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Some remarks on the spaces $N(L)$ and $N(l)$

by

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Let $N(t)$ be a non-negative Baire function. We denote by $N(L)$ the space of all real measurable functions on $(0,1)$ such that

$$\varrho_N(x)=\int_0^1 N\big(x(\xi)\big)\,d\xi<\infty.$$

By $N(l)$ we denote the space of all sequences $x=\{\xi_n\}$ such that

$$\varrho_N(x)=\sum_{n=1}^{\infty} N(\xi_n)<\infty.$$

(In this case $N(t)$ can be an arbitrary non-negative function.)

The basic properties of the spaces $N(L)$ and $N(l)$ are described in the paper [4] of S. Mazur and W. Orlicz. In this note further results regarding the spaces $N(L)$ and $N(l)$ are given. All the theorems are formulated and proved in the case of spaces $N(L)$, and the slight differences in the statement of the corresponding theorems and proofs for $N(l)$ are written in parentheses.

The sequence x_n is called *convergent to* x if $\varrho_N(x_n-x)\to 0$.

The non-negative function $N(t)$ is called *quasiconvex* if:

(a) There are constants K, C, r (C, r) such that

$$N(t+s)\leqslant\begin{cases} C[N(t)+N(s)] & \text{if} \quad N(t)+N(s)>r\ \ (<r),\\ K & \text{if} \quad N(t)+N(s)\leqslant r.\end{cases}$$

(b) For each $\varrho>0$ there are constants D_ϱ, r_ϱ such that

$$N(\omega t)\leqslant D_\varrho N(t) \qquad \text{for} \qquad |\omega|<\varrho \text{ and } |t|>r_\varrho\ \ (|t|<r_\varrho).$$

(c) $N(t_n)\to 0$ if and only if $t_n\to 0$.

S. Mazur and W. Orlicz proved in the paper [4] that if $N(t)$ is a quasi-convex function, then the set $N(L)$ ($N(l)$) is a linear space, and the formula

$$\|x\| = \inf\left\{\varepsilon : \varepsilon > 0,\ \varrho_N\left(\frac{x}{\varepsilon}\right) < \varepsilon\right\}$$

is a norm in this space such that $\|x_n - x\| \to 0$ if and only if $\varrho_N(x_n - x) \to 0$. The space $N(L)$ ($N(l)$) with this norm is complete. If $N(t)$ is an increasing function for positive t, then $\|tx\|$ is an increasing function for each x and positive t.

Two quasiconvex functions $M(t)$ and $N(t)$ are called *equivalent* if there are two positive constants a and b such that $a < N(t)/M(t) < b$.

This paper consists of five paragraphs. In the first paragraph the necessary and sufficient conditions for the existence of a bounded (1) neighbourhood of zero are presented. The theorem that every open set Z such that $c(Z) < \infty$ (see [5]) and $\sup_{x\in Z} \varrho_N(x) < \infty$ is bounded, is proved in the second paragraph. In the third paragraph I prove that the function $N(t^r)$, where $r = 1/p$, is equivalent to a convex function if and only if a p-homogeneous norm exists in $N(L)$ ($N(l)$). A necessary and sufficient condition for the existence of linear functional is presented in the fourth paragraph. In the fifth paragraph I give some examples and remarks on equivalent functions.

1. First we define the function

$$n(t) = \inf\{a : a > 0,\ N(at) \geqslant \tfrac{1}{2}N(t)\}.$$

THEOREM 1. *The following four conditions are equivalent:*

(a) *There exists in $N(L)$ ($N(l)$) a bounded neighbourhood of zero;*

(b) $\liminf_{t\to\infty} n(t) > 0$ $\quad(\liminf_{t\to 0} n(t) > 0)$;

(c) *There exists a constant a, $0 < a < 1$, such that*

$$\limsup_{t\to\infty}\frac{N(at)}{N(t)} < 1 \qquad \left(\limsup_{t\to 0}\frac{N(at)}{N(t)} < 1\right);$$

