STUDIA MATHEMATICA, T. XVIII. (1959)

On the function whose Laplace-transform is e^{-s^α}

by

J. MIKUSIŃSKI (Warszawa)

1. The function $F_\alpha(t)$, whose Laplace-transform is e^{-s^α} $(0 < \alpha < 1)$, can be explicitly written in the form

$$F_\alpha(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{tz-z^\alpha}\,dz \qquad (0 \leqslant t < \infty;\ 0 < \alpha < 1). \tag{1}$$

This function plays an important role in applications to partial equations with constant coefficients. It has been investigated by different authors, e. g. G. Doetsch [1], J. Mikusiński [2, 3, 4], H. Pollard [5], A. Wintner [6] and L. Włodarski [7].

Such values of z^α in (1) are assumed that z^α represents an analytic function in the region: $|z| > 0$, $|\arg z| < \pi$, and admits real values on the positive part of the real axis. It is easy to see that the function $\overline{F_\alpha(t)}$ whose values are conjugated to $F_\alpha(t)$ is represented by the same formula (1). This implies that $F_\alpha(t)$ is real. However, several real formulas for $F_\alpha(t)$ are known, e. g.

$$F_\alpha(t) = \frac{1}{\pi}\int_0^\infty \exp(-tr-\gamma_1 r^\alpha)\sin(\sigma_1 r^\alpha)\,dr,$$

$$F_\alpha(t) = \frac{2}{\pi}\int_0^\infty \exp(-\gamma_2 r^\alpha)\sin(\sigma_2 r^\alpha)\sin tr\,dr,$$

$$F_\alpha(t) = \frac{2}{\pi}\int_0^\infty \exp(-\gamma_2 r^\alpha)\cos(\gamma_2 r^\alpha)\cos tr\,dr$$

where

$$\gamma_1 = \cos\pi\alpha, \quad \sigma_1 = \sin\pi\alpha, \quad \gamma_2 = \cos\frac{\pi\alpha}{2}, \quad \sigma_2 = \sin\frac{\pi\alpha}{2}.$$

But it is difficult to deduce from any of the above formulas how $F_\alpha(t)$ behaves in the interval $0 < t < \infty$.

The main purpose of this paper is to prove another formula for $F_a(t)$ with a positive integrand, namely

$$(2)\qquad F_a(t) = \frac{1}{\pi}\frac{a}{1-a}\frac{1}{t}\int_0^\pi ue^{-u}d\varphi \qquad (0<t<\infty;\ 0<a<1),$$

where

$$(3)\qquad u = t^{-a/(1-a)}\left(\frac{\sin a\varphi}{\sin\varphi}\right)^{a/(1-a)}\frac{\sin(1-a)\varphi}{\sin\varphi}.$$

Since $u>0$ for $0<\varphi<\pi$, it follows trivially that $F_a(t)>0$ for $t>0$ (see [7]). From (2) we shall also deduce the following asymptotic formulae:

$$(4)\qquad F_a(t) \sim Kt^{-(2-a)/(2-2a)}\exp(-At^{-a/(1-a)}) \quad \text{for} \quad t\to 0+,$$

$$(5)\qquad F_a(t) \sim Mt^{-1-a} \quad \text{for} \quad t\to\infty.$$

The values of the constants A, K and M are:

$$(6)\qquad A = (1-a)\,a^{a/(1-a)},$$

$$(7)\qquad K = \frac{1}{\sqrt{2\pi(1-a)}}\,a^{1/(2-2a)},$$

$$(8)\qquad M = \frac{\sin a\pi}{\pi}\Gamma(1+a).$$

These results were announced without proofs in [4].

