Finally, it may be worth mentioning that our new polynomials $P_n(f; x)$, $T_n(f; x)$ etc. can also be generalized to the cases of a complex variable and of several variables. Further investigation of these polynomials is being accomplished in a joint paper of the author with L. P. Hsu, which will appear elsewhere.

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MATHEMATICS DEPARTMENT, NORTH-EAST PEOPLE'S UNIVERSITY, CHANGCHUN, CHINA

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On modular spaces

by

J. MUSIELAK and W. ORLICZ (Poznań)

In the present paper the authors investigate functionals $\varrho(x)$ defined in a real linear space X, which are called modulars. An F-norm will be introduced in certain subspaces of the space X. In the second part of this paper some examples of modulars are considered.

1. First, the following definition of a modular and a pseudomodular will be given:

1.01. Given a linear space X, a functional $\varrho(x)$ defined on X with values $-\infty < \varrho(x) \leqslant \infty$ will be called a *modular* if the following conditions hold:

A.1. $\varrho(x) = 0$ *if and only if* $x = 0$,

A.2. $\varrho(-x) = \varrho(x)$,

A.3. $\varrho(\alpha x+\beta y) \leqslant \varrho(x)+\varrho(y)$ *for every* $\alpha, \beta \geqslant 0$, $\alpha+\beta = 1$.

If $\varrho(x)$ satisfies the condition $\varrho(0) = 0$ instead of A.1, then $\varrho(x)$ will be called a *pseudomodular*.

1.02. We now give some simple properties of pseudomodulars. *Let us assume $\varrho(x)$ to be a pseudomodular on X. Then*

(a) $\varrho(x) \geqslant 0$,

(b) $\varrho(\alpha x)$ *is a non-decreasing function of* $\alpha \geqslant 0$ *for each* $x \in X$,

(c) $\varrho(\sum_{i=1}^{n} a_i x_i) \leqslant \sum_{i=1}^{n} \varrho(x_i)$ *for* $a_i \geqslant 0$, $\sum_{i=1}^{n} a_i = 1$.

Moreover, if X_ϱ denotes the set of $x \in X$ such that $\varrho(x) < \infty$, the set X_ϱ is convex and symmetric with respect to 0.

The properties (a) and (b) easily follow from A.3 and A.2; (c) is obtained by induction as follows:

$$\varrho\left(\sum_{i=1}^{n} a_i x_i\right) = \varrho\left(\sum_{i=1}^{n-1} a_i \frac{\sum_{i=1}^{n-1} a_i x_i}{\sum_{i=1}^{n-1} a_i} + a_n x_n\right) \leqslant \varrho\left(\frac{\sum_{i=1}^{n-1} a_i x_i}{\sum_{i=1}^{n-1} a_i}\right) + \varrho(x_n) \leqslant \sum_{i=1}^{n} \varrho(x_i).$$

It will be noted that for a modular $\varrho(x)$ the inequality $\varrho(x) < \varrho(y)$ does not imply in general $\varrho(ax) \leqslant \varrho(ay)$ for every real a.

1.03. *Denote by X_w a convex subset of X, symmetric with respect to 0. Let X_w^* be the set of all $x \in X$ such that $kx \in X_w$ with a positive constant k depending on x. In particular, X_ϱ^* denotes the set of all $x \in X$ such that, for a given $k > 0$, $\varrho(kx)$ is finite. It is easily seen that X_w^* are linear subspaces of X.*

1.04. We now introduce the concept of modular convergence and modular completeness. A sequence $\{x_n\} \subset X$ will be said to be:

(a) *modular convergent* or *ϱ-convergent to $x \in X$* (in symbols: $x_n \xrightarrow{\varrho} x$) if there exists a number $k > 0$ (depending on the sequence $\{x_n\}$) such that $\varrho[k(x_n - x)] \to 0$ as $n \to \infty$;

(b) *satisfying the modular Cauchy condition* if $\varrho(x_n - x_m) \to 0$ as $m, n \to \infty$.

Further, a subset $X_1 \subset X_\varrho^*$ will be said to be:

(a′) *modular complete* or *ϱ-complete* if the modular Cauchy condition implies modular convergence to an $x \in X_1$;

(b′) *strongly modular complete* or *strongly ϱ-complete* if the modular Cauchy condition implies modular convergence to an $x \in X_1$ with a constant $k > 0$ independent of the sequence $\{x_n\} \subset X_1$.