(d) *The functions $w_t(s) = N(st)/N(t)$ are equicontinuous in 0 for sufficiently large t (for sufficiently small t).*

Proof. (a) $\to$ (b). Suppose that $\liminf_{t\to\infty} n(t) = 0$ $(\liminf_{t\to 0} n(t) = 0)$. Then there exist a sequence $t_m \to \infty$ $(t_m \to 0)$ and a sequence $a_m \to 0$ such that $N(a_m t_m) \geqslant \frac{1}{2}N(t_m)$. Let ε be an arbitrary positive number and let

$$x_m(\xi) = \begin{cases} t_m & \text{for} \quad 0 \leqslant \xi < \dfrac{\varepsilon}{2N(t_m)}, \\ 0 & \text{for} \quad \xi \geqslant \dfrac{\varepsilon}{2N(t_m)} \end{cases}$$

$$\left(x_m = \{\xi_m^i\} \text{ where } \xi_m^i = \begin{cases} t_m & \text{for} \quad i < \dfrac{\varepsilon}{N(t_m)}, \\ 0 & \text{for} \quad i \geqslant \dfrac{\varepsilon}{N(t_m)} \end{cases}\right).$$

$$\varrho_N(x_m) < \varepsilon.$$

The sequence $a_m x_m$ does not converge to 0 because $\varrho_N(a_m x_m) > \frac{1}{4}\varepsilon$. Hence no open set is bounded.

(b) $\to$ (c). Suppose that for each a

$$\limsup_{t\to\infty}\frac{N(at)}{N(t)} \geqslant 1 \qquad \left(\limsup_{t\to 0}\frac{N(at)}{N(t)} \geqslant 1\right).$$

Then there exists a sequence $t_m \to \infty$ $(t_m \to 0)$ such that $N(t_m/m) \geqslant \frac{1}{2}N(t_m)$. Therefore from the definition $n(t_m) < 1/m$ and $\liminf_{t\to\infty} n(t) = 0$ $(\liminf_{t\to 0} n(t) = 0)$.

(c) $\to$ (d). If $\limsup_{t\to\infty}(N(at)/N(t)) < 1$, then there exist numbers T and $0 < \beta < 1$ such that $N(at) < \beta N(t)$ provided $t > T$. This is easy to prove by induction that $N(a^m t) < \beta^m N(t)$ whenever $a^m t > T$. Let $a^{2(m+1)} < s < a^{2m}$. We shall investigate two cases:

(i) $a^m t > T_0 = \max(T, r_1)$ where r_1 is the constant given in the condition (b) of the quasiconvexity for $\varrho = 1$; then

$$N(st) = N\left(\frac{s}{a^m}a^m t\right) \leqslant D_1 N(a^m t) \leqslant D_1\beta^m N(t).$$

(ii) $a^m t \leqslant T_0$; then

$$N(st) = N\left(\frac{s}{a^m}a^m t\right) \leqslant \sup_{0<\lambda<T_0} N(a\sqrt{s}\lambda) = M(s)$$

and we obtain

$$w_t(s) \leqslant \max(D_1\beta^m, M(s)/A)$$

where $A = \inf_{t>T_0} N(t)$, and since $M(s)$ is continuous in 0 we infer that all

(1) A set Z is called *bounded* if for every sequence $x_n \in Z$ and for every sequence of numbers t_n convergent to 0, the sequence $t_n x_n$ tends to 0.

$w_t(s)$ are equicontinuous in 0 for $t > T_0$. (The proof for the spaces $N(l)$ is somewhat easier. If $s < a^m$, then $N(st) \leqslant D_1\beta^m N(t)$ for sufficiently small t and $w_t(s) \leqslant D_1\beta^m$ for sufficiently small t).

