Some multiplicative aspects of ideal structure theory*

by

J. G. Horne, Jr. (Lexington, Kent.)

1. Introduction. While we consider some general rings in this paper, the ideal structure theory mentioned in the title is that part of the general theory which one is more apt to think of in connection with the study of Banach Algebras and of various rings of continuous functions. Most concepts are defined for arbitrary rings and a number of isolated lemmas and theorems are proved for these concepts in such rings. However, the majority of our results employ the assumption of strong-semi-simplicity and an ample supply of "local" identities. The results subsequent to section 5 use the additional assumption that the structure space is a Hausdorff space. Thus the various classes of rings considered are all sufficiently broad to include the s. s. s. G. S. algebras of Wilcox [12]. Our hypotheses have in common with those in [2] and [7] the property that they are inherited by homomorphic images of rings which possess them.

The motivation for this paper has been the desire to bring to the fore and to isolate a number of purely multiplicative aspects of ideal structure theory. In this paper, our attention is confined to those aspects which center in the notion of relative identity. Our tools are the concepts of O-ideal and prime-like ideal. We have generalized the natural definition of relative identity as given for commutative rings in [7] and obtain, incidentally, extensions of theorems 4.7 and 4.9 of that paper to arbitrary rings. The notion of relative identity is also more general (for the class of non-commutative rings) than the notion of relative unit used in [3]. Hence the notion of O-ideal is bit more general and the notion of prime-like ideal is a bit more restrictive than that used there.

The results of this paper may be summarized as follows: we obtain a number of results concerning the ideal structure (including the O-ideal structure) of s. s. s. G. S. rings which have heretofore been obtained

* Early forms of some of the ideas in this paper are to be found in the author's doctoral dissertation submitted to Tulane University. The paper [3] contains other ideas of that thesis.

Portions of this paper were presented to the American Mathematical Society on August 19, 1958, under the title "Rings with effective O-ideal structure".

either for rings of continuous functions (as in [4] and [5]), or under the assumption of an identity (as in [3], § 4). Novelties include:

(1) the definition of the topology Γ on the collection of maximal O-ideals to yield a space homeomorphic with the structure space;

(2) the use of dual O-ideals to obtain the invariance of maximal regular ideals under multiplicative isomorphisms;

(3) the theorem that any primary ideal in a s. s. s. G. S. ring is also a s. s. s. G. S. ring;

(4) the use of the previous theorem to generalize the equivalence of conditions (a) and (c) of [8], Theorem 3.3, and to prove the theorem of Shirota [11] and Civin and Yood [1] that a locally compact Hausdorff apce X is characterized by the multiplicative semigroups on $C_0(X)$ and $C_\infty(X)$ (see section 5 for a definition of these terms).

2. Notation and terminology. In general, for sets A and B, $A\backslash B$ denotes the complement of B in A. For a family of sets $\mathfrak{A}$, $\bigcup\mathfrak{A}$ denotes the set $\bigcup\{A\colon A\in\mathfrak{A}\}$ and $\bigcap\mathfrak{A}=\bigcap\{A\colon A\in\mathfrak{A}\}$.

If $\mathfrak{S}$ is a family of proper subsets of some set R, then the Stone (dual Stone) topology on $\mathfrak{A}$ is the topology $\Sigma[\Delta]$ which has the collection of sets $\mathfrak{N}(f)=\{S\in\mathfrak{S}\colon f\notin S\}[\mathfrak{D}(f)=\{S\in\mathfrak{S}\colon f\in S\}]$ as a sub-basis for its open sets. The resulting topological spaces are denoted by $\mathfrak{S}_\Sigma$ and $\mathfrak{S}_\Delta$ respectively. The sets of sub-basis elements were termed the Stone and dual Stone paratopologies respectively in [3]. However, all of the theorems of that work evidently remain valid if every occurrence of the word "paratopology" is replaced by the word "topology". For all of these theorems concern either compactness or continuity and it is well known that each of these notions can be described completely in terms of a sub-basis. We have occasion to use this fact in Theorem 4.6 for example.

If no mention is made to the contrary, the closure of a subset A of a topological space is denoted by A^-; its interior is denoted by A°.

Throughout this work, R denotes a ring. The word *ideal* always means two-sided ideal. For $f\in R$, $[f]$ denotes the principal (two-sided) ideal generated by f. As in [9], we call an ideal $J\subset R$ a *regular* ideal if R/J has an identity (such ideals are called *modular* in [6]). If $e\in R$ maps into the identity in R/J we write $e\equiv 1(J)$. Naturally, $f\equiv 0(J)$ means $f\in J$.

By the *structure space* of R is meant the collection $\mathfrak{S}(R)$ of maximal regular ideals of R together with the Stone topology. Since we have no occasion to use any other topology on $\mathfrak{S}(R)$, we find no harm in writing $\mathfrak{S}(R)$ when we mean $\mathfrak{S}(R)_\Sigma$.

Much of our work requires the restriction that R be a *strongly-semi-simple* ring (abbreviated s. s. s. *ring*). This means that $\bigcap\mathfrak{S}(R)$ is the zero ideal.

In the context of this paper, it is more usual to describe the Stone topology in terms of the notions of *hull* and *kernel*. The hull of a subset $J\subset R$ is the collection $h(J)=\{S\in\mathfrak{S}(R)\colon J\subset S\}$. The kernel of a collection $\mathfrak{F}\subset S(R)$ is the ideal $k(\mathfrak{F})=\bigcap\mathfrak{F}$. Of course the closure of a subset $\mathfrak{A}\subset\mathfrak{S}(R)$ is the set $h\big(k(\mathfrak{A})\big)$.

We now introduce some notation and conventions for the purpose of exploiting several similarities between the rings we are about to study and various rings of continuous functions. Thus for $f\in R$ we set [1] $\mathfrak{Z}(f)=\{S\in\mathfrak{S}(R)\colon f\in S\}$ and regard $\mathfrak{Z}(f)$ as the *zero set* of f. Its complement $\mathfrak{N}(f)$ is the *non-zero set* of f. The set $\mathfrak{N}(f)^-$ is the *support* of f (some authors refer to this set as the *carrier* of f — see [9], p. 84, for example). It is also natural to define the identity set $\mathfrak{E}(f)=\{S\in\mathfrak{S}(R)\colon f\equiv 1(S)\}$. Obviously both $\mathfrak{Z}(f)$ and $\mathfrak{E}(f)$ are closed sets.

For a subset $\mathfrak{F}\subset\mathfrak{S}(R)$, we find it convenient to extend the above notation and write $f\equiv 0(\mathfrak{F})$ and $f\equiv 1(\mathfrak{F})$ when we mean $f\equiv 0\big(k(\mathfrak{F})\big)$ and $f\equiv 1\big(k(\mathfrak{F})\big)$. These statements are equivalent, respectively, to the statements $f\equiv 0(S)$ and $f\equiv 1(S)$ for all $S\in\mathfrak{F}$. Also, if J is an ideal then $f\equiv 0(J)$ implies $f\equiv 0\big(h(J)\big)$ and $f\equiv 1(J)$ implies $f\equiv 1\big(h(J)\big)$.

We have frequent occasion to use Gillman's set $N(S)$ [2]. For $S\in\mathfrak{S}(R)$, $N(S)=\{f\in R\colon S\in\mathfrak{Z}(f)^\circ\}$. Then $N(S)$ is an ideal contained in S and by Theorem 3.1 of [2], $\mathfrak{S}(R)$ is a Hausdorff space if the hull of $N(S)$ is always precisely $\{S\}$. Note that in our above notation, $f\in N(S)$ if and only if $f\equiv 0(\mathfrak{U})$ for some open set $\mathfrak{U}$ containing S. Also, if $e\equiv 1(\mathfrak{U})$ for such $\mathfrak{U}$ then $e\equiv 1\big(N(S)\big)$.

Other terms are defined as they are needed.

3. Relative and local identities. The following definition of relative identity generalizes that given in [7], definition 4.3, and has many pleasant properties which other generelizations of that notion do not possess. In non-commutative rengs, it is distinct from the notion of relative unit used in [3]. Thus it leads to a different notion of O-ideal and prime-like ideal from that in [3], but it appears to be the correct one for our present purposes.

Definition 3.1. An element $e\in R$ is a *relative identity* for $f\in R$ if for every $x\in[f]$, $ex=xe=x$. An element f *has* a relative identity if such an element e exists.

Recall that the „circle product" of two elements $f,g\in R$ is defined by $f\circ g=f+g-fg$. We have the important

Lemma 3.2. *If e is a relative identity for f, and g is arbitrary, then $e\circ g$ and $g\circ e$ are relative identities for f.*

[1] Conceivably a second name for $\mathfrak{O}(f)$; however, in the remainder of the paper $\mathfrak{O}(f)$ denotes the set of maximal O-ideals containing f.

Proof. For $x \in [f]$ there are four elementary equations to be checked. The details are omitted.

It follows immediately that Lemma 4.4 of [7] is valid in arbitrary rings using the above definition of relative identity:

LEMMA 3.3. *If $f_1, \ldots, f_n$ have relative identities $e_1, \ldots, e_n$ respectively then the element $e = e_1 \circ \ldots \circ e_n$ is a relative identity for the set $\{f_1, \ldots, f_n\}$.*

Remark 3.4. Note that this common relative identity e is a "ring combination" of the individual relative identities. Hence if all the e_i belong to some ideal J then $e \in J$.

LEMMA 3.5. *Suppose that e is a relative identity for f. Then $e \equiv 1(\mathfrak{N}(f))$ (or equivalently $\mathfrak{N}(f) \subset \mathfrak{E}(e)$). Thus, if $S \in \mathfrak{S}(R)$ and $f \notin S$ then $e \equiv 1(N(S))$ and $N(S)$ is a regular ideal.*

Proof. We shall show that $\mathfrak{N}(f) \subset \mathfrak{Z}(ex-x)$ for each $x \in R$. A similar argument shows that $\mathfrak{N}(f) \subset \mathfrak{Z}(xe-x)$. Since $\mathfrak{N}(f)$ is an open set, $S \in \mathfrak{N}(f)$ implies $e \equiv 1(N(S))$ by definition of $N(S)$. Since $N(S) \subset S$ it follows that $e \equiv 1(N(f))$.

Let $x \in R$ be arbitrary. If $\mathfrak{N}(f) \not\subset \mathfrak{Z}(ex-x)$ then there exists $T \in \mathfrak{S}(R)$ such that $f \notin T$ and $ex-x \in T$. Now a maximal regular ideal is prime. Hence, there exists $y \in R$ such that $(ex-x)yf \notin T$ [10]. However, $(ex-x)yf = exyf - xyf = 0 \in T$, which is a contradiction. Hence, $\mathfrak{N}(f) \subset \mathfrak{Z}(ex-x)$.

As a partial converse we have

LEMMA 3.6. *If R is a* s. s. s. *ring and if $\mathfrak{N}(f) \subset \mathfrak{E}(e)$ then e is a relative identity for f.*

Proof. Let $S \in \mathfrak{S}(R)$ be arbitrary and suppose $x \in [f]$. If $S \notin \mathfrak{N}(f)$ then $f \in S$ so $ex-x \in S$ and $xe-x \in S$ since S is an ideal. If $S \in \mathfrak{N}(f)$, then $e \equiv 1(S)$ by hypothesis, so $ex-x \in S$ and $xe-x \in S$. Since R is a s. s. s. ring, $ex-x = 0 = xe-x$ so e is a relative identity for f.

Lemma 3.5 yields the following version of Theorem 4.7 of [7] for arbitrary rings:

THEOREM 3.7. (Kohls) *Suppose that R is a ring such that for each $S \in \mathfrak{S}(R)$ there exists $f \notin S$ which has a relative identity. Then $\mathfrak{S}(R)$ is locally compact.*

Proof. Let $S \in \mathfrak{S}(R)$. There is $f \notin S$ and $e \in R$ such that e is a relative identity for f. By Lemma 3.5, $e \equiv 1(\mathfrak{U})$ where $\mathfrak{U}$ is the open set $\mathfrak{N}(f)$. Therefore, if $J = k(\mathfrak{U})$ then R/J has an identity. By Corollary 1.2 of [2], $\mathfrak{U}^-$ is compact. Hence S has a neighborhood whose closure is compact.

The hypothesis of the previous theorem is a valuable weakening of the assumption that a ring have an identity. It was used at several points in [8]. Presently we show that in s. s. s. rings it is equivalent to a hypothesis used by Wilcox [12]. We use it extensively in the sequel (note especially section 6). Taking our cue from Loomis's reference to "local identities" ([9], p. 83) we adopt

DEFINITION 3.8. If R is a ring such that for each $S \in \mathfrak{S}(R)$ there exists $f \notin S$ which has a relative identity, then we shall call R a *local identity* ring (abbreviated l. i. *ring*).

According to the previous theorem, a l. i. ring has a locally compact structure space.

Remark 3.9. An easily proved fact which we shall have occasion to use later is that the homomorphic image of a l. i. ring is a l. i. ring.

The second condition of the following theorem is the hypothesis (B) of [12].