1.05. There result the following properties of modular convergence:

(a) *if $\varrho(x)$ is a modular, then the modular limit is uniquely determined;*

(b) *if $x_n \xrightarrow{\varrho} x$, $y_n \xrightarrow{\varrho} y$, then $\alpha x_n + \beta y_n \xrightarrow{\varrho} \alpha x + \beta y$ for every real α, β.*

1.1. In our further considerations the following conditions will be of importance:

B.1. *$a_n \to 0$ implies $\varrho(a_n x) \to 0$,*

B.2. *$\varrho(x_n) \to 0$ implies $\varrho(ax_n) \to 0$ for every real a.*

1.11. *Denote by $\bar{X}_\varrho$ the set of all $x \in X_\varrho$ such that* B.1 *holds. Then $\bar{X}_\varrho$ is convex and the linear space $\bar{X}_\varrho^*$ is closed with respect to ϱ-convergence,* as follows from the inequality $\varrho(ax) \leqslant \varrho[2a(x_n - x)] + \varrho(2ax_n)$. Hence, *if X_ϱ^* is ϱ-complete or strongly ϱ-complete, so is $\bar{X}_\varrho^*$.*

1.12. *Let X_f^* be a finite-dimensional linear subspace of $\bar{X}_\varrho^*$ and let $\varrho(x)$ be a modular. Denoting by $e_1, e_2, \ldots, e_m$ the basis of X_f^*, the necessary and sufficient condition for a sequence $x_n = \lambda_{1n} e_1 + \ldots + \lambda_{mn} e_m$ to be ϱ-convergent to $x = \lambda_1 e_1 + \ldots + \lambda_m e_m$ is $\lambda_{in} \to \lambda_i$ as $n \to \infty$ for $i = 1, 2, \ldots, m$.*

Supposing that the limit element $x = 0$, we have to prove that $\lambda_{in} \to 0$ as $n \to \infty$ for $i = 1, 2, \ldots, m$. In the contrary case we could write

$$\sum_{i=1}^{m} |\lambda_{in}| \geqslant d > 0,$$

and denoting

$$\gamma_{in} = \frac{d\lambda_{in}}{\sum_{i=1}^{m} |\lambda_{in}|} \quad \text{for} \quad i = 1, 2, \ldots, m; \; n = 1, 2, \ldots,$$

we should obtain

$$\varrho\left(k \sum_{i=1}^{m} \gamma_{in} e_i\right) \leqslant \varrho\left(k \sum_{i=1}^{m} \lambda_{in} e_i\right) = \varrho(kx_n) \to 0 \quad \text{as} \quad n \to \infty.$$

On the other hand, it may be assumed that

$$\gamma_{in} \to \gamma_i \quad \text{as} \quad n \to \infty \quad \text{for} \quad i = 1, 2, \ldots, m.$$

Then

$$\varrho\left(\frac{k}{2m} \sum_{i=1}^{m} \gamma_i e_i\right) \leqslant \sum_{i=1}^{m} \varrho[k(\gamma_i - \gamma_{in}) e_i] + \varrho\left(k \sum_{i=1}^{m} \gamma_{in} e_i\right) \to 0 \quad \text{as} \quad n \to \infty.$$

This implies $\gamma_i = 0$ and $\lambda_{in} \to 0$ as $n \to \infty$ for $i = 1, 2, \ldots, m$. The fact that if $\lambda_{in} \to 0$ as $n \to \infty$ for $i = 1, 2, \ldots, m$ then $x_n \xrightarrow{\varrho} 0$ follows from B.1.

From 1.12 there results the following statement:

1.13. *If X_f^* is a finite-dimensional linear space, $X_f^* \subset \bar{X}_{\varrho_1}^* \cap \bar{X}_{\varrho_2}^*$, $\varrho_1(x)$ and $\varrho_2(x)$ being modulars on X, then ϱ_1-convergence of a sequence $\{x_n\} \subset X_f^*$ to $x \in X_f^*$ is equivalent to ϱ_2-convergence of x_n to x.*

1.2. The object of this paper is to introduce an F-norm in a linear space by means of a modular. Similar problems have been considered by H. Nakano and his school under the additional hypothesis of convexity (see e. g. [5]) or subadditivity [5] of the modular $\varrho(x)$. Moreover, the case of semi-ordered linear spaces and that of B-norms have been chiefly investigated. Our investigation aims at obtaining some results under weaker assumptions, fitted to the structure of the spaces under consideration. Neither convexity nor subadditivity of the modular will be assumed. It is to be noted that, in the case of l^M- and L^M-spaces for example (see [6] and [2]), the assumptions of continuity and monotony of $M(u)$ suffice for $\varrho(x)$ to be a modular. In introducing the norm, a certain natural connection between the modular and the norm convergence will be required: norm convergence should imply modular convergence. Under the additional assumptions B.2, the two convergences are equivalent. A number of examples in the second part of our paper makes clear the application of the conditions introduced in the present part. The authors will return to this investigation in another paper.

1.21. *Given a modular $\varrho(x)$, let us write*

$$\|x\| = \inf\{\varepsilon > 0: \varrho(x/\varepsilon) \leqslant \varepsilon\}.$$

The functional $\|x\|$ is an F-norm in $\bar{X}^*_\varrho$, i. e.

1° $\|x\| = 0$ *if and only if* $x = 0$,

2° $\|x+y\| \leqslant \|x\|+\|y\|$,

3° $\|-x\| = \|x\|$,

4° $a_n \to a$ *and* $\|x_n - x\| \to 0$ *imply* $\|a_n x_n - ax\| \to 0$.

The norm $\|ax\|$ is a non-decreasing function of $a \geqslant 0$ for every $x \in \bar{X}^*_\varrho$. We have $\varrho(x) \leqslant \|x\|$ for $\|x\| < 1$; hence norm convergence implies modular convergence to the same limit. Moreover, if $\bar{X}^*_\varrho$ is strongly ϱ-complete, then it is complete in norm.