In particular, when $a=\frac{1}{2}$, we have $u = 1/4t\cos^2\frac{1}{2}\varphi$ and hence

$$F_{1/2}(t) = \frac{1}{4\pi t^2}\int_0^\pi \exp\left(-\frac{1}{4t\cos^2\frac{1}{2}\varphi}\right)\frac{d\varphi}{\cos^2\frac{1}{2}\varphi}.$$

After the substitution $\operatorname{tg}\frac{1}{2}\varphi = 2\sqrt{t}\,x$, the last integral becomes

$$4\sqrt{t}\exp\left(-\frac{1}{4t}\right)\int_0^\infty e^{-x^2}dx,$$

and we obtain the known formula

$$F_{1/2}(t) = \frac{1}{2\sqrt{\pi t^3}}e^{-1/4t}.$$

In this case we have $A=\frac{1}{4}$, $K=M=1/2\sqrt{\pi}$, and it is easy to verify the adequacy of the asymptotic formulae (4) and (5).

2. Now we are going to give the proof of formula (2). First we seek the points z at which $tz-z^a$ is real. For this purpose we put $z = r(\cos\varphi+i\sin\varphi)$ and $\operatorname{Im}(zt-z^a) = rt\sin\varphi - r^a\sin a\varphi = 0$. Hence

$$(9)\qquad r = \left(\frac{\sin a\varphi}{t\sin\varphi}\right)^{1/(1-a)}.$$

In the particular case $a=\frac{1}{2}$, $t=1$, we have

$$(10)\qquad r = \frac{1}{2(1+\cos\varphi)}.$$

The diagram of (10) is a parabola represented in the figure 1. In the general case (9) the diagram is similar. When φ ranges over the interval $(-\pi,\pi)$, ϱ is always positive and becomes infinite at the ends of this interval. Thus, if φ increases, the point z traces a curve $\mathcal{C}$ which goes round the origin and leaves it on the left side.

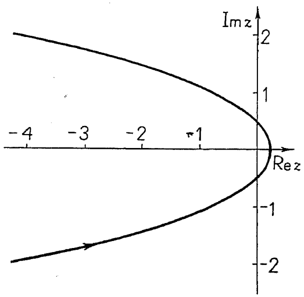

Fig. 1

In the integral (1) we can replace the path of integration (the imaginary axis) by the curve $\mathcal{C}$. Then we obtain

$$(11)\qquad F_a(t) = \frac{1}{2\pi i}\int_{\mathcal{C}} e^{tz-z^a}dz.$$

In order to justify this transformation it suffices to remark that the integral

$$\int |e^{tz-z^a}dz|,$$

taken along the semi-circumference γ: $|z|=r$, $|\arg z|<\pi/2$ tends to 0 as $r\to 0$, and that the same integral taken along each of the arcs

$$\Gamma_1:\quad |z|=R,\quad \frac{\pi}{2}<\varphi<\pi,$$

$$\Gamma_2:\quad |z|=R,\quad -\pi<\varphi<-\frac{\pi}{2},$$

tends also to 0 as $R\to\infty$. Since the integral along the contour (represented in figure 2) composed of parts of the imaginary axis and of $\mathcal{C}$, Γ_1, Γ_2, γ, vanishes, we obtain the equality of integrals (1) and (11) on letting $R\to\infty$.

It is easy to verify that along the curve $\mathcal{C}$ we have

$$zt - z^\alpha = -u \quad \text{and} \quad \frac{d\,\mathrm{Im}\,z}{d\varphi} = \frac{\alpha}{1-\alpha}\cdot\frac{u}{t},$$

where u is given by (3). Thus, we obtain from (11)

$$F_\alpha(t) = \frac{1}{2\pi i}\int_{-\pi}^{\pi} e^{-u}(d\,\mathrm{Re}\,z + i\,d\,\mathrm{Im}\,z)$$

$$= \frac{1}{2\pi}\cdot\frac{\alpha}{1-\alpha}\cdot\frac{1}{t}\int_{-\pi}^{\pi} e^{-u}u\,d\varphi,$$

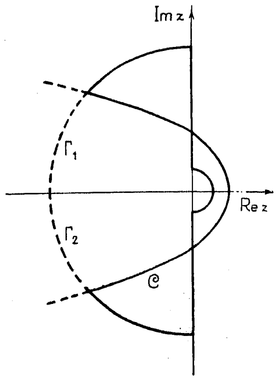

Fig. 2

since the imaginary part must vanish. To complete the proof of (2) it is now sufficient to remark that u is an even function of φ.