THEOREM 3.10. *Let R be a* s. s. s. *ring. Then the following conditions are equivalent:*

(1) *R is a* l. i. *ring.*

(2) *Every $S \in \mathfrak{S}(R)$ is contained in an open set whose closure has regular kernel.*

Proof. Suppose that R is a l. i. ring and let $S \in \mathfrak{S}(R)$. Then there exists $f \notin S$ and $e \in R$ such that e is a relative identity for f. By Lemma 3.5, $\mathfrak{N}(f) \subset \mathfrak{E}(e)$, so $\mathfrak{N}(f)^- \subset \mathfrak{E}(e)$, since $\mathfrak{E}(e)$ is a closed set. Thus $e \equiv 1(\mathfrak{N}(f)^-)$, or what is the same thing, $e \equiv 1(k(\mathfrak{N}(f)^-))$. Thus S is contained in the open set $\mathfrak{N}(f)$ whose closure has regular kernel.

Assume, conversely, that (2) holds and let $S \in \mathfrak{S}(R)$. Then there is an open set $\mathfrak{U}$ containing S and an element $e \in R$ such that $e \equiv 1(k(\mathfrak{U}^-))$; that is, $\mathfrak{U} \subset \mathfrak{E}(e)$. Since $S \in \mathfrak{U}$, S does not belong to the closed set $\mathfrak{F} = (\mathfrak{S}(R) \backslash \mathfrak{U}$, so $S \not\supset k(\mathfrak{F})$. Thus there exists $f \in k(\mathfrak{F}) \backslash S$, and for such f, $\mathfrak{N}(f) \subset \mathfrak{U} \subset \mathfrak{E}(e)$. By Lemma 3.6, e is a relative identity for f. Hence R is a l. i. ring.

The next result is used in the remainder of the paper as well as to obtain immediately a version of another result in [7] for arbitrary rings:

THEOREM 3.11. *Suppose that $\mathfrak{F} \subset \mathfrak{S}(R)$ is compact and J is an ideal. If for each $S \in \mathfrak{F}$ there exists $f \notin S$ such that J contains a relative identity for f, then there exists $e \in J$ such that $e \equiv 1(N(S))$ for every $S \in \mathfrak{F}$. Hence, if R is a* l. i. *ring then the ideals $J(\mathfrak{F}) = \bigcap \{N(S)\colon S \in \mathfrak{F}\}$ and $k(\mathfrak{F})$ are regular ideals.*

Proof. In virtue of the assumption concerning relative identities and the compactness of F, there exists a finite set of elements $f_1, \ldots, f_n \in R$ and corresponding relative identities $e_1, \ldots, e_n \in J$ such that $\mathfrak{F} \subset \bigcup_{i=1}^{n} \mathfrak{N}(f_i)$. Now $e = e_1 \circ \ldots \circ e_n \in J$, and by Lemma 3.3, e is a relative identity for the set $\{f_1, \ldots, f_n\}$. By Lemma 3.5, $e \equiv 1(N(S))$ for each $S \subset \mathfrak{F}$. The first part of the final conclusion is obtained by taking $J = R$ and the second part follows from it since $N(S) \subset S$.

Remark 3.12. Our use of $J(\mathfrak{F})$ in the previous theorem agrees with that in [12] where $J(\mathfrak{F})$ was defined to consist of the collection of elements f such that $f \equiv 0(\mathfrak{U})$ for some open set $\mathfrak{U} \supset \mathfrak{F}$. But this collection obviously coincides with the set $\bigcap \{N(S)\colon S \in \mathfrak{F}\}$. In particular, if $\mathfrak{F} = \{S\}$ then $J(\mathfrak{F}) = N(S)$.

It is apparent from Theorem 3.11 that if R is a strongly-semi-simple l. i. ring and $\mathfrak{S}(R)$ is compact, then R has an identity. The proof of the converse proposition which appears in [7] is valid in such rings. Hence we have the following version of Theorem 4.9 of that paper.

THEOREM 3.13. (Kohls) *Let R be a strongly-semi-simple* l. i. *ring. Then $\mathfrak{S}(R)$ is compact if and only if R has an identity.*

Next is an analogue of Lemma 25 C of [9], which we use extensively in the sequel.

LEMMA 3.14. *Let R be a* l. i. *ring. If $\mathfrak{F}_0 \subset \mathfrak{S}(R)$ is compact and disjoint from the hull $h(J)$ of an ideal J then there exists $e \in J$ such that $e \equiv 1(\mathfrak{F}_0)$.*

Proof. By theorem 3.11, $\mathfrak{F}_0$ has a regular kernel. By [9], p. 62, for example, there exists $e \in J$ such that $e \equiv 1\big(k(\mathfrak{F}_0)\big)$. Therefore, $e \equiv 1(\mathfrak{F}_0)$.

Remark 3.15. It follows from the previous lemma that if R is a l. i. ring then compact sets in $\mathfrak{S}(R)$ are closed. For if $\mathfrak{F} \subset \mathfrak{S}(R)$ is compact and $S \notin \mathfrak{F}$, then $S \not\supset k(\mathfrak{F})$: Simply choose $x \notin S$. By the lemma, there exists $e \in S$ such that $e \equiv 1(\mathfrak{F})$. Thus $ex - x \in \big(k(\mathfrak{F})\backslash S\big)$.

Now consider the set $J(\infty)$ defined in [12] to consist of the set of elements f such that $\mathfrak{N}(f)$ is contained in a compact set $\mathfrak{F}$. By the remark, $\mathfrak{N}(f)^- \subset \mathfrak{F}$, so f has compact support. Certainly if f has compact support then $f \in J(\infty)$. Hence we have

Remark 3.16. In l. i. rings the set $J(\infty)$ is the set of elements having compact support.

Finally, we use Lemma 3.14 to show that strongly-semi-simple l. i. rings possess an abundance of divisors of zero.

THEOREM 3.17. *Suppose that R is a strongly-semi-simple* l. i. *ring and let $S \in \mathfrak{S}(R)$. Then the members of $N(S)$ are divisors of zero.*

Proof. Suppose $f \in N(S)$. Then $S \notin \mathfrak{N}(f)^-$, so $J = k\big(\mathfrak{N}(f)^-\big)$ is an ideal whose hull is disjoint from $\{S\}$. By the previous lemma, there exists $e \in J$ such that $e \equiv 1(S)$. Thus ef and fe both belong to $\bigcap \mathfrak{S}(R)$, while $e \neq 0$. Since R is a s. s. s. ring, $ef = fe = 0$, so f is a divisor of zero.

Remark 3.18. According to [13], the group algebra of a group which is a product of a compact group and a locally compact abelian group is a s. s. s. G. S. algebra and hence a strongly-semi-simple l. i. ring

in virtue of Theorem 3.10. The previous theorem yields an abundance of divisors of zero in such algebras. On the other hand, Żelazko [14] has given a construction of a divisor of zero in an arbitrary (non-trivial) group algebra.

4. O-ideals and prime-like ideals. The notion of relative identity leads to the relations

$$O = \{(f, e)\colon e \text{ is a relative identity for } f\},$$

and

$$O^2 = \{(f, e)\colon \text{for some } e',\ fOe' \text{ and } e'Oe\}.$$

Obviously both relations are transitive and $O^2 \subset O$.

Warning. A difference is to be noted in the above definition of O and of the relation bearing this designation in [3]. The present relation evidently includes that in [3] and the two coincide in commutative rings.

In any case, we have the notions of O-ideal and O^2-ideal as special cases of definition 1.2 of [3], and as we shall see, most of the results listed there for O-ideals are valid for the present meaning of this term. In particular, it is easy to see that a subset I of a ring is an O-ideal (O^2-ideal) if and only if I is an ideal which is O-directed (O^2-directed) in the sense of [3], § 2, for example. Thus the notions of O-ideal and O^2-ideal coalesce in rings. In virtue of remark 3.4 we have counterparts of Lemma 4.7 and Theorem 4.8, respectively, of [3]. The proofs there suffice here with minor changes and are omitted.

THEOREM 4.1. *A subset I of a ring is an O-ideal if and only if I is an ideal which satisfies the condition*

(1) *$f \in I$ implies there exists $e \in I$ such that fOe (fO^2e).*

THEOREM 4.2. *If $\mathfrak{I}$ is a collection of O-ideals then the set $\bigvee \mathfrak{I}$ of finite sums taken from $\bigcup \mathfrak{I}$ is an O-ideal. In particular, if I_1 and I_2 are O-ideals then $I_1 + I_2$ is an O-ideal.*

The relation O also yields the notion of prime-like and prime-like (O^2) ideal ([3], § 2). The fact that there may exist elements which possess no relative identities raises the need for the following extension of these ideas.

DEFINITION 4.3. Let R be a ring and A a fixed subset of R. An ideal $P \subset R$ is prime-like (prime-like (O^2)) *with respect to* A if whenever (f, e, k) is a triple satisfying $f \notin P$, fOe (fO^2e), $ke \in P$ and $k \in A$, then $k \in P$. The collections of all such ideals which are proper subsets of R are denoted by $\mathfrak{P}(R, A)$ and $\mathfrak{P}_2(R, A)$ respectively.

Evidently a prime-like (prime-like (O^2)) ideal of R is simply a prime-like (prime-like (O^2)) ideal with respect to R (and hence with respect to every subset of R). Every prime-like ideal is prime-like (O^2) and a sufficient condition for the converse to hold is given in [3], § 5, for commutative rings.

If P is a prime ideal in R then it is a prime-like ideal. For suppose f, k, $r \in R$ are elements such that $f \notin P$, fOe and $ke \in P$. Let $g \in R$ be arbitrary. Then $fgk \in P$, for $fgk = fgke$, since fOe. Therefore $k \in P$. Hence we have proved

THEOREM 4.4. *Every prime ideal in an arbitrary ring is prime-like.*

To see that prime-like ideals have a certain irreducible quality, see Lemma 10.2.

To indicate yet another source of prime-like ideals we have the following theorem. Recall that an ideal P is completely prime if f, $g \notin P$ imply $fg \notin P$.

THEOREM 4.5. *If R is a strongly-semi-simple* l. i. *ring and $S \in \mathfrak{S}(R)$ then $N(S)$ is a prime-like (O^2) ideal. If the members of $\mathfrak{S}(R)$ are completely prime then $N(S)$ is prime-like.*

Proof. If $f \notin N(S)$ then $S \in \mathfrak{N}(f)$. If fOe' then by Lemma 35, $\mathfrak{N}(f)^- \subset \mathfrak{E}(e')$. Thus $S \in \mathfrak{N}(e')$. If ke' belongs to $N(S)$, then there exists an open set $\mathfrak{U}$ such that $S \in \mathfrak{U} \subset \mathfrak{Z}(ke')$. Now if members of $\mathfrak{S}(R)$ are completely prime, $\mathfrak{Z}(ke') = \mathfrak{Z}(k) \cup \mathfrak{Z}(e')$, so $S \in \mathfrak{U} \cap \mathfrak{N}(e') \subset \mathfrak{Z}(k) \cap \mathfrak{N}(e') \subset \mathfrak{Z}(k)$. Hence $k \in N(S)$ and the final statement has been proved.

If we assume instead that fO^2e, then there exists e' as above so that $e'Oe$. Thus we have the additional result $\mathfrak{N}(e')^- \subset \mathfrak{E}(e)$. Combined with the above statements this yields $S \in \mathfrak{N}(e') \subset \mathfrak{E}(e)$. In other words, $e \equiv 1(\mathfrak{N}(e'))$, so $e \equiv 1(N(S))$. Hence if $ke \in N(S)$ then obviously $k \in N(S)$ and the first statement is proved.

The proof of Theorem 3.5 and its Corollary 3.6 of [3] requires only trivial modification to be valid for the collections $\mathfrak{P}(R, A)$ and $\mathfrak{P}_2(R, A)$ defined above. We have already mentioned that the word "paratopology" appearing in [3] can be replaced by "topology". Hence we can state the following theorem, a part of which is used in section 6 to obtain the compactness of the maximal O-ideal space of certain rings which may possess no identity.

THEOREM 4.6. *Let R be an arbitrary ring and A a fixed subset of R. Then the collections $\mathfrak{P}(R, A)$ and $\mathfrak{P}_2(R, A)$ are compact in the dual Stone topology.*

5. The operation L. Associated with the relation O is the operation L. This operation is obviously akin to the operation L in [3] and we shall show presently that it is intimately related to the operation $\mathfrak{j}$ of Loomis [9] and J of Wilcox [12].

DEFINITION 5.1. For a subset J of a ring R we set
$$L(J) = \{f \in R\colon fOe \text{ for some } e \in J\}.$$

We have occasion to use the operation L^2 defined by $L^2(J) = L(L(J))$, or equivalently $L^2(J) = \{f \in R\colon fO^2e \text{ for some } e \in J\}$.

The following elementary propositions are useful:

(i) L and L^2 are monotone functions; if J is an ideal then $L^2(J) \subset L(J) \subset J$ while $L^2(J)$ and $L(J)$ are ideals. (The proof of the latter statement uses remark 3.4, of course.)

