{F‘}\varphi_l\langle \text{F‘}\varphi_m\text{F‘}\varphi_k\rangle\} &\equiv \underbrace{\varepsilon(l, r) = 0}_{\text{(D5.}\alpha)} \& \underbrace{\langle \text{F‘}\varphi_m\text{F‘}\varphi_k\rangle \in \text{F‘}\varphi_r}_{\text{(D6.2)}} \\
&\quad \& (t)_{t<r}[\underbrace{\varepsilon(t, r) = 0}_{\text{(D5.}\alpha)} \equiv \underbrace{\text{F‘}\varphi_t = \text{F‘}\varphi_l}_{\text{(D5.5)}} \\
&\quad \vee \underbrace{\text{F‘}\varphi_t = \langle \text{F‘}\varphi_m\text{F‘}\varphi_k\rangle}_{\text{(D5.2)}}].
\end{aligned}
$$

Proof of Lemma D8.

$$
\text{(D8.0)}\ \ \text{F‘}\varphi_i \in \text{F‘K}_1\text{‘}\varphi_j \cdot \mathfrak{Cnv}_3(\text{F‘K}_2\text{‘}\varphi_j) \equiv \underbrace{\varepsilon\big(i, \varkappa_1(j)\big) = 0}_{\text{(D2.}\alpha)} \& \underbrace{\text{F‘}\varphi_i \in \mathfrak{Cnv}_3(\text{F‘K}_2\text{‘}\varphi_j)}_{1}.
$$

$$
\begin{aligned}
\text{(D8.1)}\ \ \text{F‘}\varphi_i \in \mathfrak{Cnv}_3(\text{F‘K}_2\text{‘}\varphi_j) &\equiv (EX)(EY)(EZ)[\text{F‘}\varphi_i = \langle XYZ\rangle \\
&\quad \& \langle XZY\rangle \in \text{F‘K}_2\text{‘}\varphi_j] \\
&\equiv (Ek)(El)(Em)_{k,l,m<i,\varkappa_2(j)}[\underbrace{\text{F‘}\varphi_i = \langle \text{F‘}\varphi_k\text{F‘}\varphi_l\text{F‘}\varphi_m\rangle}_{\text{(D7.2)}} \\
&\quad \& \underbrace{\langle \text{F‘}\varphi_k\text{F‘}\varphi_m\text{F‘}\varphi_l\rangle \in \text{F‘K}_2\text{‘}\varphi_j}_{\text{(D7.3)}}].
\end{aligned}
$$

4. The least function operator. Probably the most fundamental tool of hierarchy theory is the choice operator. More than anything else it is the nature of the choice operators available that sets the pattern into which the theorems of hierarchy theory fall. Insofar as the selection of an object from a set containing exactly one object is concerned, there is great uniformity among the hierarchies based on quantification of variables of different types. The cause of this is given in the following two fundamental equivalences which hold for the unique object operator ι, when it is defined, for all types t.

$$
(\mathrm{U}_1)\ \ \iota\alpha^t P(\mathfrak{a}, \alpha^t) = \varphi^t \equiv P(\mathfrak{a}, \varphi^t),
$$

$$
\begin{aligned}
(\mathrm{U}_2)\ \ \iota\alpha^{t+1} P(\mathfrak{a}, \alpha^{t+1})(\psi^t) = \chi^t &\equiv (E\varphi^{t+1})[P(\mathfrak{a}, \varphi^{t+1}) \& \varphi^{t+1}(\psi^t) = \chi^t] \\
&\equiv (\varphi^{t+1})[P(\mathfrak{a}, \varphi^{t+1}) \to \varphi^{t+1}(\psi^t) = \chi^t].
\end{aligned}
$$