3. In this paragraph we give the proof of formula (4).

The initial terms of the power series for the function

$$w(\varphi) = \frac{\sin\alpha\varphi}{\sin\varphi} \tag{12}$$

are

$$w(\varphi) = \alpha + \frac{\alpha}{6}(1-\alpha^2)\varphi^2 + \ldots$$

Hence we find

$$[w(\varphi)]^{\alpha/(1-\alpha)} = \alpha^{\alpha/(1-\alpha)} + \frac{1}{6}(1+\alpha)\,\alpha^{1/(1-\alpha)}\varphi^2 + \ldots$$

Similarly

$$\frac{\sin(1-\alpha)\varphi}{\sin\varphi} = (1-\alpha) + \frac{\alpha}{6}(1-\alpha)(2-\alpha)\varphi^2 + \ldots$$

From the last two formulae it follows that the expansion of (3) has the form

$$u = t^{-\alpha/(1-\alpha)}(A + B\varphi^2 + \ldots),$$

where A is given by (6), and

$$B = \tfrac{1}{2}(1-\alpha)\,\alpha^{1/(1-\alpha)}.$$

Given any ε $(0 < \varepsilon < \frac{1}{2})$ there exists a number φ_0 $(0 < \varphi_0 < \pi)$ such that

$$t^{-\alpha/(1-\alpha)}\big(A + (B-\varepsilon)\varphi^2\big) = v_1 < u < v_2 = t^{-\alpha/(1-\alpha)}\big(A + (B+\varepsilon)\varphi^2\big) \tag{13}$$
$$(0 \leqslant \varphi \leqslant \varphi_0).$$

Hence

$$\int_0^{\varphi_0} v_1 e^{-v_2}d\varphi = I_1 < I = \int_0^{\varphi_0} u e^{-u}d\varphi.$$

We have

$$I_1 > At^{-\alpha/(1-\alpha)}e^{-At^{-\alpha/(1-\alpha)}}\int_0^{\varphi_0}\exp\big(-t^{-\alpha/(1-\alpha)}(B+\varepsilon)\varphi^2\big)d\varphi.$$

Putting for brevity

$$\Phi(t) = t^{-\alpha/(2-2\alpha)}\exp(-At^{-\alpha/(1-\alpha)})$$

and substituting

$$t^{-\alpha/(2-2\alpha)}\sqrt{B+\varepsilon}\cdot\varphi = \psi,$$

we get

$$I_1 > \Phi(t)\frac{A}{B+\varepsilon}\int_0^{\Omega} e^{-\psi^2}d\psi \qquad (\Omega = t^{-\alpha/(2-2\alpha)}\varphi_0\sqrt{B+\varepsilon}).$$

Hence

$$\liminf_{t\to 0+}\frac{I}{\Phi(t)} \geqslant \frac{A}{\sqrt{B+\varepsilon}}\cdot\frac{\sqrt{\pi}}{2}. \tag{14}$$

Similarly, starting from the inequality

$$I < I_2 = \int_0^{\varphi_0} v_2 e^{v_1}d\varphi$$

we find

$$\limsup_{t\to 0+}\frac{I}{\Phi(t)} \leqslant \frac{A + (B+\varepsilon)\varphi_0^2}{\sqrt{B-\varepsilon}}\,\frac{\sqrt{\pi}}{2}.$$

For given ε, we can fix φ_0 so small as to have $(B+\varepsilon)\varphi_0^2 < \varepsilon$; then

$$\limsup_{t\to 0+}\frac{I}{\Phi(t)} \leqslant \frac{A+\varepsilon}{\sqrt{B-\varepsilon}}\,\frac{\sqrt{\pi}}{2}. \tag{15}$$