(ii) An O-ideal is contained in an ideal J if and only if it is contained in $L^2(J)$. Hence if M is an O-ideal, $L^2(M) = L(M) = M$.

(iii) If I and J are ideals then $L(I \cap J) = L(I) \cap L(J)$ so $L^2(I \cap J) = L^2(I) \cap L^2(J)$.

One interest in L^2 is seen in the next theorem whose easy proof is omitted. The hypothesis that O^2 is dense was used in [3] and appears again in section 6. It is shown to be available in the class of rings studied in section 7.

THEOREM 5.2. *If R is a ring in which O^2 is dense and J is an ideal then $L^2(J)$ is an O-ideal.*

A certain amount of interest attaches to the set $L(R)$, as the following remarks and theorms show. If R is itself an O-ideal (for example, if R has an identity) then $L(R) = R$. If R is the ring $C_\infty(X)$ of complex-valued continuous functions which vanish at infinity on a locally compact Hausdorff space X then $L(R)$ is the subring $C_0(X)$ consisting of those functions having compact support (see [1] for example; also see Theorem 5.6 of this paper).

Possibly of more general interest is

THEOREM 5.3. *Let R be an arbitary ring and let J be an ideal. If $J \not\supset L(R)$ then J is contained in a regular ideal.*

Proof. Choose $f \in L(R) \setminus J$ and let $e \in R$ be such that fOe. Define $A(f) = \{z \in R\colon \text{for all } y \in [z],\ yf \text{ and } fy \in J\}$. Evidently $A(f)$ is an ideal containing J. Further, $e \equiv 1(A(f))$. For suppose $x \in R$ and suppose $y \in [ex - x]$ has the form $y = u(ex-x)v$ where u, v may denote either integers or elements of R. Then $(u(ex-x)v)f = uexvf - uxvf = uxvf - uxvf = 0 \in J$. It is now apparent that for arbitary $y \in ex - x$, $yf = fy = 0 \in J$. Therefore $ex - x \in A(f)$. A similar argument shows $xe - x \in A(f)$, so $e \equiv 1(A(f))$ and the theorem is proved.

The hypothesis of the following corollary is satisfied, for example, by the ring $C_\infty(X)$ referred to above. This follows from [8], Corollary 3.6, and the remark above that $L(C_\infty(X)) = C_0(X)$ (observe the difference in notation between the present paper and [8]).

Corollary 5.3.1. *If $L(R)$ is contained in no maximal ideal then every maximal ideal is regular.*

We have the following more specialized result concerning $L(R)$.

Theorem 5.4. *A ring R is a* l. i. *ring if and only if the hull of $L(R)$ is empty.*

Proof. If R is a l. i. ring and $S \in \mathfrak{S}(R)$ then there exists $f \notin S$ and $e \in R$ such that fOe. Thus $f \in L(R) \backslash S$; since S is arbitrary, $h\big(L(R)\big) = \emptyset$.

Conversely, if $h\big(L(R)\big) = \emptyset$ then for each $S \in \mathfrak{S}(R)$, $L(R) \not\subset S$. Therefore there exists $f \notin S$ and $e \in R$ such that fOe. Again since S is arbitrary, we have that R is a l. i. ring.

Remark 5.5. In connection with the previous theorem, see Theorem 1.1 (a) of [12]. According to Theorem 3.10 above, the ring considered in [12] is a l. i. ring and as we shall see shortly, $J(\infty)$ may be identified with the collection of elements which possess relative identities (i. e., with $L(R)$ when R designates the ring).

A relation can now be established between L, the operation j of Loomis ([9], p. 84) and the operation $\mathfrak{J}$ of Wilcox ([12], definition 1.3). indeed, in l. i. rings, $j = \mathfrak{J}$. For $\mathfrak{F} \subset \mathfrak{S}(R)$, $j(\mathfrak{F})$ is the set of elements $f \in R$ such that $\mathfrak{N}(f)^-$ is compact and disjoint from $\mathfrak{F}$, while $\mathfrak{J}(\mathfrak{F}) = J(\mathfrak{F}) \cap J(\infty)$ where $J(\mathfrak{F})$ is the set discussed in Remark 3.12 above and $J(\infty)$ is the set of elements mentioned in Remark 3.16. By the latter remark, if $f \in J(\infty)$ then $\mathfrak{N}(f)^-$ is compact, and if $f \in J(\mathfrak{F})$ then $\mathfrak{F} \subset \mathfrak{Z}(f)^\circ$, whence $\mathfrak{F} \cap \mathfrak{N}(f)^- = \emptyset$. Thus $\mathfrak{J}(\mathfrak{F}) \subset j(\mathfrak{F})$. If $f \in j(\mathfrak{F})$ then $\mathfrak{N}(f)^-$ is compact so $f \in J(\infty)$. Also $\mathfrak{F} \cap \mathfrak{N}(f)^- = \emptyset$, so $\mathfrak{F} \subset \mathfrak{Z}(f)^\circ$ and $f \in J(\mathfrak{F})$. The identity $j = \mathfrak{J}$ follows.

We now relate L to j:

Theorem 5.6. *Suppose that R is a strongly-semi-simple* l. i. *ring. Let $\mathfrak{F} \subset \mathfrak{S}(R)$ be closed. Then $j(\mathfrak{F}) = L\big(k(\mathfrak{F})\big)$. Thus $L(R)$ is the set of elements having compact support.*

Proof. If $\mathfrak{N}(f)^-$ is compact and disjoint from $\mathfrak{F}$ then by Lemma 3.14 there exists $e \in k(\mathfrak{F})$ such that $e \equiv 1\big(\mathfrak{N}(f)^-\big)$. By Lemma 3.6, fOe, so $f \in L\big(k(\mathfrak{F})\big)$.

If $e \in k(\mathfrak{F})$ is a relative identity for f then $e \equiv 1\big(\mathfrak{N}(f)^-\big)$ by Lemma 3.5, so $\mathfrak{N}(f)^-$ is compact by Corollary 1.2 of [2]. Also, $\mathfrak{N}(f)^- \subset \mathfrak{E}(e)$ so $\mathfrak{N}(f)^- \cap \mathfrak{F} = \emptyset$. Hence $f \in j(\mathfrak{F})$ and we have $j(\mathfrak{F}) = L\big(k(\mathfrak{F})\big)$.

The final conclusion is obtained by taking $\mathfrak{F}$ to be the null set, and invoking the definition of $j(\mathfrak{F})$.

Corollary 5.6.1. *With R as above, $L(R) = J(\infty)$ and for $S \in \mathfrak{S}(R)$, $L(S) = N(S) \cap L(R)$.*

Proof. The first assertion of the corollary is simply the second conclusion of the theorem, together with the definition of $J(\infty)$. For the assertion concerning S, let $\mathfrak{F} = \{S\}$. By the first conclusion of the theorem, $L(S) = j(\mathfrak{F})$ since $k(\mathfrak{F}) = S$. But $j(\mathfrak{F}) = \mathfrak{J}(\mathfrak{F}) = J(\mathfrak{F}) \cap J(\infty)$. From the definition of $J(\mathfrak{F})$ and $N(S)$ we have $J(\mathfrak{F}) = N(S)$. In summary, $L(S) = J(\mathfrak{F}) \cap J(\infty) = N(S) \cap L(R)$.

The previous theorem also yields the following which allows a simpler definition of maximal O-ideal in the rings of chief interest to us here.

Theorem 5.7. *If R is a strongly-semi-simple* l. i. *ring then $L^2(R) = L(R)$.*

Proof. The inclusion $L^2(R) \subset L(R)$ always holds. If $f \in L(R)$ then $\mathfrak{N}(f)^-$ is compact by the previous corollary, and since R is a l. i. ring, $h\big(L(R)\big) = \emptyset$ by Theorem 5.4. Therefore $\mathfrak{N}(f)^- \cap h\big(L(R)\big) = \emptyset$, so by Lemma 3.14, there exists $e' \in L(R)$ such that $e' \equiv 1\big(\mathfrak{N}(f)^-\big)$. By Lemma 3.6, fOe', and since $e' \in L(R)$, there exists $e \in R$ such that $e'Oe$. Hence fO^2e and $f \in L^2(R)$.

By induction we can of course give meaning to L^n for any positive integer n. If for such n, $L^n(R)$ is an O-ideal then $L^n(R)$ is the (unique) largest O-ideal in R. Thus, in order to avoid trivialities we adopt the following definition which is a modification of that given in [3].

Definition 5.8. An O-ideal M of a ring R is a *proper* O-ideal if $M \neq L^n(R)$ for any integer n. A *maximal* O-ideal is a proper O-ideal M which is contained in no other proper O-ideal. The collection of all maximal O-ideals of a ring R is denoted by $\mathfrak{M}(R)$.

Remark 5.9. Evidently in strongly-semi-simple l. i. rings $L^n(R) = L(R)$ for every positive integer n in virtue of the previous theorem. Thus in such rings, a proper O-ideal is simply an O-ideal M such that $M \neq L(R)$.

As a consequence of Theorem 5.3, every proper O-ideal is contained in a regular ideal and, hence, by Zorn's lemma, in a member of $\mathfrak{S}(R)$. That is, if M is an O-ideal there exists $S \in \mathfrak{S}(R)$ such that $M \subset S$. Hence $M \subset L^2(S)$. Now if O^2 is dense, $L^2(S)$ is an O-ideal, so if M is a maximal O-ideal then $M = L^2(S)$. We have proved the

Theorem 5.10. *In an arbitrary ring R a proper O-ideal is contained in a maximal regular ideal. If O^2 is dense then for every maximal O-ideal M there exists $S \in \mathfrak{S}(R)$ such that $M = L^2(S)$.*

6. On the assumption O^2 is dense.

In this section we generalize essentially all of the results in [3], § 4, in so far as those results concern rings. There seems to be little doubt that corresponding generalizations exist for the class of semirings considered there. The generalizations are in the direction of weakening the assumption of an identity. In place of it we use the assumption

(A) *The hull of* $L^2(R)$ *is empty.*

This assumption is easily seen to be equivalent to the assumption

(A)′ R *is a* l. i. *ring such that* $L^2(R) = L(R)$.

For if (A)′ holds then the hull of $L^2(R)$ is empty by Theorem 5.4. Conversely, suppose (A) holds. By Theorem 5.3, $L^2(R) \supset L(R)$ so $L^2(R) = L(R)$ since $L(R)$ always contains $L^2(R)$. Thus, the hull of $L(R)$ is also empty, so by Theorem 5.4 again, R is a l. i. ring.

In virtue of Theorem 5.10, condition (A)′ (and hence condition (A)) is satisfied in strongly-semi-simple l. i. rings. It is also satisfied if $L(R) = R$ and hence if R has an identy (for if R has an identity 1 then for every $f \in R$, there is $e = 1$ such that fOe). Our arguments evidently remain valid if the relation O^2 is everywhere replaced by a stronger (= smaller) relation for which Theorem 6.1 below is true. Now the relation used in [3] is such a relation if in definition 4.2 of [3] one takes $\varphi(h, f, g) = 1-g$ and $h_0 = 0$. Thus the results of this section do indeed generalize the results of that paper as far as those results concern rings.

THEOREM 6.1. *If* e_1, e_2, u_1 *and* u_2 *are elements of a ring* R *and* $e_1O^2e_2O^2u_1O^2u_2$ *then* $(u_1-e_2)O^2(u_2-e_1)$.

Proof. We mention that one should prove, as a preliminary result, that this theorem is valid if the relation O^2 is replaced by O throughout. The details are straightforward and are omitted. Now assume the hypotheses of the theorem. Then there exist elements f, g and $v \in R$ such that e_1OfOe_2Og and gOu_1OvOu_2. Applying the preliminary result to the chain $e_1OfOvOu_2$, we have $(v-f)O(u_2-e_1)$ and applying this result to the chain fOe_2Ou_1Ov, we have $(u_1-e_2)O(v-f)$. Hence $(u_1-e_2)O^2(u_2-e_1)$.

Now let $\mathfrak{P}_2(R)$ denote the collection $\mathfrak{P}_2(R, L(R))$ of all proper ideals which are prime-like (O^2) with respect to $L(R)$. We can infer from Theorem 4.5 that $\mathfrak{P}_2(R)_\Delta$ is compact. Let $\mathfrak{L}(R) = \{L^2(P) : P \in \mathfrak{P}_2(R)\}$. If $\mathfrak{L}(R)$ is given the dual Stone topology then L^2 obviously becomes a continuous function. Hence $\mathfrak{L}(R)_\Delta$ is compact. Curiously, perhaps, so is $\mathfrak{L}_0(R) = \mathfrak{L}(R)\backslash\{L(R)\}$. For suppose $\mathfrak{F}$ is a proper closed subset of $\mathfrak{L}_0(R)$. Then $\mathfrak{F} = \mathfrak{L}_0(R) \cap \mathfrak{F}'$ for some proper closed subset $\mathfrak{F}'$ of $\mathfrak{L}(R)$. Therefore $L(R) \notin \mathfrak{F}'$ since subsets of $\mathfrak{L}(R)$ which contain $L(R)$ are dense. Therefore, $\mathfrak{F}' \subset \mathfrak{L}_0(R)$ and $\mathfrak{F} = \mathfrak{F}'$, so $\mathfrak{F}$ is compact. If every proper closed subset of a topological space is compact then the space is compact. Hence $\mathfrak{L}_0(R)_\Delta$ is compact.