These equivalences characterize the graphs of $\lambda\mathfrak{a}\iota\alpha^t P(\mathfrak{a}, \alpha^t)$ and its values, showing that $\lambda\mathfrak{a}\ \iota\alpha^t P(\mathfrak{a}, \alpha^t)$ is of the same category as P while its values are $\Sigma_1^t \cap \Pi_1^t$ in P. An example of the uniformity of structure induced by these equivalences is given by the analogous normal forms, $\lambda n U^0\big(\iota y T^0(e, n, y)\big)$ and $\lambda n U^1\Big(\iota\alpha(y)\, T^1\big(e, n, \bar{\alpha}(y)\big)\Big)$ with (particular or general) primitive recursive U^t, T^t for $\Sigma_1^0 \cap \Pi_1^0$ (recursive) and $\Sigma_1^1 \cap \Pi_1^1$ (hyperarithmetical) functions, respectively (cf. [11]).

But of even greater importance than a unique object operator is an operator that will select an object from a set containing an arbitrary number (>0) of objects. Such an operator is characteristically obtained by well-ordering the universe of objects by a predicate $<$. A least (relative to $<$) object operator μ is thus induced, for which the following fundamental equivalences hold:

$$
(\mathrm{L}_1)\ \ \mu\alpha^t P(\mathfrak{a}, \alpha^t) = \varphi^t \equiv P(\mathfrak{a}, \varphi^t) \& \overline{(E\alpha^t)}_{\alpha^t<\varphi^t} P(\mathfrak{a}, \alpha^t),
$$

$$
\begin{aligned}
(\mathrm{L}_2)\ \ \mu\alpha^{t+1} P(\mathfrak{a}, \alpha^{t+1})(\psi^t) = \chi^t &\equiv (E\varphi^{t+1})[P(\mathfrak{a}, \varphi^{t+1}) \& \overline{(E\alpha^{t+1})}_{\alpha^{t+1}<\varphi^{t+1}} P(\mathfrak{a}, \alpha^{t+1}) \\
&\quad \& \varphi^{t+1}(\psi^t) = \chi^t] \\
&\equiv (\varphi^{t+1})[P(\mathfrak{a}, \varphi^{t+1}) \& \overline{(E\alpha^{t+1})}_{\alpha^{t+1}<\varphi^{t+1}} P(\mathfrak{a}, \alpha^{t+1}) \\
&\quad \to \varphi^{t+1}(\psi^t) = \chi^t].
\end{aligned}
$$

These equivalences characterize the graphs of $\lambda\mathfrak{a}\mu\alpha^t P(\mathfrak{a}, \alpha^t)$ and its values. But here, unlike the case with ι, the complexity of the new factor $(E\alpha^t)_{\alpha^t<\varphi^t}P(\mathfrak{a}, \alpha^t)$, or its negation, must be evaluated. To a great extent this can be done by establishing the category of $<$ and using the following fundamental quantifier interchange devices (8):

$$(\mathrm{D}^0)\quad (Ey_1)_{y_1<y}(x)P(\mathfrak{a}, y_1, x) \equiv (x)(Ey_1)_{y_1<y}P\big(\mathfrak{a}, y_1, (x)_{y_1}\big)\,,$$

$$(\mathrm{D}^1)\quad (E\beta_1)_{\beta_1<\beta}(\alpha)P(\mathfrak{a}, \beta_1, \alpha) \equiv (\alpha)(E\beta_1)_{\beta_1<\beta}(i)P\big(\mathfrak{a}, \beta_1, \lambda n\alpha(p_i^n)\big)\,,$$

$$(\mathrm{D}^{t+2})\quad (E\beta_1^{t+2})_{\beta_1^{t+2}<\beta^{t+2}}(\alpha^{t+2})P(\mathfrak{a}, \beta_1^{t+2}, \alpha^{t+2})$$
$$\equiv (\alpha^{t+2})(E\beta_1^{t+2})_{\beta_1^{t+2}<\beta^{t+2}}(\varphi^{t+1})P\big(\mathfrak{a}, \beta_1^{t+2}, \lambda\psi^{t+1}\alpha^{t+2}(\lambda\chi^t p_{\varphi^{t+1}(\chi^t)}^{\psi^{t+1}(\chi^t)})\big)\,,$$

for any type t, predicate P, and Ω_t well-ordering $<$ of the range of x, β, β^{t+2}, respectively. (For notation cf. [12], p. 538.)