(d) $\to$ (a). Let the functions $w_t(s)$ be equicontinuous for $t > T$. Let x_m be an arbitrary sequence such that $\varrho_N(x_m) < K$. Let

$$A_m = \{\xi\colon\ |x_m(\xi)| > T\}.$$

We write

$$x'_m(\xi) = \begin{cases} x_m(\xi) & \text{for} \quad \xi \in A, \\ 0 & \text{for} \quad \xi \notin A, \end{cases}$$

and we put $x''_m = x_m - x'_m$. Let s_m be an arbitrary sequence of numbers convergent to 0. Since $\varrho_N(s_m x''_m) \leqslant D_1 N(s_m T)$ and $\varrho_N(s_m x'_m) \leqslant w(s_m) K$, where $w(s) = \sup\limits_{t>T} w_t(s)$, the sequence x_m is bounded.

2. Let A be a neighbourhood of zero such that $tA \subset A$ (²) for all numbers t, $|t| < 1$. By the *concavity module $c(A)$ of the neighbourhood A* we mean the infimum of the set of all positive numbers s such that $A+A \subset sA$ (see [5]). If $c(A) < \infty$, then for every positive integer k there exists a number K such that $A^k = \underbrace{A \oplus \ldots \oplus A}_{k\text{-fold}} \subset KA$.

THEOREM 2. *If A is an open set such that $c(A) < \infty$ and $\sup\limits_{x \in A} \varrho_N(x) < \infty$, then A is bounded.*

Proof. Since every quasiconvex function is equivalent to an even function increasing for positive t (see [4], 2.6 and 1.61), we can suppose that $N(t)$ is an increasing function for positive t, and $N(t) = N(-t)$. If there exists no bounded neighbourhood of zero in $N(L)$ ($N(l)$), then by Theorem 1 there exists a sequence $t_m \to \infty$ ($t_m \to 0$) such that $n(t_m) \to 0$. Let $p = \inf\limits_{x \in A} \varrho_N(x)$ and $r = \sup\limits_{x \in A} \varrho_N(x)$ and $k = [4r/p+1]$.

Let

$$x^i_m(\xi) = \begin{cases} t_m & \text{for} \quad \dfrac{ip}{2N(t_m)} < \xi < \dfrac{(i+1)p}{2N(t_m)}, \\ 0 & \text{otherwise} \end{cases}$$

$$\left(x^i_m = \xi^i_{m,n} \quad \text{where} \quad \xi^i_{m,n} = \begin{cases} t_m & \text{for} \quad \left[\dfrac{ip}{N(t_m)}+1\right] < n < \left[\dfrac{(1+i)p}{N(t_m)}\right], \\ 0 & \text{otherwise} \end{cases}\right),$$

$$i = 1, 2, \ldots, k.$$

(²) By tA we denote the set of elements tx where $x \in A$. By $A \oplus B$ we denote the set of sum $x+y$, where $x \in A$ and $y \in B$.

Let $x_m = x_1 + \ldots + x_k$. We have $\varrho_N(x^i_m) < p$, whence $x^i_m \in A$. On the other hand, $\varrho_N(x_m) > 2r$ and $2n(t_m)x_m \notin A$. And since $n(t_m) \to 0$, we have $c(A) = \infty$. This contradiction establishes the theorem.

COROLLARY. *If the space $N(L)$ ($N(l)$) is a B_0-space, then it is a B-space* (see [4], 1.9 and 2.9).

3. S. Mazur and W. Orlicz have proved ([4], 1.9 and 2.9) that the space $N(L)$ ($N(l)$) is a B-space if and only if $N(t)$ is equivalent to a convex function. This result can be generalized in the following way:

THEOREM 3. *A p-homogeneous norm in the space $N(L)$ ($N(l)$) exists if and only if the function $N^*(t) = N(t^r)$, where $r = 1/p$, is equivalent to a convex function.*

Proof. Necessity. If there exists a p-homogeneous norm $\|x\|$ in X, then the set $A_\varepsilon = \{x\colon \|x\| < \varepsilon\}$ is p-convex in Landsberg's sense (³). The continuation of this proof is analogous to the proof of S. Mazur and W. Orlicz (see [4], 1.9 and 2.9).