The next two lemmas constitute steps in the direction of showing that $\mathfrak{L}_0(R) = \mathfrak{M}(R)$.

LEMMA 6.2. *Suppose that* S *is an ideal in a ring* R *and* q, $u \in R$ *satisfy* $q \equiv 1(S)$ *and* qO^2u. *Then, for all* $x \in L^2(R)$, *we have* $ux-x$ *and* $xu-x \in L^2(S)$.

Proof. There exist elements v, w_1 and $w_2 \in R$ such that $qOvOu$ and xOw_1Ow_2. It is not difficult to show that $(ux-x)O(w_2-vw_2)O(w_1-qw_1)$ with similar relations holding for $xu-x$. By hypothesis, $w_1-qw_1 \in S$, so $ux-x$ and $xu-x \in L^2(S)$ by definition of $L^2(S)$.

LEMMA 6.3. *Suppose that* R *satisfies condition* (A) *and that* O^2 *is dense. Then for each* $S \in S(R)$ *there exist* $q \in R$, $u \in L(R)$ *such that* $q \equiv 1(S)$, *and* qO^2u.

Proof. Since the hull of $L^2(R)$ is empty, there exists $f \notin S$ and $u' \in R$ such that fO^2u'. Since O^2 is dense, there exists q, $u \in R$ such that $fO^2qO^2uO^2u'$. Thus $u \in L(R)$. Further, fO^2q implies fOq, so $q \equiv 1(N(S))$ by Lemma 3.5. Since $N(S) \subset S$, $q \equiv 1(S)$ so the proof of the lemma is complete.

The proofs of the following results possess many similarities with their counterparts in [3]. Indeed, it would be possible to give a list of modifications necessary to make arguments there valid under present conditions. However, the list is long so it seems advisable to include most details.

THEOREM 6.4. *Suppose that* R *satisfies condition* (A) *and that* O^2 *is dense. Then for each* $S \in \mathfrak{S}(R)$, $L^2(S)$ *is a maximal* O*-ideal.*

Proof. We have that $L^2(R)$ and $L^2(S)$ are O-ideals by Theorem 5.2. Thus, since $L(R) = L^2(R)$, $L(R)$ is an O-ideal and $L^2(S)$ is a proper O-ideal. For if $L^2(S)$ is not a proper O-ideal then $L^2(S) = L(R)$, so $L(R) \subset L^2(S) \subset S$, which is contrary to condition (A).

Suppose that M is an O-ideal which contains $L(S)$. If $M \subset S$ then $M = L^2(S)$ since $L^2(S) \subset M = L^2(M) \subset L^2(S)$.

If $M \not\subset S$ then there exist elements f, q, e_1, e_2, u_1, and $u_2 \in M\backslash S$ such that $fO^2qO^2e_1O^2e_2$ and $e_2O^2u_1O^2u_2$. Now on the one hand, we have seen that $f \notin S$ and fO^2q imply $q \equiv 1(S)$. On the other hand, $(u_1-e_2)O^2(u_2-e_1)$ by theorem 6.1. Now $q(u_2-e_1) = 0 \in S$. Since S is a prime ideal, S is a prime-like (O^2) ideal by Theorem 4.4. Therefore $u_2-e_1 \in S$, so $u_1-e_2 \in L^2(S) \subset M$. However, $e_2 \in M$, so $u_1 \in M$. But this implies $M = L(R)$. For let $x \in L(R)$. Therefore $x \in L^2(R)$, so by Lemma 6.2, $u_1x-x \in L^2(S) \subset M$. Since $u_1 \in M$, $u_1x \in M$, and hence $x \in M$. Thus $M = L(R)$, so $L^2(S)$ is not contained in any other proper O-ideal. Hence $L^2(S)$ is a maximal O-ideal.

The previous theorem can be extended to yield the result $\mathfrak{M}(R) = \mathfrak{L}_0(R)$ with the aid of the following

Theorem 6.5. *Suppose that R satisfies condition* (A) *and that O^2 is dense. Let P be an ideal in R which is prime-like* (O^2) *with respect to $L(R)$. Then $L^2(P) = L(R)$, or $L^2(P)$ is a maximal O-ideal. Hence if $\mathfrak{S}(R)$ is not empty, every such P contains a maximal O-ideal.*

Proof. According to Theorem 5.3, either $P \subset S$ for some $S \in \mathfrak{S}(R)$, or $L(R) \subset P$. In the first case, $P \supset L^2(S)$. For otherwise there exist elements f, e_1 and $e_2 \in L^2(S)\setminus P$ such that $fO^2e_1O^2e_2$. By Lemma 6.3, there exists $q \in R$, $u' \in L(R)$ such that $q \equiv 1(S)$ and qO^2u'. Since $L(R)$ is an O-ideal and $e_2 \in L^2(S) \subset L(R)$, there exists $u \in R$ such that $u'O^2u$ and e_2O^2u. Then $e_1(u-e_2) = 0 \in P$. Since $e_1 \in L(R)$, we have $u-e_2 \in P \subset S$. Therefore, $u \in S$, since $e_2 \in L^2(S) \subset S$. This is a contradiction, since $q \equiv 1(S)$ and qO^2u imply $u \equiv 1(S)$. We conclude that in this case, $L^2(S) \supset L^2(P) \supset L^2(S)$, so $L^2(P)$ is a maximal O-ideal by the previous theorem.

In the second case, $L(R) \subset P$. Therefore $L(R) = L^2(P)$ since $L(R) = L^2(R)$. If $\mathfrak{S}(R)$ is not empty then R contains maximal O-ideals by the previous theorem again, and these are always contained in $L(R)$. Thus in this case too, P contains a maximal O-ideal.

One immediate consequence of Theorem 6.5 is that every proper O-ideal which is prime-like (O^2) with respect to $L(R)$ is maximal. An argument similar to one employed in [3], § 4, shows that the converse holds. Since some of the details must be changed, we include the complete proof of this latter statement.

Theorem 6.6. *Suppose that R satisfies condition* (A) *and that O^2 is dense. Then the class of proper O-ideals which are prime-like* (O^2) *with respect to $L(R)$ coincides with $\mathfrak{M}(R)$.*

Proof. It only remains to show that every maximal O-ideal M is prime-like (O^2) with respect to $L(R)$. Suppose $f \notin M$ and fO^2e. It is easy to see that the set $\{x \in R\colon xO^2e\}$, which we might here call $L^2(e)$, is an O-ideal which includes f. By virtue of Theorem 4.2, $M+L^2(e)$ is an O-ideal. Since M is maximal, $M+L^2(e) = L(R)$. By Theorem 5.10, $M = L^2(S)$ for some $S \in \mathfrak{S}(R)$. By Lemma 6.3 there are elements $q \in R$ and $u \in L(R)$ such that $q \equiv 1(S)$ and qO^2u. Thus for some $m \in M$ and $e' \in L^2(e)$, $m+e' = u$.

Now suppose $k \in L(R)$ and $ke \in M$. Then $ke' \in M$, since $e' = ee'$, so $ku \in M$. Since $k \in L(R)$ and $L(R) = L^2(R)$, $ku-k \in L^2(S) = M$ by Lemma 6.2. Hence $k \in M$ and we have shown that M is prime-like (O^2) with respect to $L(R)$.

Finally we have the long promised

Theorem 6.7. *Assume condition* (A) *and that O^2 is dense. Then $\mathfrak{M}(R) = \mathfrak{L}_0(R)$ so $M(R)$ is dual Stone compact.*

Proof. We observed earlier in this section that $\mathfrak{L}_0(R)$ is dual Stone compact so the equality $\mathfrak{M}(R) = \mathfrak{L}_0(R)$ yields the final conclusion immediately.

If $M \in \mathfrak{L}_0(R)$ then $M = L^2(P)$ for some proper ideal P which is prime-like (O^2) with respect to $L(R)$ and further $M \neq L(R)$. Therefore by Theorem 6.5, $L^2(P) \in \mathfrak{M}(R)$ so $M \in \mathfrak{M}(R)$. Conversely, if $M \in \mathfrak{M}(R)$ then $M \neq L(R)$ and $M = L^2(S)$ for some $S \in \mathfrak{S}(R)$ by Theorem 5.10. Since S is prime, we have that S is prime-like by Theorem 4.2 so $S \in \mathfrak{P}_2(R)$. Thus $M \in \mathfrak{L}_0(R)$ and the proof is complete.

The following theorem and its corollary are analogues, respectively, of Theorem 4.25 and its corollary in [3]. The proofs there are applicable here with a few changes which we indicate.

Theorem 6.8. *Suppose that R is a ring such that $\mathfrak{M}(R)_\Delta$ is compact and $\bigcap \mathfrak{M}(R)$ is zero. Then for every ideal $J \subset R$, $J = \bigcap\{M+J\colon M \in \mathfrak{M}(R)\}$.*

Proof. If $f \in \bigcap\{M+J\colon M \in \mathfrak{M}(R)\}$ then for each $M \in \mathfrak{M}(R)$, there exists $m \in M$ and $g \in J$ such that $f = m+g$. Thus there exists $e \in M$ such that $(f-g)e = f-g$, or more briefly, $f \in fe+J$. Hence $\mathfrak{M}(R) \subset \bigcap\{\mathfrak{O}(e)\colon f \in fe+J\}$. The compactness of $M(R)_\Delta$ yields finite sets $e_1, \ldots, e_n \in R$ and $g_1, \ldots, g_n \in J$ such that $f = fe_i+g_i$ for $i = 1, \ldots, n$ and every $M \in \mathfrak{M}(R)$ contains some e_i. The argument given in [3] is now applicable since we have assumed that the only element belonging to every $M \in \mathfrak{M}(R)$ is zero. Thus $f \in J$. The reverse inclusion is immediate.

Corollary 6.8.1. *Suppose that R satisfies the conditions of the previous theorem. If J is an O-ideal then $J = \bigcap\{M\colon M \in \mathfrak{M}(R)$ and $J \subset M\}$.*

Proof. As in the proof in [3], we can write $J = I' \cap I''$ where $I' = \bigcap\{M+J\colon M \in \mathfrak{M}(R)$ and $J \subset M\}$, and $I'' = \bigcap\{M+J\colon M \in \mathfrak{M}(R)$ and $J \not\subset M\}$. Now if M is an O-ideal then $M+J$ is an O-ideal by Theorem 4.2 above, so if M is a maximal O-ideal and $M \not\supset J$ then $M+J = L^n(R)$ for some positive integer n. The integer is the same regardless of the particular M so $I'' = L^n(R)$ and $I' \cap I'' = I'$. Finally if $J \subset M$ then $M+J = M$, so the desired conclusion follows.

7. s. s. s. G. S. rings.

If R is a s. s. s. G. S. algebra in the sense of [12] then R has a Hausdorff structure space, and, by Theorem 3.10 above, is also a l. i. ring. Accordingly we adopt

Definition 7.1. A l. i. ring which has a Hausdorff structure space is termed a G. S. *ring*.

Unless the contrary is mentioned, we assume in the remainder of this section that R denotes a fixed s. s. s. G. S. ring.

In addition to being a locally compact Hausdorff space, $\mathfrak{S}(R)$ is of course a regular space. This fact enters the proof of the following lemma and two subsequent theorems in an entirely familiar way, enabling us to abbreviate the proofs of the theorems somewhat.

LEMMA 7.2. *Suppose that R is a* s. s. s. G. S. *ring. If $f \in L(R)$ and $\mathfrak{F} \subset S(R)$ is closed and disjoint from $\mathfrak{N}(f)^-$ then there exists $e \in L(R)$ such that $\mathfrak{N}(f)^- \cap \mathfrak{E}(e)$ and $\mathfrak{N}(e)^- \cap \mathfrak{F} = \emptyset$.*

Proof. There are open sets $\mathfrak{U}$ and $\mathfrak{V}$ with compact closures such that $\mathfrak{N}(f)^- \subset \mathfrak{U} \subset \mathfrak{U}^- \subset \mathfrak{V}$ and $\mathfrak{V}^- \cap \mathfrak{F} = \emptyset$. By Lemma 3.14, there exists $e \in R$ such that $e \equiv 1(\mathfrak{U}^-)$ and $e \equiv 0(\mathfrak{S}(R)\backslash\mathfrak{V})$, so $\mathfrak{N}(f)^- \subset \mathfrak{E}(e)$ and $\mathfrak{N}(e)^- \subset \mathfrak{F} = \emptyset$. In particular, e has compact support, so, by the second conslusion of Theorem 5.6, $e \in L(R)$.

According to Theorem 5.7, $L(R) = L^2(R)$. Therefore R satisfies condition (A)$'$ and, hence, condition (A) of the previous section. Thus the following theorem implies that all of the results of that section are valid here.