When $t = 0$ the category of $<$ is established by the following triviality:

(W^0) *There exists an Ω_0 (i. e., an ω) $\Sigma_1^0 \cap \Pi_1^0$ well-ordering $<$ of N.*

Thus when $t = 0$ applications of (D^0) and (W^0) (together with straightforward techniques) permit one to make the best possible evaluation of $(E\alpha^t)_{\alpha^t<\varphi^t}P(\mathfrak{a}, \alpha^t)$ for P in $\Sigma_{k+2}^0 \cap \Pi_{k+2}^0$, Σ_{k+1}^0, or Π_{k+2}^0.

Now if we let C be $\emptyset$, $\mathfrak{a}$ be φ, and $R(\varphi, \beta_1, \alpha, x)$ be $\varphi(x) = \beta_1(x)$ in (C^1) we obtain

(W^1) (A special case of (C^1)). *There exists an Ω_1 $\Sigma_2^1 \cap \Pi_2^1$ well-ordering $<$ of N^N.*

Hence, by our main theorem, we have the following

COROLLARY. (A)$\to$(W^1),

which establishes the category of $<$, under the assumption of (A), when $t = 1$.

Thus when $t = 1$ applications of (D^1) and (W^1) (together with straightforward techniques) permit one to make the best possible evaluations of $(E\alpha^t)_{\alpha^t<\varphi^t}P(\mathfrak{a}, \alpha^t)$ for P in $\Sigma_{k+3}^0 \cap \Pi_{k+3}^0$, Σ_{k+2}^0, or Π_{k+3}^0.

But just as (D^0) and (W^0) do not alone permit the best evaluation of $(E\alpha^0)_{\alpha^0<\varphi^0}P(\mathfrak{a}, \alpha^0)$ for P in $\Sigma_1^0 \cap \Pi_1^0$ or Π_1^0, so also (D^1) and (W^1) do not alone permit the best evaluation of $(E\alpha^1)_{\alpha^1<\varphi^1}P(\mathfrak{a}, \alpha^1)$ for P in $\Sigma_1^1 \cap \Pi_1^1$, Σ_1^1, Π_1^1, $\Sigma_2^1 \cap \Pi_2^1$, or Π_2^1. For both of these situations something further is needed.

(8) A similar device (cf. [6], (2)) played a principal role in [6].

When $t = 0$ this "something further" is provided by the following well-known theorem:

(C^0) *There exists an Ω_0 (i. e. an ω) well-ordering $<$ of N such that for any subset C of N^N and any predicate P primitive recursive in functions in C the set $\hat{\mathfrak{a}}\hat{y}\,(Ey_1)_{y_1<y}P(\mathfrak{a}, y_1)$ is in $\Sigma_1^0[C] \cap \Pi_1^0[C]$.*

When $t = 1$ this "something further" is provided, under the assumption of (A), by the analog of (C^0), viz. (C^1).

Whether or not (C^1) is an actual improvement over (W^1) we do not know: that is, whether or not (W^1) implies (C^1) is an open problem.

Now using (D^1) and (C^1) (together with straightforward techniques of [13]) the following table can be computed. It shows the effect of the bounded (relative to $<$) existential (or, by duality, universal) function quantifier and of the least (relative to $<$) function operator of the classes of the analytical hierarchy.