Putting

$$\int_0^\pi ue^{-u}d\varphi = I+J,$$

we have

$$J = \int_{\varphi_0}^\pi ue^{-u}d\varphi.$$

We are going to show that

(16) $$\lim_{t\to 0+}\frac{J}{\Phi(t)} = 0.$$

The derivative of the function (12) is

$$w'(\varphi) = \frac{a\sin\varphi\cos a\varphi - \cos\varphi\sin a\varphi}{\sin^2\varphi}.$$

Hence we find

$$\frac{d}{d\varphi}\left(\sin^2\varphi\frac{dw}{d\varphi}\right) = (1-a^2)\sin\varphi\sin a\varphi > 0.$$

Since

$$\lim_{\varphi\to 0+}\sin^2\varphi\frac{dw}{d\varphi} = 0,$$

this implies that $dw/d\varphi > 0$ in $0<\varphi<\pi$. Thus w is increasing in $0<\varphi<\pi$. Hence u, given by (3), is also increasing. Therefore, in view of (13),

$$u > v_0 = t^{-a/(1-a)}\big(A+(B-\varepsilon)\varphi_0^2\big) \quad \text{for} \quad \varphi_0\leqslant\varphi<\pi.$$

For small values of t we have $v_1 > 1$. Since the function xe^{-x} is decreasing in $1<x<\infty$, this implies

$$J < \int_{\varphi_0}^\pi v_0e^{-v_0}d\varphi = Ct^{-a/(1-a)}\exp\big(-Ct^{-a/(1-a)}\big)\cdot(\pi-\varphi_0),$$

where $C = A+(B-\varepsilon)\varphi_0^2$. Since $C > A$, (16) follows.

Since ε can be chosen arbitrarily, formulae (14), (15), (16) together imply that

$$\int_0^\pi ue^{-u}d\varphi \sim \frac{A}{\sqrt{B}}\frac{\pi}{2}\Phi(t) \quad \text{for} \quad t\to 0+.$$

This is equivalent to (4); to see this, it suffices to take in account the form of the constants A, B, K and of the function $\Phi(t)$.

4. It remains to prove (5).

By (12), we have

$$\lim_{\varphi\to\pi}(\pi-\varphi)w(\varphi) = \sin a\pi.$$

Hence by (3)

$$\lim_{\varphi\to\pi-}(\pi-\varphi)^{1/(1-a)}u = (\sin a\pi)^{1/(1-a)}t^{-a/(1-a)}.$$

Given any two numbers a, b such that

$$0 < a < (\sin a\pi)^{1/(1-a)} < b,$$

there is a number φ_0 $(0<\varphi_0<\pi)$ such that

$$at^{-a/(1-a)} < (\pi-\varphi)^{1/(1-a)}u < bt^{-a/(1-a)} \quad \text{for} \quad \varphi_0\leqslant\varphi<\pi;$$

the number φ_0 of course does not depend on t.

Hence

$$at^{-a/(1-a)}\int_{\varphi_0}^\pi(\pi-\varphi)^{-1/(1-a)}\exp\big[-bt^{-a/(1-a)}(\pi-\varphi)^{-1/(1-a)}\big]$$

$$= J_1 < J = \int_{\varphi_0}^\pi ue^{-u}du.$$

Substituting

$$bt^{-a/(1-a)}(\pi-\varphi)^{-1/(1-a)} = x,$$

we find

$$J_1 = ab^{-a}(1-a)t^{-a}\int_{x_0}^\infty x^{a-1}e^{-x}dx,$$

where

$$x_0 = bt^{-a/(1-a)}(\pi-\varphi_0)^{-1/(1-a)}.$$

Hence

(17) $$\liminf_{t\to\infty} t^aJ \geqslant (1-a)ab^{-a}\Gamma(a).$$

Similarly

$$J < J_2 = bt^{-a/(1-a)}\int_{\varphi_0}^\pi(\pi-\varphi)^{-1/(1-a)}\exp\big[-at^{-a/(1-a)}(\pi-\varphi)^{-1/(1-a)}\big]$$