THEOREM 7.3. *Suppose that R is a* s. s. s. G. S. *ring. Then O^2 is dense.*

Proof. If fO^2e then there exists $e' \in R$ such that $fOe'Oe$. Thus $\mathfrak{N}(f)^- \subset \mathfrak{E}(e') \subset \mathfrak{N}(e') \subset \mathfrak{E}(e)$, so $\mathfrak{N}(f)^- \subset \mathfrak{E}(e)^0$. Now apply the lemma three times, beginning with the pair $\mathfrak{N}(f)^-$ and $\mathfrak{F} = \mathfrak{S}(R)\backslash\mathfrak{E}(e)^0$. This yields elements e_1, e_2 and e_3 such that $fOe_1Oe_2Oe_3Oe$. Thus $fO^2e_2O^2e$, so O^2 is dense.

In particular, the hypotheses of Theorem 6.8 are satisfied.

THEOREM 7.4. *Suppose that R is a* s. s. s. G. S. *ring. Then $\mathfrak{M}(R)_\Delta$ is compact and $\bigcap \mathfrak{M}(R)$ is zero.*

Proof. The compactness of $M(R)_\Delta$ is immediate from Theorem 6.7. If $f \neq 0$ then there exists $S \in \mathfrak{S}(R)$ such that $f \notin S$, since R is a s. s. s. ring. Certainly $f \notin L^2(S)$, so, since $L^2(S) \in \mathfrak{M}(R)$ by Theorem 6.4, $f \notin \bigcap \mathfrak{M}(R)$. Hence $\bigcap \mathfrak{M}(R)$ is zero.

Actually, the set $L^2(S)$ can be replaced by the simpler set $L(S)$ in this setting. More generally, we have

THEOREM 7.5. *Suppose that R is a* s. s. s. G. S. *ring. If J is a kernel then $L(J) = L^2(J)$, so $L(J)$ is an O-ideal.*

Proof. Let $\mathfrak{F} = h(J)$ and suppose fOe with $e \in J$. Obviously $\mathfrak{N}(f)^- \cap \mathfrak{F} = \emptyset$, and since $f \in L(R)$, $\mathfrak{N}(f)^-$ is compact by Theorem 5.6. By Lemma 7.1, applied twice, there exist elements $e_1, e_2 \in R$ such that $\mathfrak{N}(f)^- \subset \mathfrak{E}(e_1)$, $\mathfrak{N}(e_1)^- \subset \mathfrak{E}(e_2)$ and $\mathfrak{N}(e_2) \cap \mathfrak{F} = \emptyset$. The two inclusions yield fOe_1Oe_2 by Lemma 3.6, and the final assertion implies $e_2 \equiv 0(\mathfrak{F})$. Thus we have both fO^2e_2 and $e_2 \in J$, so $f \in L^2(J)$. Hence $L(J) \subset L^2(J)$. The reverse inclusion always holds, so the theorem is proved.

Now recall Theorem 5.6 above and the discussion just preceding it. As a consequence of that theorem and Theorem 7.5, we can assert that

if $\mathfrak{F}$ is a hull then $j(\mathfrak{F})$ and, hence, $\mathfrak{J}(\mathfrak{F})$ are O-ideals. In addition, setting $J = k(\mathfrak{F})$, we have $h(L(J)) = h(J) = \mathfrak{F}$. For the proof of Theorem 1.2 (b) of [12] is clearly valid for s. s. s. G. S. rings, so $h(\mathfrak{J}(\mathfrak{F})) = \mathfrak{F}$. By Theorem 5.6, $L(J) = \mathfrak{J}(\mathfrak{F})$, so $h(L(J)) = \mathfrak{F}$. We have proved

THEOREM 7.6. *Suppose that R is a* s. s. s. G. S. *ring. Suppose $\mathfrak{F}$ is a hull and let $J = k(\mathfrak{F})$. Then $j(\mathfrak{F})$ $(= \mathfrak{J}(\mathfrak{F}))$ is the O-ideal $L(J)$ and $h(L(J)) = \mathfrak{F}$.*

We also have the following sharpened version of Lemma 3.14.

THEOREM 7.7. *Suppose that R is a* s. s. s. G. S. *ring. If $\mathfrak{F} \subset \mathfrak{S}(R)$ is compact and disjoint from the hull $h(J)$ of an ideal J then there exists $e \in J$ such that $e \equiv 1(J(\mathfrak{F}))$.*

Proof. Since $\mathfrak{J}(\mathfrak{F}) \subset J(\mathfrak{F}) \subset k(\mathfrak{F})$, it follows from the previous theorem that $h(J(\mathfrak{F})) = \mathfrak{F}$. According to Theorem 3.11, $J(\mathfrak{F})$ is a regular ideal. Thus $J(\mathfrak{F})$ is a regular ideal whose hull is disjoint from the hull of J. Hence by the lemma of [9], p. 62, which we have used once before, there exists $e \in J$ such that $e \equiv 1(J(\mathfrak{F}))$.

An element $f \in R$ *belongs locally* to an ideal J at $S \in \mathfrak{S}(R)$ provided there exists a neighborhood $\mathfrak{U}$ of S and $g \in J$ such that $f \equiv g(\mathfrak{U})$ ([9], p. 85; [12], definition 2.2). This is evidently the same as saying $f \in N(S)+J$. In like manner, f belongs locally to J "at infinity" if $f \in L(R)+J$. Theorem 6.8 above yields an alternate proof of Theorem 25 E of [9] and Theorem 2.2 of [12].

THEOREM 7.8. *Suppose that R is a* s. s. s. G. S. *ring. Suppose that J is an ideal and $f \in R$. If f belongs locally to J at every point of $h(J)$ and at infinity then $f \in J$.*

Proof. We shall show that these hypotheses on f imply $f \in L(S)+J$ for every $S \in \mathfrak{S}(R)$. As we have seen, $L(S) \in \mathfrak{M}(R)$, and since $L(S) = L^2(S)$, every member of $\mathfrak{M}(R)$ has this form by Theorem 5.10. We will have shown then that $f \in M+J$ for every $M \in \mathfrak{M}(R)$. By Theorem 7.4, the hypotheses of Theorem 6.8 are satisfied, so by the latter theorem, $f \in J$.

Note first that if $S \in \mathfrak{S}(R)$ and $f \in (N(S)+J) \cap (L(R)+J)$ then $f \in L(S)+J$. For there exist elements g, $g' \in J$, $s \in N(S)$ and $s' \in L(R)$ such that $f = s+g$ and $f = s'+g'$. There exists $e \in L(R)$ such that $(f-g')e = f-g'$, or $f = (f-g')e+g'$. Using the equality $f = s-g$, we have $f = (s-g-g')e+g' = se+g''$ where $g'' \in J$. But $se \in N(S) \cap L(R)$, so $se \in L(S)$ by Corollary 5.6.1. Hence $f \in L(S)+J$.

Now the hypotheses on f imply $f \in (N(S)+J) \cap (L(R)+J)$ for every $S \in h(J)$. If $S \notin h(J)$ then $J \not\subset S$. By Theorem 3.11, $N(S)$ is a regular ideal and as we have mentioned before, the kernel of $N(S)$ is $\{S\}$. Therefore $N(S)+J = R$. But we still have $f \in L(R)+J$, so by the result in the previous paragraph, $f \in L(S)+J$, and the proof of the theorem is complete.

We list next a set of conditions which, in the presence of strong-semi-simplicity, are equivalent to the assumption that R is a G. S. ring. Since a ring with identity is automatically a l. i. ring, the significance of the assumption that $\mathfrak{S}(R)$ is a Hausdorff space is thrown into sharper focus. To avoid trivialities we assume that $\mathfrak{S}(R)$ contains at least two elements.

THEOREM 7.9. *Suppose that R is a* s. s. s. *ring. Then these are equivalent*:

(1) *R is a* G. S. *ring.*

(2) *Every member of $\mathfrak{S}(R)$ contains a maximal O-ideal and every maximal O-ideal is contained in a unique member of $\mathfrak{S}(R)$.*

(3) *For every $S \in \mathfrak{S}(R)$, the hull of $L(S)$ is $\{S\}$.*

(4) *As restricted to $\mathfrak{S}(R)$, L is a one-to-one map into $\mathfrak{M}(R)$.*

Proof. (1)$\Rightarrow$(2): The first part of (2) follows from Theorem 6.4. The second assertion follows from theorems 5.10, 6.7 and 7.6.

(2)$\Rightarrow$(3): Suppose $S \in \mathfrak{S}(R)$. By hypothesis, S contains a maximal O-ideal $M(S)$ which is contained in no other member of $\mathfrak{S}(R)$. Now $M(S) \subset L(S) \subset S$, so the hull of $L(S)$ is $\{S\}$.

(3)$\Rightarrow$(1): First observe that R is a l. i. ring. For let $S \in \mathfrak{S}(R)$. By the assumption listed above, there exists $S' \in \mathfrak{S}(R)$ such that $S' \neq S$. By hypothesis, $L(S') \not\subset S$, so there exist elements f, $e \in S' \backslash S$ such that $f O e$. Hence R is a l. i. ring. The hypotheses of Corollary 5.6.1 are now satisfied, so $L(S) \subset N(S)$. Using (3) again, the hull of $L(S)$ is $\{S\}$ so the hull of $N(S)$ is $\{S\}$. By Theorem 3.1 of [2], $\mathfrak{S}(R)$ is a Hausdorff space, so R is a G. S. ring.

(1)$\Rightarrow$(4): The assertion that L maps $\mathfrak{S}(R)$ into $\mathfrak{M}(R)$ follows from Theorem 6.4 and Theorem 7.5. That L is one-to-one follows from Theorem 7.6.

(4)$\Rightarrow$(3): Let $S, T \in \mathfrak{S}(R)$ and suppose $L(S) \subset T$. Since L maps $\mathfrak{S}(R)$ into $\mathfrak{M}(R)$, $L(S)$ and $L(T)$ are maximal O-ideals, so $L(S) = L(T)$. Therefore, $S = T$ since L is one-to-one, so the hull of $L(S)$ is $\{S\}$.

The proof of the theorem is complete.

COROLLARY 7.9.1. *Suppose that R is a* s. s. s. *ring such that $L(R) = R$. (For example, suppose that R is a* s. s. s. *ring with identity.) Then each of the conditions* (2)-(4) *of the theorem is equivalent to the assertion that $\mathfrak{S}(R)$ is a Hausdorff space.*

Proof. This immediate, since if $L(R) = R$ then $h(L(R)) = \emptyset$, so R is a l. i. ring by Theorem 5.4.

Remark 7.10. We mentioned in section 3 that the homomorphic image of a strongly-semi-simple l. i. ring is also such a ring. As in the proof of Theorem 3.2 of [2], it is apparent that the homomorphic image of a ring with Hausdorff structure space also has a Hausdorff structure space. Using the previous corollary, we see that the ring Z of integers, for example, is "quite far" from being the homomorphic image of a ring with Hausdorff structure space. For if so then Z would have to contain enough relative identities to distinguish between each pair of maximal ideals.

8. The topology Γ on $\mathfrak{M}(R)$.

If R is a s. s. s. G. S. ring then L is a one-to-one function from $\mathfrak{S}(R)$ to $\mathfrak{M}(R)$. However, L is not a homeomorphism between the structure space and $\mathfrak{M}(R)_\Delta$ in general. For $\mathfrak{M}(R)_\Delta$ is compact, while, by Theorem 3.13, $\mathfrak{S}(R)$ is compact if and only if R has an identity. We now define a topology on $\mathfrak{M}(R)$ by modifying the closure operation of Δ. With this topology, L becomes a homeomorphism. We continue to use "$^-$" to denote both Stone and dual Stone closure. Of course, if $\mathfrak{A} \subset \mathfrak{S}(R)$, $\mathfrak{A}^-$ is the closure of $\mathfrak{A}$ in $\mathfrak{S}(R)$, while if $\mathfrak{A} \subset \mathfrak{M}(R)$ then $\mathfrak{A}^-$ is the Δ-closure of $\mathfrak{A}$ in $\mathfrak{M}(R)$.

DEFINITION 8.1. If $\mathfrak{B} \subset \mathfrak{M}(R)$ then $\mathfrak{B}^c$ is defined to be the collection of all $M \in \mathfrak{M}(R)$ such that for some $g \in L(R)$, depending on M, $M \in \big(\mathfrak{B} \backslash \mathfrak{D}(g)\big)^-$. (Recall that $\mathfrak{D}(g) = \{M' \in \mathfrak{M}(R):\ g \in M'\}$.)

It is perhaps not obvious that c is a closure operation for a topology. We prove that this is so in the process of proving the following theorem. In anticipation of this, we denote the resulting topology by Γ and the corresponding topological space by $\mathfrak{M}(R)_\Gamma$.