P		$\hat{\mathfrak{a}}\hat{\beta}(E\beta_1)_{\beta_1<\beta}P(\mathfrak{a}, \beta_1)$	$\lambda\psi\mu\varphi P(\varphi, \psi)$ $\mu\varphi P(\varphi)$
$\Sigma_1^1 \cap \Pi_1^1$		$\Sigma_2^1 \cap \Pi_2^1$	$\Sigma_2^1 \cap \Pi_2^1$
Σ_1^1		$\Sigma_2^1 \cap \Pi_2^1$	$\Sigma_2^1 \cap \Pi_2^1$
Π_1^1		$\Sigma_2^1 \cap \Pi_2^1$	$\Sigma_2^1 \cap \Pi_2^1$
$\Sigma_k^1 \cap \Pi_k^1$	$k \geqslant 2$	$\Sigma_k^1 \cap \Pi_k^1$	$\Sigma_k^1 \cap \Pi_k^1$
Σ_k^1	$k \geqslant 2$	Σ_k^1	$\Sigma_{k+1}^1 \cap \Pi_{k+1}^1$
Π_k^1	$k \geqslant 2$	Π_k^1	$\Sigma_{k+1}^1 \cap \Pi_{k+1}^1$

Because Theorem 1 was given in relative form and (D^1) relativizes, the table relativizes to any C.

Using only the categories Σ_k^1, Π_k^1 $\Sigma_k^1 \cap \Pi_k^1$ as entries it can be shown that the table, columned as above, cannot be improved. Thus, unlike the uniformity through all types of the tables for ι, the tables for μ show a fundamental and far-reaching divergence already between types 0 and 1. It is this divergence which has been responsible for many of the "strange" results which have been collecting for many years in descriptive set theory, both in the classical, and in the more recent effective, branches. The most celebrated example is, of course, the flop-over of the separation principles (cf. [3]). What part this phenomenon has to play in the failure of the analog of Post's representation theorem (cf. [6]) is under investigation.

Any one of many results could be used to show that there cannot be much better Ω_1 well-orderings of N^N. Perhaps the easiest is this: any Ω_1 well-ordering predicate for N^N must be nonmeasurable (cf. [17]), whereas all $\Sigma_1^1 \cup \Pi_1^1$ (indeed all $\Sigma_1^1[N^N] \cup \Pi_1^1[N^N]$) predicates are measurable. As a matter of fact all of the rather extensive subclasses of $\Sigma_2^1[N^N] \cap \Pi_2^1[N^N]$ which have been studied (cf. [6], [18]) contain only measurable sets.

Preliminary and as yet incomplete investigations now underway on hierarchies based on quantification of higher type indicate that the μ-tables for types $t > 1$ are, under the assumption of (A), like the μ-table for $t = 0$. Thus it appears that one has access to a choice operator already at the first level at higher types, and so that of all the finite types the second ($t = 1$) stands out as peculiar. This would mean, for example, that under the assumption of the axiom of constructibility the separation results (cf. [3]) are uniform according to the kind of the outermost quantifier for all types t ($t \geqslant 0$) and levels k ($k \geqslant 1$) except for the lone case when $t = 1$, $k = 1$.

Three of the most important hierarchies still missing today are hierarchies of $\Sigma_1^0 \cap \Pi_1^0$, $\Sigma_2^1 \cap \Pi_2^1$, and $\Sigma_1^2 \cap \Pi_1^2$ functions from N into N. On the other hand nice hierarchies for $\Sigma_1^1 \cap \Pi_1^1$ functions (Suslin-Kleene theorem) and $\Sigma_2^0 \cap \Pi_2^0$ functions (unpublished work of the author) are known. A great amount of effort has been put into finding a $\Sigma_1^0 \cap \Pi_1^0$ hierarchy (cf. the promising work of Axt [7]), and a $\Sigma_2^1 \cap \Pi_2^1$ hierarchy (cf. the more or less related work of Grzegorczyk, Kantorovič, Kleene, Kolmogorov, Kondô, Kreider, Livenson, Lorenzen, Luzin, Lyapunov, Mostowski, Myhill, Rogers, Selivanowskiĭ, Spector, Wang, the author, and others), but as yet there is no published work in the direction of a "$\Sigma_1^2 \cap \Pi_1^2 =$ hyperanalytical" theorem. We note that all three of these missing hierarchies are for "μ-classes", i. e. the classes at which the choice operator first appears in the respective type.