and just as before

(18) $$\limsup_{t\to\infty} t^aJ \leqslant (1-a)a^abΓ(a).$$

Putting

$$\int_0^{\pi} ue^{-u}d\varphi = I+J$$

we have

$$I = \int_0^{\varphi_0} ue^{-u}d\varphi.$$

We are going to prove that

(19) $$\lim_{t\to\infty} t^a I = 0.$$

Since the function $w(\varphi)$ is continuous in $(0, \pi)$ and bounded at $\varphi = 0$, there is a number $N > 0$ such that $w(\varphi) < N$ for $0 < \varphi < \varphi_0$. Thus $I < Nt^{-a/(1-a)}\varphi_0$ and this implies (21).

Since a and b can be chosen arbitrarily close to $(\sin a\pi)^{1/(1-a)}$, it follows from (17), (18), (19) that

$$\lim_{t\to\infty} t^a \int_0^{\pi} ue^{-u}d\varphi = (1-a)\sin\pi a\Gamma(a).$$

This is equivalent to (5).

References

[1] G. Doetsch, *Handbuch der Laplace-Transformation I*, p. 163.

[2] J. Mikusiński, *Sur les fonctions exponentielles du calcul opératoire*, Studia Math. 12 (1951), p. 208.

[3] — *Sur la croissance de la fonction opérationnelle* $\exp(-s^a\lambda)$, Bull. Ac. Pol. Sci., Cl. III, 4.7 (1956), p. 423-425.

[4] — *Sur la fonction dont la transformée de Laplace est* e^{-s^a}, ibidem, à paraitre.

[5] H. Pollard, *The representation of* e^{-x^λ} *as a Laplace integral*, Bull. Amer. Math. Soc. 52 (1946), p. 908-910.

[6] A. Wintner, *Cauchy's stable distributions and an explicit formula of Mellin*, Amer. J. Math. 78 (1956), p. 819-861.

[7] L. Włodarski, *Une remarque sur une classe de fonctions exponentielles du calcul opérationnel*, Studia Math. 13 (1953), p. 188-189.

Reçu par la Rédaction le 12. 9. 58

STUDIA MATHEMATICA, T. XVIII. (1959)

Consistency theorems for Banach space analogues of Toeplitzian methods of summability

by

A. ALEXIEWICZ and W. ORLICZ (Poznań)

We deal in this paper with the generalized Toeplitz sequence-to--sequence transformations from one Banach space X into another Y; these transformations will be called in conformity with the case of numerical sequences *methods of summability*. One instance of such methods, namely those involving the strong limits, has recently been introduced by Robinson [6] and Melvin-Melvin [4], who derived the Toeplitzian conditions for permanency.

One of the non-trivial results in the theory of summability of numerical sequences is the bounded consistency theorem, stating, roughly speaking, that if two Toeplitzian methods are consistent for convergent sequences and if every bounded sequence summable by the first method is summable by the second, both methods are consistent for bounded sequences [3].

It is the purpose of this paper to prove the bounded consistency theorem in the case of sequence-to-sequence transformations in Banach spaces. Our method consists in considering the spaces of bounded summable sequences as two-norm spaces; in these spaces a notion γ of limit arises in a natural way, leading to the class of continuous distributive functionals called the γ-linear functionals. Essential for the success of our method is the fact that the spaces we are dealing with are such that the limit of any pointwise convergent sequence of γ-linear functionals is γ-linear, which is not the case in all the two-norm spaces.

The authors are indebted to Mr. Bogdanowicz for valuable simplifications and corrections.

1. Preliminaries. We shall deal in this paper with the following methods of summability of sequences of Banach spaces. We are given two Banach spaces X and Y and a system $A = \{A_{i\nu}\}$ of linear operators