THEOREM 8.2. *Suppose that R is a* s. s. s. G. S. *ring. Then the mapping L: $\mathfrak{S}(R) \to \mathfrak{M}(R)_\Gamma$ is a homeomorphism.*

Proof. We show first that if $\mathfrak{F} \subset \mathfrak{S}(R)$ is compact, $\mathfrak{A} \subset \mathfrak{F}$ and $S_o \in \mathfrak{A}^-$ then $L(S_o) \in L(\mathfrak{A})^c$. Next we show that if $L(S_o) \in L(\mathfrak{A})^c$ ($\mathfrak{A} \subset \mathfrak{S}(R)$ now arbitrary) then $S \in \mathfrak{A}^-$. Since $\mathfrak{S}(R)$ is locally compact and since c is obviously a monotone function, it follows that for arbitrary $\mathfrak{A} \subset \mathfrak{S}(R)$, $L(\mathfrak{A}^-) = L(\mathfrak{A})^c$. Thus c is a closure operation in a topology Γ for $\mathfrak{M}(R)$ and L is a homeomorphism.

Now suppose that $\mathfrak{F} \subset \mathfrak{S}(R)$ is compact and $S_o \in \mathfrak{A}^- \subset \mathfrak{F}$. By Theorem 7.7, there exists $e \in L(R)$ such that $e \equiv 1\big(N(S)\big)$ for all $S \in \mathfrak{A}^-$. Obviously $\mathfrak{A} = \{S \in \mathfrak{S}(R): S \in \mathfrak{A}$ and $e \notin L(S)\}$ so $L(\mathfrak{A}) = L(\mathfrak{A}) \backslash \mathfrak{D}(e)$. Suppose $L(S) \notin L(\mathfrak{A})^c$. Then there are elements f, e_1 and $e_2 \in L(S_o) \backslash \bigcup L(\mathfrak{A})$ such that $f O^2 e_1 O^2 e_2$. Now $e_1(e-e_2) = e_1 e - e_1 \in \cap\{N(S):\ S \in \mathfrak{A}\}$. Since $f \in L(R)$ but $f \notin \bigcup L(\mathfrak{A})$, we have $f \notin \bigcup\{N(S):\ S \in \mathfrak{A}\}$. Each $N(S)$ is prime-like (O^2) by Theorem 4.5, so $e - e_2 \in \bigcap\{N(S):\ S \in \mathfrak{A}\} \subset \bigcap\{S:\ S \in \mathfrak{A}\} \subset S_o$. However,

$e_2 \in L(S_o) \subset S_o$ so $e \in S_o$, which is a contradiction since $e \equiv 1(S_o)$. Hence $L(S_o) \in L(\mathfrak{A})^c$.

Now let $\mathfrak{A} \subset \mathfrak{S}(R)$ be arbitrary and suppose $L(S_o) \in L(\mathfrak{A})^c$. Then there exists $g \in L(R)$ such that $L(S_o) \subset \bigcup (L(\mathfrak{A}) \setminus \mathfrak{D}(g))$. Assume $S_o \notin \mathfrak{A}^-$, and let $J = k(\mathfrak{A}^-)$. By Theorem 7.6, $h(L(J)) = \mathfrak{A}^-$, and as we have seen before, $h(N(S_o)) = \{S_o\}$. Therefore by Theorem 7.7, there exists $e \in L(J)$ such that $e \equiv 1(N(S_o))$. In particular, for each $S \in \mathfrak{A}^-$, $e \in L(S)$ since $L(J) \subset S$. Also $eg - g \in L(R) \cap N(S_o) = L(S_o)$. Since $L(S_o) \subset \bigcup (L(\mathfrak{A}) \setminus \mathfrak{D}(g))$, there exists $S \in \mathfrak{A}$ such that $eg - g \in L(S)$ and $g \notin L(S)$. However, if $S \in \mathfrak{A}$ then $e \in L(S)$, so $eg \in L(S)$. Hence $g \in L(S)$ which is a contradiction. We conclude that $S_o \in \mathfrak{A}^-$ and the proof of the theorem is complete.

For every $\mathfrak{B} \subset \mathfrak{M}(R)$, $\mathfrak{B}^c \subset \mathfrak{B}^-$, so $\Delta \subset \Gamma$. If R has an identity 1 then $\Delta = \Gamma$ since we can take $g = 1$ in definition 8.1. Suppose, conversely, that $\Delta = \Gamma$. Then $\mathfrak{M}(R)_\Gamma$ is compact, so, by the theorem, $\mathfrak{S}(R)$ is compact. Hence, by Theorem 3.13, R has an identity. We have proved

THEOREM 8.3. *Suppose that R is a* s. s. s. G. S. *ring. Then $\Delta \subset \Gamma$, and equality holds if and only if R has an identity.*

From the definition of Γ, it is apparent that if R_1 and R_2 are two s. s. s. G. S. rings which are multiplicatively isomorphic then $M(R_1)_\Gamma$ and $M(R_2)_\Gamma$ are homeomorphic. Hence we have the following theorem. This theorem generalizes Corollary 3.3 of [1] since, by Example 1 of [12] a commutative regular Banach algebra is a G. S. algebra.

THEOREM 8.4. *Is R_1 and R_2 are* s. s. s. G. S. *rings whose multiplicative semigroups are isomorphic then their structure spaces are homeomorphic.*

In the next section we see that an even stronger result is true.

9. Dual O-ideals. The notion of dual O-ideal is to some extent of independent interest, but it is used here to show that if R_1 and R_2 are s. s. s. G. S. rings and χ is a multiplicative isomorphism between them, then $\chi(S) \in \mathfrak{S}(R_2)$ for each $S \in \mathfrak{S}(R_1)$. This provides an extension of a part of Corollary 2.3 of [4].

DEFINITION 9.1. A *dual O-ideal* in a ring R is an O^{-1}-ideal, where $O^{-1} = \{(e, f) \colon fOe\}$. We shall refer to a dual O-ideal more briefly as a *K-set*.

Thus a subset $K \subset R$ is a K-set provided

(1) $k \in K$ and kOk' imply $k' \in K$,

(2) $k_1, k_2 \in K$ imply there exists $m \in K$ such that mOk_1 and mOk_2 [3].

Since the relation O is transitive, an induction argument shows that if $k_1, \ldots, k_n$ is any finite subset of a K-set K then there exists $m \in K$ such that mOk_i, for $i = 1, \ldots, n$. Since every element of a ring is a relative identity for the zero element, it follows from condition (1) that if a K-set K contains the zero element then $K = R$.

By a *proper* K-set is meant simply a K-set which is a proper subset of R. A *maximal* K-set is a proper K-set which is contained in no other such K-set.

One source of K-sets is the collection of sets $K(S) = \{k \in R\colon S \in \mathfrak{E}(k)^o\}$ where $S \in \mathfrak{S}(R)$.

THEOREM 9.2. *Suppose that R is a* s. s. s. G. S. *ring. If $S \in \mathfrak{S}(R)$ then $K(S)$ is a maximal K-set.*

Proof. Suppose that $k \in K(S)$ and kOk'. Now kOk' implies $\mathfrak{N}(k) \subset \mathfrak{E}(k')$, by Lemma 3.6. Since $k \in K(S)$, $S \in \mathfrak{E}(k)^o$, so $S \in \mathfrak{E}(k')^o$. Thus $k' \in K(S)$, so $K(S)$ satisfies (1) above.

If $k_1, k_2 \in K(S)$, then there exists an open set $\mathfrak{U} \subset \mathfrak{S}(R)$ such that $S \in \mathfrak{U} \subset \mathfrak{E}(k_1) \cap \mathfrak{E}(k_2)$. Now there exists an open set $\mathfrak{V}$ with compact closure such that $S \in \mathfrak{V} \subset \mathfrak{V}^- \subset \mathfrak{U}$. By Lemma 3.14, there exists $m \in R$ such that $m \equiv 1(\mathfrak{V}^-)$ and $m \equiv 0(\mathfrak{S}(R) \setminus \mathfrak{U})$. By definition of $K(S)$, $m \in K(S)$, and by Lemma 3.6, mOk_1 and mOk_2. Hence $K(S)$ is a K-set.

Next assume that K is a K-set which properly contains $K(S)$. Then there exists $k \in K$ such that $S \in (\mathfrak{S}(R) \setminus \mathfrak{E}(k))^-$. Now there are elements $m_1, m_2 \in K$ such that $m_1 O m_2 O k$, so $\mathfrak{N}(m_1)^- \subset \mathfrak{E}(m_2) \subset \mathfrak{N}(m_2) \subset \mathfrak{E}(k)$. It follows that $S \in \mathfrak{Z}(m_1)^o$, so $m_1 \in N(S)$. Since $m_1 \in L(R)$, we have $m_1 \in L(S)$. By using Lemma 3.14 and the local compactness of $S(R)$ we can, in a now familiar fashion, find $k' \in K(S)$ such that $\mathfrak{N}(k') \subset \mathfrak{Z}(m_1)^o$. There must exist $m \in K$ such that mOk' and mOm_1. Thus $\mathfrak{N}(m) \subset \mathfrak{E}(k') \cap \mathfrak{E}(m_1)$ $\subset N(k') \cap N(m_1) = \emptyset$. By the strong-semi-simplicity of R, $m = 0$. That is, $0 \in K$, so $K = R$. Hence $K(S)$ is a maximal K-set.

THEOREM 9.3. *Suppose that R is a* s. s. s. G. S. *ring. If K is a proper K-set then $K \subset K(S)$ for some $S \in \mathfrak{S}(R)$. Hence if K is a maximal K-set then $K = K(S)$.*

Proof. It is evident that if mOk then $\mathfrak{E}(m) \subset \mathfrak{E}(k)^o$ since $\mathfrak{E}(m) \subset \mathfrak{N}(m) \subset \mathfrak{E}(k)$. Now if $k \in K$ then $\mathfrak{E}(k) \neq \emptyset$, for there is $m \in K$ such that mOk. Since K is a proper K-set, $m \neq 0$. Thus $\mathfrak{N}(m) \neq \emptyset$ since R is a s. s. s. ring, and hence $\mathfrak{E}(k) \neq \emptyset$. If $k_1, \ldots, k_n$ is any finite subset of K then there is $m \in K$ such that mOk_i for $i = 1, \ldots, n$, so $\mathfrak{E}(m) \subset \bigcap_{i=1}^{n} \mathfrak{E}(k_i)^o$. Now $k'Ok$ implies $k' \in L(R)$. Thus $\mathfrak{N}(k')^-$ and, hence, $\mathfrak{E}(k')$ are compact. Therefore there exists $k' \in K$ such that $\mathfrak{E}(k')$ is compact. Hence there exists $S \in \mathfrak{S}(R)$ such that $S \in \bigcap \{\mathfrak{E}(k)\colon k \in K\}$. According to above results, $S \in \mathfrak{E}(k)^o$ for each $k \in K$, so $K \subset K(S)$.

It is an easy matter, given a maximal K-set K and $S \in \mathfrak{S}(R)$, to decide when $K = K(S)$.

THEOREM 9.4. *Suppose that R is a* s. s. s. G. S. *ring. Let K be a maximal K-set in R and let $S \in \mathfrak{S}(R)$. Then $K = K(S)$ if and only if $K \cap L(S) = \emptyset$.*

Proof. By the previous theorem, $K = K(T)$ for some $T \in \mathfrak{S}(R)$. If $S = T$ it is obvious that $K(T) \cap L(S) = \emptyset$. If $S \neq T$ then by combining the fact that $\mathfrak{S}(R)$ is a locally compact Hausdorff space with Lemma 3.14, we can obtain $e \in L(S)$ such that $T \in \mathfrak{E}(e)^o$. Hence $L(S) \cap K(T) \neq \emptyset$.

We conclude the preliminaries to the main theorem of this section with the

LEMMA 9.5. *Suppose that R is a* s. s. s. G. S. *ring and let $S \in \mathfrak{S}(R)$. Then $R \backslash S = \{f \in R\colon [f] \cap K(S) \neq \emptyset\}$.*

Proof. Certainly if $[f] \cap K(S) \neq \emptyset$ then $f \notin S$. Suppose, conversely, that $f \notin S$. As we have seen, $N(S)$ is a regular ideal whose hull is $\{S\}$, so $[f] + N(S) = R$. Now there exists $k \in K(S)$, so there are elements $g \in [f]$ and $s \in N(S)$ such that $g + s = k$. Since $\mathfrak{E}(k) \cap \mathfrak{Z}(s) \subset \mathfrak{E}(k-s)$, we have $g = k - s \in K(S)$, so $[f] \cap K(S) \neq \emptyset$.