It is important to note, both in this connection and independently, that in view of the μ-tables the analogy which has been so much referred to between the $\Sigma_1^0 \cap \Pi_1^0$ (recursive) and $\Sigma_1^1 \cap \Pi_1^1$ (hyperarithmetical) may in some directions become overshadowed by a firmer analogy between $\Sigma_1^0 \cap \Pi_1^0$ and $\Sigma_2^1 \cap \Pi_2^1$.

The extent to which the assumption of (A) can be relaxed in the applications of Theorem 1, preliminary indications of which were given and discussed in [4], is to be the subject of a forthcoming paper.

5. Aplications. 5.1. Our first application of the preceding results is to show, as promised in [3], that the same treatment of the separation principles that worked at the third and higher levels of the C-analytical hierarchies works also at the second level.

Following the outline of [3], we wish to show that the argument of Case IV now applies when $k = 2$. We are given two sets X, Y in $\Sigma_2^1[C]$ and we are to find two sets X_1, Y_1 in $\Sigma_2^1[C]$ such that $X_1 \subseteq X$, $Y_1 \subseteq Y$, $X_1 \cup Y_1 = X \cup Y$, and $X_1 \cap Y_1 = \emptyset$. By hypothesis, for some R, S recursive in functions in C,

$$X = \hat{\alpha}(E\gamma)(\beta)(Ex)R(\alpha, \gamma, \beta, x),$$
$$Y = \hat{\alpha}(E\gamma)(\beta)(Ex)S(\alpha, \gamma, \beta, x).$$

Just as before we set

$$X_1 = \hat{\alpha}(E\gamma)[(\beta)(Ex)R(\alpha, \gamma, \beta, x) \,\&\, (\overline{E\gamma_1})_{\gamma_1<\gamma}(\beta)(Ex)S(\alpha, \gamma_1, \beta, x)],$$

with Y_1 being defined analogously. Now it follows from the relativized table that the second factor of the matrix for X_1 is in $\Sigma_2^1[C] \cap \Pi_2^1[C]$, so $X_1 \in \Sigma_2^1[C]$, as required. The rest of the proof is the same as before.

This proof depends of course on (A), and a proof not depending on (A) is known (cf. [3] Case III). Nevertheless we find it of considerable interest to see that the same proof that works for $k \geqslant 3$ also works for $k = 2$. (Note, however, that for $k \geqslant 3$ (W^1) sufficed, whereas for $k = 2$ the full force of (C^1) appears to be used.)

5.2. Our second application of the main theorem is in the theory of bases. A subclass B of N^N is a basis for a category $\mathcal{C}$ of predicates on N^N if and only if

$$(P)\big[P \in \mathcal{C} \rightarrow [(E\alpha)P(\alpha) \rightarrow (E\alpha)[\alpha \in B \,\&\, P(\alpha)]]\big].$$

Kleene has shown (cf. [14], XXVI) that the class of $\Sigma_1^1 \cap \Pi_1^1$ functions is not a basis for Π_0^1 ($= \Pi_1^0$; cf. [13], 3.5). On the other hand, by the method of [13], 5.5, the class of functions whose graphs are the difference of two Σ_1^1 sets (and, a fortiori, the class of $\Sigma_2^1 \cap \Pi_2^1$ functions) is a basis for Π_0^1. The questions naturally arise then what classes are bases for Π_k^1 when $k > 0$. We answer these questions here, under the assumption of (A), with the following:

THEOREM 2. (A)$\rightarrow$[*the class of* $\Sigma_{k+1}^1 \cap \Pi_{k+1}^1$ *functions is a basis for* Π_k^1 *if* $k > 0$].

Proof. Immediate from the table for μ.

In the following theorem, which does not depend on (A), we refine the study of bases by a consideration of the collection of all bases for a category of predicates on N^N.