Sufficiency. We can suppose that $N(t)$ is an increasing function for positive t and $N(t) = N(-t)$ (see [4], 1.61 and 2.84). Now we define a one-to-one transformation U of the space $N(L)$ ($N(l)$) onto the space $N^*(L)$ ($N^*(l)$):

$$U(x(\xi)) = |x(\xi)|^p \operatorname{sign} x(\xi) \qquad \left(U(\{\xi_n\}) = |\xi_n|^p \operatorname{sign} \xi_n\right).$$

This transformation has the following properties:

(a) $$\varrho_{N^*}(U(x)) = \varrho_N(x),$$

(b) $$\varrho_{N^*}(U(x+y)) \leqslant \varrho_{N^*}(|U(x)|+|U(y)|),$$

(c) $$\varrho_N(tx) = \varrho_{N^*}(t^p U(x)).$$

Since $N^*(t)$ is equivalent to a convex function, the space $N^*(L)$ ($N^*(l)$) is a B-space (see [4], 1.9 and 2.9), and we infer that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\varrho_{N^*}(y_i) < \delta$ then

$$\varrho_{N^*}\left(\frac{y_1+\ldots+y_n}{n}\right) < \varepsilon.$$

Let $\varrho_N(x_i) < \delta$ ($i = 1, 2, \ldots, n$). Then

$$\varrho_N\left(\frac{x_1+\ldots+x_n}{n}\right) = \varrho_{N^*}\left(\frac{U(x_1+\ldots+x_n)}{n}\right)$$
$$\leqslant \varrho_{N^*}\left(\frac{|U(x_1)|+\ldots+|U(x_n)|}{n}\right) < \varepsilon.$$

(³) A set A is called *p-convex in Landsberg's sense* if from $x, y \in A$ and $a, b > 0$, and $a^p + b^p = 1$ it follows that $ax+by \in A$ (see [3]).

Hence there exists an open set A which is p-convex in Landsberg's sense and such that $\sup_{x\epsilon A} \varrho_N(x) < \infty$. By Theorem 2 this set is bounded and thus there exists a p-homogeneous norm in $N(L)$ $(N(l))$ (see [3]).

COROLLARY 1. *There exists no q-homogeneous norm in the space L^p (l^p) for $q > p$.*

COROLLARY 2. *There exists such a space $N(L)$ that there exists a q-homogeneous norm in $N(L)$ for each $q < p$, but there is no p-homogeneous norm.*

In fact for

$$N(t) = \frac{N(a_k)}{a_k^{p-1/k}} t^{p-1/k} \quad \text{for} \quad a_k \leqslant t < a_{k+1},$$

where the sequence a_k is chosen in such way that $N(a_k)/a_k^p < 1/k$, $N(L)$ has the desired property.

Remark. Let $N(t)$ be an increasing quasiconvex function for positive t and $N(-t) = N(t)$. If there are bounded open sets in $N(L)$ $(N(l))$, then it follows from condition (b) of Theorem 1 that $N^{-1}(2t)/N^{-1}(t)$ is bounded for sufficiently large t (for sufficiently small t). Hence, since $N^{-1}(t)$ is increasing for positive t, the function $N^{-1}(t)$ is quasiconvex (see [4], 1.62 and 2.71) and the space $N^{-1}(L)$ $(N^{-1}(l))$ is linear.

On the other hand, since $N(t)$ is a quasiconvex function, $N(2t)/N(t)$ is bounded for sufficiently large t (for sufficiently small t) and there are open bounded sets in the space $N^{-1}(L)$ $(N^{-1}(l))$. It follows from Theorem 3 that there exists a number $r \geqslant 1$ such that $N^{-1}(t^r)$ is equivalent to a convex function $M(t)$, whence $N(t)^p$, where $p = 1/r$, is equivalent to the function $M^{-1}(t)$, which is a concave function.

4. THEOREM 4. *In order that there exist linear functionals in the space $N(L)$ it is necessary and sufficient that*

$$\liminf_{t\to\infty} \frac{N(t)}{t} > 0.$$

Proof. There is a linear functional in $N(L)$, provided there exists an open convex set not identical with the whole space (4). Hence we will seek the necessary and sufficient condition for the existence of an open convex set not identical with the whole space.