Finally we have

THEOREM 9.6. *Suppose that R_1 and R_2 are* s. s. s. G. S. *rings and suppose that $\chi\colon R_1 \to R_2$ is a multiplicative isomorphism onto. If $S_1 \in \mathfrak{S}(R_1)$ then $\chi(S_1) \in \mathfrak{S}(R_2)$.*

Proof. We know that $L(S_1)$ is a maximal O-ideal in R_1 and $K(S_1)$ is a maximal K-set in R_1. Therefore $\chi(L(S_1))$ is a maximal O-ideal in R_2 and $\chi(K(S_1))$ is a maximal K-set in R_2. Hence $\chi(L(S_1)) = L(S_2)$ for some $S_2 \in \mathfrak{S}(R_2)$ and $\chi(K(S_1)) = K(T_2)$ for some $T_2 \in \mathfrak{S}(R_2)$. Since $L(S_1) \cap K(S_1) = \emptyset$ we have $L(S_2) \cap K(T_2) = \emptyset$, so $S_2 = T_2$, by Theorem 9.4, and hence $\chi(K(S_1)) = K(S_2)$. Now $R_2 \backslash \chi(S_1) = \chi(R_1 \backslash S_1)$, so $R_2 \backslash \chi(S_1) = \{\chi(f_1)\colon \chi([f_1]) \cap \chi(K(S_1)) \neq \emptyset\}$, by the lemma. Evidently $\chi(f_1) = [\chi(f_1)]$, so $R_2 \backslash \chi(S_1) = \{f_2 \in R_2\colon [f_2] \cap K(S_2) \neq \emptyset\}$. Applying the lemma again, we see that the right hand set is $R_2 \backslash S_2$. Summarizing, $R_2 \backslash \chi(S_1) = R_2 \backslash S_2$, so $\chi(S_1) = S_2$ and the proof of the theorem is complete.

10. Concerning primary ideals.

Results in this section fall into two categories. The first theorem generalizes Theorem 1 of [5]. In the remainder of the section we show that a primary ideal of a s. s. s. G. S. ring is a s. s. s. G. S. ring and obtain as corollaries a well known result concerning $C_\infty(X)$ and a generalization of a part of Theorem 3.3 of [8].

In this paper, a *primary ideal* is an ideal which is contained in *at most* one member of $S(R)$. This meaning for the term is thus slightly more general than that in [9] or [12], for example.

THEOREM 10.1. *Suppose that R is a* s. s. s. G. S. *ring. Then for an ideal $P \subset R$ these are equivalent:*

(1) *P contains a maximal O-ideal.*

(2) *P is a primary ideal.*

(3) *P is a prime-like (O^2) ideal with respect to $L(R)$.*

Proof. (1)$\Rightarrow$(2): If M is a maximal O-ideal then $M = L(S)$ for some $S \in \mathfrak{S}(R)$. Now the hull of $L(S)$ is $\{S\}$, so every maximal O-ideal is primary. An ideal which contains a primary ideal is a primary ideal, so (1)$\Rightarrow$(2).

(2)$\Rightarrow$(1): Let P be a primary ideal and let S be an arbitrary maximal regular ideal. Then $P \subset S$ or $L(R) \subset L(S) + P$. For suppose $P \not\subset S$. Then $L(S) + P$ is contained in no member of $\mathfrak{S}(R)$ since $L(S)$ is a primary ideal. Therefore by Theorem 5.3, $L(R) \subset L(S) + P$. From Theorem 6.8 and the proof of Theorem 7.8, we infer that $P = \bigcap \{L(S) + P\colon S \in \mathfrak{S}(R)\}$. Thus if P is contained in no member of $\mathfrak{S}(R)$ then $P \supset L(R)$, while if $P \subset S_o$ for some $S_o \in \mathfrak{S}(R)$ then $P \supset (L(S_o) + P) \cap L(R) \supset L(S_o)$. In either case, P contains a maximal O-ideal.

(3)$\Rightarrow$(1): This follows from Theorem 6.5, since, as we noted in section 7, the hypotheses of that theorem are satisfied in s. s. s. G. S. rings.

(1)$\Rightarrow$(3): If P contains a maximal O-ideal then $P \supset L(S)$ for some $S \in \mathfrak{S}(R)$. Suppose $f \notin P$, fO^2e, $k \in L(R)$ and $ke \in P$. Now $f \in L(R)$, so $f \notin N(S)$ since $f \notin L(S)$. There exists $e' \in R$ such that $fOe'Oe$, and $e' \notin S$ since otherwise $f \in L(S) \subset P$. By Lemma 9.5, there exists $g \in [e']$ such that $g \equiv 1(N(S))$. Therefore $kg - k \in N(S) \cap L(R) = L(S) \subset P$. However, $kg = keg$, so $kg \in P$ and hence $k \in P$. We conclude that P is a prime-like (O^2) ideal with respect to $L(R)$.

Our next objective is to show that a primary ideal P of a s. s. s. G. S. ring R is a s. s. s. G. S. ring. The first step is to show that the correspondence $\varphi\colon M \to M \cap L(P)$ is one-to-one from $\mathfrak{M}(R) \backslash \{L(P)\}$ onto $\mathfrak{M}(P)$. The next series of lemmas, of some interest in themselves, lead up to this fact.

LEMMA 10.2. *Suppose that J is a prime-like (O^2) ideal with respect to $L(R)$, and I_1, I_2 are O-ideals such that $J \supset I_1 \cap I_2$. Then $J \supset I_1$ or $J \supset I_2$.*

Proof. Assume $J \not\supset I_1$ and let $f \in I_2$ be arbitrary. There are elements $e, g \in I_1 \backslash J$ such that gO^2e. Now $fe \in I_1 \cap I_2 \subset J$, and $f \in L(R)$. Therefore $f \in J$ since J is prime-like (O^2) with respect to $L(R)$. Hence $I_2 \subset J$.

LEMMA 10.3. *Suppose that I is a proper O-ideal and M_1, M_2 are maximal O-ideals. If $I \supset M_1 \cap M_2$ then $I \subset M_1$ or $I \subset M_2$. Hence, if I is a maximal O-ideal then $I = M_1$ or $I = M_2$.*

Proof. There is a maximal O-ideal M such that $M \supset I$, so $M \supset M_1 \cap M_2$. By Theorem 6.6, M is a prime-like (O^2) ideal with respect to $L(R)$, so by the previous lemma, we have $M \supset M_1$ or $M \supset M_2$. Since M_1 and M_2 are maximal, either $M = M_1$ or $M = M_2$, so $M_1 \supset I$ or $M_2 \supset I$.

In the future, we denote the fact that a given set A is a proper subset of some set B by writing $A < B$.

LEMMA 10.4. *Suppose that I is an O-ideal and M_1, M_2 are maximal O-ideals. If $M_1 \cap M_2 < I$ then $I = M_1$ or $I = M_2$.*

Proof. We have $I \subset M_1$ or $I \subset M_2$, by the previous lemma. However, not both inclusions can hold since the inclusion $I \supset M_1 \cap M_2$ is proper. Suppose $I \subset M_1$ but $I \not\subset M_2$. Then M_1 is the only maximal O-ideal containing I. For if M is a maximal O-ideal and $M \supset I$ then $M \supset M_1 \cap M_2$. By the previous lemma again, $M = M_1$ or $M = M_2$, and since $I \not\subset M_2$, we have $M = M_1$. By Corollary 6.8.1, I is the intersection of all the maximal O-ideals containing it. Hence, $I = M_1$.

Suppose that $J \subset R$ is a sub-ring of R and $I \subset J$. We say that I is an ideal, O-ideal, maximal O-ideal, etc. *in* J if I is an ideal, O-ideal, maximal O-ideal, etc. respectively, as a subset of the ring J. Obviously if I is an ideal (O-ideal) (in R) then I is an ideal (O-ideal) in J. Under certain conditions the converse holds:

LEMMA 10.5. *Let R be an arbitrary ring and suppose that J is an ideal. If $I \subset L(J)$ and I is an ideal in J then I is an ideal; if I is an O-ideal in J then I is an O-ideal.*

Proof. Let $f \in I$ and $h \in R$. By hypothesis, there exists $e \in J$ such that $f O e$. Hence $hf = e(hf) = (eh)f$ and $fh = (fh)e = f(he)$. Since J is an ideal, $eh \in J$ and $he \in J$. Since I is an ideal in J, $(eh)f \in I$ and $f(he) \in I$. Hence hf and $fh \in I$, so I is an ideal.

Next note that if $e \in J$ and e is a relative identity for f as an element in J, then e is a relative identity for f. For suppose g belongs to the principal ideal generated by f in R. If g has the form $g = xfy$, where x, y may denote either integers or elements of R, then $e(xfy) = e(x(efe)y) = e(xe)f(ey) = (xe)f(ey)$, since xe and $ey \in J$, and since e is a relative identity for f as an element of J. But $(xe)f(ey) = x(efe)y = xfy$, so $e(xfy) = xfy$. A similar argument shows that $(xfy)e = xfy$. Since members of $[f]$ are sums of such g, we see that e is a relative identity for f.

It now follows that if I is an O-ideal in J then $I \subset L(J)$, so I is an ideal. The previous paragraph also shows that I satisfies condition (1) of Theorem 4.1, so I is an O-ideal.

Remark 10.6. If J is an ideal in an arbitrary ring R, then according to the proof of the previous lemma, $L(J)$ is the same, whether J is thought of as a subset of R or as a ring in its own right.

Now let P_o be a fixed primary ideal in R. Let $L(P_o)$ be denoted by M_o and let P_o^* denote R, if P_o is contained in no member of $\mathfrak{S}(R)$, or, otherwise, the unique member of $\mathfrak{S}(R)$ which contains P_o. Let $\mathfrak{S}_o(R)$ denote the subset $\mathfrak{S}(R)\setminus\{P_o^*\}$ as well as the subspace of $\mathfrak{S}(R)$ formed with this set. Finally, let $\mathfrak{M}_o(R)_T$ denote the subspace of $\mathfrak{M}(R)_T$ formed with the subset $\mathfrak{M}_o(R) = \mathfrak{M}(R)\setminus\{M_o\}$. The proposition that P_o is a s.s.s.G.S. ring if R is such a ring is proved with the aid of the following

THEOREM 10.7. *Suppose that R is a* s. s. s. G. S. *ring. Then the mapping φ: $M \to M \cap M_o$ is one-to-one from $\mathfrak{M}_o(R)$ onto $\mathfrak{M}(P_o)$.*

Proof. There are two cases to consider: Either $M_o = L(R)$, or M_o is a maximal O-ideal. In the first case, the conclusion is immediate. For if $M \in \mathfrak{M}_o(R)$ then $M \subset L(R) = M_o$, so $\varphi(M) = M$. If M' is a maximal O-ideal in P_o then $M' \neq M_o$. Also M' is an O-ideal in R by the previous lemma. Since every O-ideal in R is contained in $L(R) = M_o \subset P_o$, M' is a maximal O-ideal in R. Hence $M' \in \mathfrak{M}_o(R)$, so φ maps onto $\mathfrak{M}(P_o)$.

Assume that M_o is a maximal O-ideal, and suppose $M \in \mathfrak{M}_o(R)$. On the one hand, $M \not\subset P_o$. For if $M \subset P_o$ then $M + M_o \subset P_o$. By Theorem 4.2, $M + M_o$ is an O-ideal, so, by maximality, $M + M_o = L(R)$. But then $M_o = L(P_o) \supset L(R)$, which is a contradiction. Now, on the other hand, $M \cap M_o < M_o$, so $M \cap M_o$ is a proper O-ideal in P_o. Suppose that I is an O-ideal in P_o such that $I > M \cap M_o$. By Lemma 10.5, I is an O-ideal in R, so by Lemma 10.4, $I = M$ or $I = M_o$. The first equality is impossible since, as we saw above, $M \not\subset P_o$. Therefore $I = M_o$, and we conclude that $\varphi(M) \in \mathfrak{M}(P_o)$.

If $M' \in \mathfrak{M}(P_o)$ then $M' < M_o$ and M' is an O-ideal in R. By Corollary 6.8.1, M' is the intersection of all of the members of $\mathfrak{M}(R)$ which contain it, so there exists $M \neq M_o$ such that $M' \subset M \cap M_o$. Surely $M \cap M_o$ is an O-ideal in P_o so $M' = M \cap M_o$, by maximality, and hence φ maps onto $\mathfrak{M}(P_o)$.

Suppose $M_1, M_2 \in \mathfrak{M}_o(R)$. If $M_1 \cap M_o = M_2 \cap M_o$ then $M_1 \supset M_2 \cap M_o$ and $M_2 \supset M_1 \cap M_o$. By Lemma 10.2, $M_1 \supset M_2$ and $M_2 \supset M_1$, so $M_1 = M_2$ and φ is a one-to-one mapping.

THEOREM 10.8. *Suppose that R is a* s. s. s. G. S. *ring and P_o is a primary ideal in R. Then the correspondence ψ: $S \to S \cap P_o$ is a homeomorphism from $\mathfrak{S}_o(R)$ onto $\mathfrak{S}(P_o)$, and P_o is a* s. s. s. G. S. *ring. Finally, the correspondence φ of the previous theorem is a homeomorphism from $\mathfrak{M}_o(R)_T$ onto $\mathfrak{M}(P_o)_T$.*

Proof. It follows from well known (and easily proved) properties of maximal regular ideals in arbitrary rings that ψ maps $\mathfrak{S}_o(R)$ into and indeed onto $\mathfrak{S}(P_o)$ (where, for this result, P_o need be simply an arbitrary ideal in R). Suppose $S_1, S_2 \in S_o(R)$ and $S_1 \cap P_o = S_2 \cap P_o$. According to proposition (ii) of section 5, $L(S_1 \cap P_o) = L(S_1) \cap L(P_o)$ and $L(S_2 \cap P_o) = L(S_2) \cap L(P_o)$. Hence $L(S_1) = L(S_2)$ by the previous theorem and ψ is a one-to-one mapping.