THEOREM 3. *For any* k: *for any* ψ *in* $\Sigma_{k+1}^1 \cap \Pi_{k+1}^1$ *every basis for* Π_k^1 *contains a function in which* ψ *is primitive recursive; the intersection of all bases for* Π_k^1 *is contained in the class of* $\Sigma_{k+1}^1 \cap \Pi_{k+1}^1$ *functions.*

Proof. For the first part of the theorem simply consider $\hat{\varphi}\varphi = \psi$ as a Σ_{k+1}^1 set, i. e. as the projection of a Π_k^1 set of ordered pairs of each of which ψ is the first element.

The second part of the theorem is an easy consequence of (U_2).

As far as bases for the categories of Σ_k^1 and $\Sigma_k^1 \cap \Pi_k^1$ predicates are concerned we have the following results:

THEOREM 4. *The intersection of all bases for $\Sigma_0^1 \cap \Pi_0^1$ (or for Σ_0^1) is the empty class, but this class is not a basis; any dense class of functions (e. g. the class of functions with values 0 from some point on) is a basis.*

The intersection of all bases for $\Sigma_1^1 \cap \Pi_1^1$ (or for Σ_1^1) is the class of $\Sigma_1^1 \cap \Pi_1^1$ functions, but this class is not a basis; the class of functions recursive in O (and, a fortiori, the class of $\Sigma_2^1 \cap \Pi_2^1$ functions) is a basis.

The intersection of all bases for $\Sigma_k^1 \cap \Pi_k^1$ (or for Σ_k^1) $(k \geqslant 2)$ is the class of $\Sigma_k^1 \cap \Pi_k^1$ functions, and, under the assumption of (A), *this intersection is itself a basis.*

Proof. Straightforward from the results of [13] and [14], the method of proof of Theorem 3, and the table for μ.

5.3. A third application of our main theorem is the solution, under the assumption of (A), of a problem of Tarski (cf. [24]). A separate discussion of this is to appear with related material in [5].

5.4. A fourth application of our main theorem is in the study of comparability. Kleene and Post demonstrated (cf. [16]) the existence of incomparable degrees ($\Sigma_1^0 \cap \Pi_1^0$-degrees). That is, they constructed two functions neither of which was $\Sigma_1^0 \cap \Pi_1^0$ (i. e. recursive) in the other. More recently Spector [23] has demonstrated the existence of incomparable hyperdegrees ($\Sigma_1^1 \cap \Pi_1^1$-degrees), and also the existence of incomparable equivalence classes relative to still more general operations. We now show that his results cannot be extended, within Gödel set theory, to $\Sigma_2^1 \cap \Pi_2^1$-degrees.

THEOREM 5. (A)$\to$[*all $\Sigma_2^1 \cap \Pi_2^1$-degrees are comparable*] (⁹).

Proof. Let α, β be any elements of N^N. Assume (A). Then $\alpha, \beta \in \mathrm{L}$, and $\mathrm{Or}`\alpha \leqslant \mathrm{Or}`\beta < \Omega_1$ or $\mathrm{Or}`\alpha \leqslant \mathrm{Or}`\beta < \Omega_1$. Assume $\mathrm{Or}`\alpha \leqslant \mathrm{Or}`\beta < \Omega_1$. Then for some k

$$\alpha(a) = b \equiv \langle ba \rangle \in \mathrm{F}`\chi_k,$$

$$\text{where} \quad \chi = \big(\mu\psi\big[M\big((\psi)_0, (\psi)_1\big) \,\&\, (Ei)[\omega \times \omega \cdot \mathrm{F}`(\psi)_{0_i} = \beta]\big]\big)_0 .$$

By an application of Lemmas A and B to the scope of $\mu\psi$, followed by an application of the relativized table for μ, followed by an application

(⁹) Prof. Myhill, in a letter received 17 June 1958, called our attention to the following conjecture of Gödel: all $\Sigma_2^1 \cap \Pi_2^1$-degrees represented by constructible sets are comparable. Our theorem, the proof of which follows in part a suggestion of Prof. Myhill's letter, gives the consistency of a stronger proposition. We present a proof of the conjecture itself in the paper mentioned at the end of Section 4.

of the proof of Lemma A (cf. (A.1)), we have that the graph of α, and hence that α, is $\Sigma_2^1 \cap \Pi_2^1$ in β.