Necessity. Let ε be an arbitrary positive number. Suppose that

$$\liminf_{t\to\infty} \frac{N(t)}{t} = 0.$$

(4) It is a well-known fact (see for example [2]).

Then there exists a sequence $t_m \to \infty$ such that $N(t_m)/t_m \to 0$. Let $k = [N(t_m)/\varepsilon]+1$ and

$$x_m^i(\xi) = \begin{cases} t_m & \text{for} \quad \dfrac{i-1}{k} \leqslant \xi < \dfrac{i}{k} \\ 0 & \text{for} \quad \xi < \dfrac{i-1}{k} \text{ and } \xi \geqslant \dfrac{i}{k} \end{cases} \qquad (i = 1, 2, \ldots, k).$$

We have $\varrho_N(x_m^i) < \varepsilon$, and it can easily be checked that

$$y_m(\xi) = \frac{1}{k}\sum_{i=1}^{k} x_m^i(\xi) \equiv \frac{t_m}{k} \to \infty.$$

Thus we have proved that every constant function belongs to the set $A = \operatorname{conv}\{x\colon \varrho_N(x) < \varepsilon\}$. In a similar way we can prove that every simple function belongs to A. Thus A is dense in the whole space and, because of being open, $A = N(L)$. Since ε is arbitrary, it follows that there is no linear functional in $N(L)$.

Sufficiency. If $\liminf_{t\to\infty}(N(t)/t) > 0$, then there are two positive constants μ and T such that $N(t) > \mu|t|$ for $|t| > T$. We are going to show that $A = \operatorname{conv}\{x\colon \varrho_N(x) < 1\} \neq N(L)$. Let $x_1, \ldots, x_n$ be arbitrary elements such that $\varrho_N(x_i) < 1$ $(i = 1, 2, \ldots, n)$. Let $B_i = \{\xi\colon |x_i(\xi)| \leqslant T\}$; we write

$$x_i'(\xi) = \begin{cases} x_i(\xi) & \text{for} \quad \xi \epsilon B_i, \\ 0 & \text{for} \quad \xi \notin B_i, \end{cases} \quad \text{and} \quad x_i'' = x_i - x_i'.$$

We write

$$x_0' = \frac{x_1' + \ldots + x_n'}{n} \quad \text{and} \quad x_0'' = \frac{x_1'' + \ldots + x_n''}{n}.$$

Thus, since $|x_i'(\xi)| \leqslant T$, we have $|x_0'(\xi)| \leqslant T$ and since $\mu\int_0^1 |x_i''(\xi)|\,d\xi < 1$, it follows that $\int_0^1 |x_0''(\xi)|\,d\xi < 1/\mu$, and $\operatorname{ess\,inf}|x_0''(\xi)| < 1/\mu$. Therefore $z(\xi) \equiv T + 1/\mu \notin A$.

4. A lot of examples can be derived from the following construction.

Let $N(t)$ and $N'(t)$ be two quasiconvex functions increasing for positive t, and let $N(t) < N'(t)$ $(N(t) > N'(t))$.

We take the family of the functions $M_p(t) = pN(t)N'(t)$. This family has the following properties:

1. Every function $M_p(t)$ is quasiconvex (⁵).

2. All functions $M_p(t)$ are increasing for positive t.

3. Every equation $M_p(t) = N(t)$ and $M_p(t) = N'(t)$ has one and only one positive solution.

We choose the sequence t_m in the following way: t_1 is an arbitrary number such that $t_1 > r$ $(t_1 < r)$, where r is a constant given in the definition of quasiconcavity.

Suppose that we have defined t_{2k}, and let us consider the function $M_{p_k}(t)$ such that $M_{p_k}(t_{2k}) = N(t_{2k})$. We define t_{2k+1} as the solution of the equation $M_{p_k}(t) = N'(t)$, and t_{2k+2} as the solution of the equation $N(t) = N'(t_{2k+1})$.