Obviously ψ is a continuous function from $\mathfrak{S}_o(R)$ to $\mathfrak{S}(P_o)$. To see that ψ is actually a homeomorphism, suppose $S' \cap P_o \supset \bigcap \{S \cap P_o \colon S \in \mathfrak{A}\}$, where $\mathfrak{A} \subset \mathfrak{S}_o(R)$. If $S' \not\supset \bigcap \mathfrak{A}$ then there exists $e \in R$ such that $e \equiv 1(S')$ and $e \equiv 0(\mathfrak{A})$. Since $S' \neq P_o^*$, there exists $e' \in L(P_o)$ such that $e' \equiv 1(S')$. Then $ee' \in \bigcap \{S \cap P_o \colon S \in \mathfrak{A}\}$, so $ee' \in S'$. However, $ee' \equiv 1(S')$, which is a contradiction. Hence $S' \supset \bigcap A$ and ψ is a homeomorphism.

Using the first paragraph of this proof and Theorem 10.7, it is now an easy matter to prove that every maximal regular ideal in P_o contains a maximal O-ideal in P_o and that every maximal O-ideal in P_o is contained in a unique maximal regular ideal in P_o. We omit the details.

Certainly P_o is a s. s. s. ring since R is a s. s. s. ring and P_o is an ideal in R. Therefore P_o is a s. s. s. G. S. ring, by Theorem 7.9.

Finally, φ is a homeomorphism. For let L_o denote the correspondence between the maximal regular ideals of P_o and the maximal O-ideals of P_o. Then L_o is a homeomorphism by Theorem 8.2, and evidently $\varphi = L_o \psi L^{-1}$ where L^{-1} is restricted to $\mathfrak{M}_o(R)$. By Theorem 8.2, again, L is a homeomorphism and hence so is L^{-1}. We have just shown that ψ is a homeomorphism. Hence φ is a homeomorphism. The proof of the theorem is complete.

The equivalence between conditions (1) and (2) of the following theorem are obviously analagous to the equivalence of conditions (a) and (c) of [8], Theorem 3.3. Subsequent to the proof of this theorem we show that the present equivalence implies that in [8].

THEOREM 10.9. *Let R be a* s. s. s. G. S. *ring and let S_o be a member of $\mathfrak{S}(R)$. Then the following conditions are equivalent for an ideal $J \subset R$:*

(1) $L(S_o) \subset J \subset S_o$.

(2) *The mapping $\psi\colon S' \to S' \cap J$ is a homeomorphism from $\mathfrak{S}(S_o)$ onto $\mathfrak{S}(J)$.*

(3) *The mapping $\varphi\colon M' \to M' \cap L(J)$ is a homeomorphism from $\mathfrak{M}(S_o)_\Gamma$ onto $\mathfrak{M}(J)_\Gamma$.*

Proof. If at any time it is known that J is a primary ideal then (2) and (3) are immediately equivalent. For then both J and S_o are s. s. s. G. S. rings by the previous theorem. Hence, $\mathfrak{S}(S_o)$ is homeomorphic with $\mathfrak{M}(S_o)_\Gamma$ and $\mathfrak{S}(J)$ is homeomorphic with $\mathfrak{M}(J)_\Gamma$ under homeomorphisms which we might denote by L_1 and L_2 respectively. Thus $L_2\psi = \varphi L_1$, and the equivalence of (2) and (3) follows.

Now if $J \supset L(S_o)$ then J is a primary ideal by Theorem 10.1. By the previous theorem again, ψ is a homeomorphism from $(\mathfrak{S}(S_o) \setminus \{J^*\})$ onto $\mathfrak{S}(J)$. If $S_o \supset J$ then $(\mathfrak{S}(S_o)/\{J^*\}) = \mathfrak{S}(S_o)$ so (1) implies (2).

In showing that $(2) \Rightarrow (1)$, consider first the special case $J \subset S_o$ and suppose $J \not\supset L(S_o)$. Then $J \subset S'$ for some $S' \in \mathfrak{S}(S_o)$ by Theorem 5.3.

But then $S' \cap J = J \notin \mathfrak{S}(J)$ so ψ does not map $\mathfrak{S}(S_o)$ into $\mathfrak{S}(J)$. We conclude that if $J \subset S_o$ then $(2) \Rightarrow (1)$.

Now let J satisfy the hypothesis of (2) but be otherwise arbitrary. If we set $J' = J \cap S_o$ then the mapping $\psi'\colon S' \to S' \cap J'$ is a homeomorphism from $\mathfrak{S}(S_o)$ onto $\mathfrak{S}(J')$. For if $S' \in \mathfrak{S}(S_o)$ then $S' = T \cap S_o$ for some $T \in \mathfrak{S}(R)$, so $S' \cap J' = (T \cap S_o) \cap (S_o \cap J) = (T \cap S_o) \cap J = S' \cap J$. Thus ψ' is a homeomorphism since ψ is. By the work of the previous paragraph, $J' = J \cap S_o \supset L(S_o)$, so $J \supset L(S_o)$. In particular, J is a primary ideal, so we have already the equivalence of (2) and (3). Now if $J \not\subset S_o$ then $J \supset L(R)$, again by Theorem 5.3. But this implies that $\varphi\colon M' \to M' \cap L(J)$ is not a homeomorphism from $M(S_o)_\Gamma$ onto $M(J)_\Gamma$ which is in violation of (3) and, hence, of (2). For $L(S_o) \notin M(S_o)$, and yet $L(S_o) \in M(J)$ when $J \supset L(R)$. More specifically, if $M' \in \mathfrak{M}(S_o)$ then $M' = M \cap L(S_o)$ for some $M \in \mathfrak{M}(R)$ such that $M \neq L(S_o)$. Thus $M' \cap L(J) = (M \cap L(S_o)) \cap L(J) = M \cap L(S_o) \neq L(S_o)$. Hence we have $L(S_o) \subset J \subset S_o$ and $(2) \Rightarrow (1)$.

Finally, condition (3) implies that J is a primary ideal so $(3) \Rightarrow (1)$. For if not, then $J \subset S$ and $J \subset T$ for distinct elements $S, T \in \mathfrak{S}(R)$. Therefore $L(J) \subset L(S) \cap L(T)$, so $\varphi(L(S) \cap L(S_o)) = (L(S) \cap L(S_o)) \cap L(J) = L(S_o) \cap L(J) = (L(T) \cap L(S_o)) \cap L(J) = \varphi(L(T) \cap L(S_o))$. If both $L(T) \cap L(S_o)$ and $L(S) \cap L(S_o)$ are distinct from $L(S_o)$, we have distinct elements of $\mathfrak{M}(S_o)$ mapping into the same element in $\mathfrak{M}(J)$. If, on the other hand, $L(T) \cap L(S_o) = L(S_o)$, for example, then $T = S_o$ so $J \subset S_o$ and $L(J) \subset L(S_o)$. Therefore $L(S_o) \cap L(J) = L(J) \in \mathfrak{M}(J)$, while $(L(S) \cap L(S_o)) \cap L(J) = L(S_o) \cap L(J)$. That is, φ does not map into $\mathfrak{M}(J)$. In either case, we achieve a contradiction of (3). We conclude that J is a primary ideal and infer that (3) is equivalent to (2).

If X is a topological space, let $C(X)$ denote the ring of all continuous complex-valued functions on X. Let $C^*(X)$ denote the ring of bounded members of $C(X)$. For $x \in X$, let S_x^* denote the maximal ideal in $C^*(X)$ which is associated with x (i. e. $S_x^* = \{f \in C^*(X)\colon f(x) = 0\}$). Let $M_x^* = L(S_x^*)$. Now assume X is a locally compact Hausdorff space. The terms $C_\infty(X)$ and $C_o(X)$ have already been mentioned in section 5. Let $X^* = X \cup \{\infty\}$ denote the one point compactification of X with adjoined point $\{\infty\}$. If $x \in X^*$, let $S_x = \{f \in C(X^*)\colon f(x) = 0\}$ and $M_x = L(S_x)$. It is straight forward to show that $C_\infty(X)$ is isomorphic to S_∞.

Now it follows from Theorem 5.10 of [3] that the correspondence $\mu\colon x \to M_x$ is a homeomorphism from X^* onto $M(C(X^*))_\Delta$. By Theorem 8.3, $M(C(X^*))_\Delta = M(C(X^*))_\Gamma$. By Theorem 10.8, the correspondence $\varphi\colon M \to M \cap L(S_\infty)$ is a homeomorphism from $(\mathfrak{M}(C(X^*)) \setminus \{L(S_\infty)\})_\Gamma$ onto $M(S_\infty)_\Gamma$. A purely topological argument shows that μ, restricted to X, is a homeomorphism from X onto $(\mathfrak{M}(C(X^*)) \setminus L(S_\infty))_\Gamma$. Therefore $\varphi\mu$ is a homeomorphism from X onto $M(S_\infty)_\Gamma$. Now we can evidently "identify" $\mathfrak{M}(S_\infty)$ with $\mathfrak{M}(C_\infty(X))$ and the set $M_x \cap L(S_\infty)$ with the set $M_x^* \cap C_0(X)$. The previous comments may be summarized by saying that the correspondence $x \to M_x^* \cap C_0(X)$ is a homeomorphism from X onto $\mathfrak{M}(C_\infty(X))_\Gamma$. By Theorem 8.2, the inverse of L is a homeomorphism from $\mathfrak{M}(C_\infty(X))_\Gamma$ onto $\mathfrak{S}(C_\infty(X))$. We can also identify $S_x \cap S_\infty$ with $S_x^* \cap C_\infty(X)$. These latter comments can now be summarized by saying that the correspondence $x \to S_x^* \cap C_\infty(X)$ is a homeomorphism from X to $\mathfrak{S}(C_\infty(X))$.

By combining the results of the previous paragraph with Theorem 10.9, we obtain the equivalence between conditions (a) and (c) of [8], Theorem 3.3. The condition (3) of the corollary is an additional result.

COROLLARY 10.9.1. *Let X be a locally compact Hausdorff space and let J be an ideal in $C^*(X)$. Then these are equivalent:*

(1) $C_0(X) \subset J \subset C_\infty(X)$.

(2) *The correspondence $x \to S_x^* \cap J$ is a homeomorphism from X onto $\mathfrak{S}(J)$.*

(3) *The correspondence $x \to M_x^* \cap L(J)$ is a homeomorphism from X onto $\mathfrak{M}(J)_\Gamma$.*

In virtue of Theorem 8.4 the above results also yield a result obtained by Shirota [11] and Civin and Yood [1].

COROLLARY 10.9.2. *Let X be a locally compact Hausdorff space and let J be an ideal such that $C_0(X) \subset J \subset C_\infty(X)$. Then X is characterized by the multiplicative semigroup of J.*

Added in proof. I have been told that Andrunakiewitsch has some results analogus to Theorem 6.6 and Corollary 6.8.1.

References

[1] P. Civin and B. Yood, *Ideals in multiplicative semi-groups of continuous functions*, Duke Math. J. 23 (1956), p. 325-334.

[2] L. Gillman, *Rings with Hausdorff structure space*, Fund. Math. 45 (1957), p. 1-16.

[3] J. G. Horne, Jr., *On the ideal structure of certain semirings and compactification of topological spaces*, Trans. Amer. Math. Soc. 90, p. 408-430.

[4] — *C_∞-ideals in $C(X)$*, Proc. Amer. Math. Soc. 9, p. 511-518.

[5] — *On primary ideals in $C(X)$*, Proc. Amer. Math. Soc. 10, p. 158-163.

[6] N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloquium Publications, vol. 37, Providence 1956.

[7] C. W. Kohls, *The space of prime ideals of a ring*, Fund. Math. 45 (1957), p. 17-27.

[8] — *Ideals in rings of continuous functions*, Fund. Math. 45 (1957), p. 28-50.

[9] L. H. Loomis, *An introduction to abstract harmonic analysis*, New York 1953.

[10] N. H. Mc Coy, *Prime ideals in general rings*, Amer. J. Math. 71 (1949), p. 823-833.

[11] T. Shirota, *A generalization of a theorem of I. Kaplansky*, Osaka Math. J. 4 (1952), p. 121-132.

[12] A. B. Wilcox, *Some structure theorems for a class of Banach Algebras*, Pac. J. Math. 6 (1956), p. 177-192.

[13] — *A note on certain group algebras*, Proc. Amer. Math. Soc. 7 (1956), p. 874-879.

[14] W. Żelazko, *On the divisors of zero of the group algebra*, Fund. Math. 45 (1957), p. 99-102.

UNIVERSITY OF KENTUCKY
LEXINGTON, KENTUCKY

Reçu par la Rédaction le 21. 10. 1958