5.5. As a fifth application of our main theorem we now inquire as to the complexity of a set composed of exactly one "representation" for each ordinal number less than some fixed ordinal number. By a *representation* of an ordinal number ν we mean a function α such that $\lambda ij\; \alpha(i, j) = 0$ is a well-ordering of N of type ν.

First, without using (A), we have the following (¹⁰):

THEOREM 6. *There exists a Π_1^1 set $W_{\omega_1}^*$ composed of exactly one representation for each ordinal number $< \omega_1$.*

Proof.

$$W_{\omega_1}^* = \hat{\varphi}\Big[W(\varphi) \,\&\, \varphi \in \Sigma_1^0 \cap \Pi_1^0 \,\&\, \overline{(E\psi)}\Big[Q(\psi, \varphi) \,\&\, (E\chi)(i)\Big[(Ek)[\chi(k) = i]$$
$$\& (j)\big[\varphi(i, j) = 0 \equiv \psi\big(\chi(i), \chi(j)\big) = 0\big]\Big]\Big]\Big],$$

where Q is the well-ordering of recursive functions induced by their least Gödel numbers. Since $W \in \Pi_1^1$ (cf. (B)), it follows by the techniques of [13] that $W_{\omega_1}^* \in \Pi_1^1$. The proof is completed, if the original generative definition of ω_1 is intended, by the Markwald-Spector result (cf. e. g. [22]) that the generative and relational concepts of constructive ordinal coincide.

Similarly we have the following:

THEOREM 7. (A)$\to$[*there exists a $\Sigma_2^1 \cap \Pi_2^1$ set $W_{\Omega_1}^*$ composed of exactly one representation for each ordinal number $< \Omega_1$*].

Proof. $W_{\Omega_1}^*$ is defined like $W_{\omega_1}^*$, except that "$\varphi \in \Sigma_1^0 \cap \Pi_1^0$" is dropped and "$Q(\psi, \varphi)$" replaced by "$\psi < \varphi$". The proof then follows easily using the bounded quantifier table. (Note that although our table was developed for 1-place functions, by the use of recursive pairing operators it applies also to 2-place functions.)

5.6. Further applications of our main theorem are to appear in [23] and [11].

6. The classification of $\mathrm{L} \cdot N^N$.

From Lemmas A and B we can also deduce the following result, which does not depend on (A):

THEOREM 8. $\mathrm{L} \cdot N^N \in \Sigma_2^1$, *i. e. the class of constructible functions from N into N is a Σ_2^1 class.*

(¹⁰) The analogous problem, for ordinal notations in place of ordinal representations, was considered by Spector in [22], p. 159.

Proof. By Gödel's results that $\alpha \in \mathrm{L} \to \mathrm{Od}{}^{\epsilon}\alpha < \Omega_1$ (similar to [9] 12.2), $\omega \times \omega \in \mathrm{L}$ ([9] 10.11, 11.42, 9.88, 9.64), $X, Y \in \mathrm{L} \to X \cdot Y \in \mathrm{L}$ ([9] 9.61), it follows that

$$\mathrm{L} \cdot N^N = \hat{a}(E\varphi)(E\varepsilon)\big[M(\varphi, \varepsilon) \,\&\, (Ei)[\omega \times \omega \cdot \mathrm{F}{}^{\epsilon}\varphi_i = \alpha]\big]. \tag{6.0}$$

The theorem then follows by Lemmas A and B.

No hierarchic lower bound on the complexity of $\mathrm{L} \cdot N^N$ can be proved to exist, at least within Gödel set theory, because (A), which is consistent, implies that $\mathrm{L} \cdot N^N = N^N \in \Sigma_1^0 \cap \Pi_1^0$. On the other hand we conjecture that $\mathrm{L} \cdot N^N \in \Pi_2^1$ is not provable in Gödel set theory.

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