We define the function $M(t)$ as follows:

$$M(t) = \begin{cases} M_{p_k}(t) & \text{for} \quad t_{2k} \leqslant t < t_{2k+1} \ (t_{2k+1} \leqslant t < t_{2k}), \\ N'(t_{2k+1}) & \text{for} \quad t_{2k+1} \leqslant t < t_{2k+2} \ (t_{2k+2} \leqslant t < t_{2k+1}), \\ N'(t) & \text{for} \quad |t| < t_1 \ (|t| > t_1), \end{cases}$$

$$M(-t) = M(t).$$

It can easily be checked that:

1. The function $M(t)$ is increasing for positive t,
2. $N(t) \leqslant M(t) \leqslant N'(t)$ $(N'(t) \leqslant M(t) \leqslant N(t))$,
3. $t_m \to \infty$ $(t_m \to 0)$,
4. $M(t) \geqslant M_{p_k}(t)$ for $t < t_{2k}$ and $M(t) \leqslant M_{p_k}(t)$ for $t > t_{2k}$.

We shall prove that $M(t)$ is quasiconvex. Let t and t' be two positive constants and $t_{2k} \leqslant t+t' < t_{2k+1}$ $(t_{2k+1} \leqslant t+t' < t_{2k})$. Then

$$M(t+t') \leqslant M_{p_k}(t+t') \leqslant 2C^2[M_{p_k}(t)+M_{p_k}(t')] \leqslant 2C^2[M(t)+M(t')].$$

Let $t_{2k+1} \leqslant t+t' < t_{2k+2}$ $(t_{2k+2} \leqslant t+t' < t_{2k+1})$. If t or t' is greater than t_{2k+1} (t_{2k+2}) the proof is trivial. In the other case $M(t+t') \leqslant M_{p_k}(t+t')$ $(M(t+t') \leqslant M_{p_{k-1}}(t+t'))$ and the proof is similar to that in the first case.

(⁵) Let $N(t)$ and $N'(t)$ be two increasing functions; then

$$[N(t)-N(t')][N'(t)-N'(t')] \geqslant 0,$$

$$N(t)N'(t)+N(t')N'(t') \geqslant N(t)N'(t')+N(t')N'(t).$$

Hence

$$\begin{aligned} M_p(t+t') &= pN(t+t')N'(t+t') \leqslant pC[N(t)+N(t')]C[N'(t)+N'(t')] \\ &\leqslant 2C^2p[N(t)N'(t)+N(t')N'(t')] = 2C^2[M_p(t)+M_p(t')]. \end{aligned}$$

Let $N(t) = t$, $N'(t) = t^2+t$ $(N'(t) = t+t^2,\ N(t) = t^2)$. Then the function $M(t)$ constructed in the described way has the following properties:

1. $M(t)$ is not equivalent to any concave or convex function, whence $M(L)$ $(M(l))$ is not a B-space.

2. $\liminf\limits_{t\to\infty} n(t) = 0$, but $\limsup\limits_{t\to\infty} n(t) \neq 0$ (similarly for $t \to 0$).

3. There exists a non-trivial linear functional in $M(L)$.

Remark 1. Let $N(t)$ be a quasiconvex function, and let the constant C given in the definition of the quasiconvexity be equal to 1; then there exists a concave function $M(t)$ which is equivalent to $N(t)$.

The proof is similar to that of Theorem 1 in paper [1].

Remark 2. There exists a space $M(L)$ in which there exists a bounded neighbourhood of zero and a non-trivial linear functional, and which is a not a B-space.

In this case we construct $M(t)$ as before with a small modification: we define t_{2k+2} as a solution of the equation $N(t) = N'(t_{2k+1})\sqrt{t/t_{2k+1}}$ and we define

$$M(t) = \begin{cases} M_{p_k}(t) & \text{for} \quad t_{2k} \leqslant t \leqslant t_{2k+1}, \\ N(t_{2k+1}) & \text{for} \quad t_{2k+1} \leqslant t < t_{2k+2}, \\ N'(t) & \text{for} \quad |t| < t_1; \end{cases}$$

$$M(t) = M(-t).$$

References

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