It will be noted that $\varrho(x/a) = a$ for an $a > 0$ implies $\|x\| = a$ and that, if $\varrho(ax)$ is a continuous function of $a \geqslant 0$ for every $x \in \bar{X}^*_\varrho$, then $\varrho(x/\|x\|) = \|x\|$ for every $x \neq 0$, $x \in \bar{X}^*_\varrho$. Moreover, *if $\varrho(x)$ is a pseudomodular, then $\|x\|$ is an F-pseudonorm; if $\varrho(x) = 0$ implies $\varrho(2x) = 0$ and $\bar{X}^*_\varrho$ is strongly ϱ-complete, then it is complete in pseudonorm.*

Conditions 1° and 3° and the monotony of $\|ax\|$ as a function of $a \geqslant 0$ being trivial, we now prove the triangle inequality 2°. Given any $\varepsilon > 0$ and $x, y \in \bar{X}^*_\varrho$, we write $a = \|x\|+\frac{1}{2}\varepsilon$, $\beta = \|y\|+\frac{1}{2}\varepsilon$. Then $\varrho(x/a) \leqslant a$, $\varrho(y/\beta) \leqslant \beta$, and

$$\varrho\left(\frac{x+y}{a+\beta}\right) = \varrho\left(\frac{a}{a+\beta}\cdot\frac{x}{a} + \frac{\beta}{a+\beta}\cdot\frac{y}{\beta}\right) \leqslant \varrho\left(\frac{x}{a}\right)+\varrho\left(\frac{y}{\beta}\right) \leqslant a+\beta.$$

Hence $\|x+y\| \leqslant a+\beta$ and the triangle inequality follows. To prove 4°, let us first note that

(a′) $a_n \to 0$ implies $\|a_n x\| \to 0$,

(b′) $\|x_n\| \to 0$ implies $\|ax_n\| \to 0$ for any real a,

these conditions being easy to verify by the basic inequalities

$$\varrho\left(\frac{x}{\lambda\|x\|}\right) \begin{cases} > \lambda\|x\| & \text{for} \quad \lambda < 1, \\ \leqslant \lambda\|x\| & \text{for} \quad \lambda > 1, \end{cases}$$

where $x \neq 0$. Applying the triangle inequality $\|a_n x_n - ax\| \leqslant \|a_n(x_n - x)\| + \|(a_n - a)x\|$ and the monotony of $\|ax\|$, and using (a′) and (b′), we obtain 4°. Now, let us assume $\bar{X}^*_\varrho$ to be strongly ϱ-complete and let $\|x_n - x_m\| \to 0$ as $m, n \to \infty$. Then $\varrho[(x_n - x_m)/\varepsilon] \to 0$ for any $\varepsilon > 0$. The strong ϱ-completeness of $\bar{X}^*_\varrho$ implies the existence of an $\hat{x}_\varepsilon \in \bar{X}^*_\varrho$ such that $\varrho[k(x_n/\varepsilon - \hat{x}_\varepsilon)] \to 0$ as $n \to \infty$. Obviously $\varrho[k(x_n - \varepsilon\hat{x}_\varepsilon)] = \varrho[k\varepsilon(x_n/\varepsilon - \hat{x}_\varepsilon)] \to 0$ as $n \to \infty$ for any $0 < \varepsilon \leqslant 1$. On the other hand, $\varrho[k(x_n - \hat{x}_1)] \to 0$. Hence $\varrho[k(\hat{x}_\varepsilon - \hat{x}_1/\varepsilon)/2] = 0$ and $\varrho[k(x_n - \hat{x}_1)/4\varepsilon] \to 0$ as $n \to \infty$ for any $\varepsilon > 0$; thus $\|x_n - \hat{x}_1\| \to 0$.

1.22. The following theorem establishes the uniqueness of the norm $\|x\|$:

Let $\varrho(x)$ be a modular and let $\|\ \|'$ and $\|\ \|''$ be two complete F-norms in $\bar{X}^*_\varrho$ such that norm convergence of a sequence of elements of $\bar{X}^*_\varrho$ to zero implies modular convergence of this sequence to zero. Then the norms $\|\ \|'$ and $\|\ \|'$ are equivalent in the sense that $\|x_n - x\|' \to 0$ if and only if $\|x_n - x\|'' \to 0$.

We consider the operation $U(x) = x$ from $\langle\bar{X}^*_\varrho, \|\ \|'\rangle$ to $\langle\bar{X}^*_\varrho, \|\ \|''\rangle$. Assuming $\|x_n - x\|' \to 0$ and $\|U(x_n) - y\|'' \to 0$. we have $\varrho(x_n - x) \to 0$ and $\varrho(x_n - y) \to 0$. The modular limit being unique, this implies $x = y$. Then the Banach closed graph theorem implies $U(x)$ to be linear. Theorem 1.22 follows immediately.

1.3. In the following theorem, condition B.2 is of importance:

1.31. *Modular convergence is equivalent to norm convergence in a subset X_w of $\bar{X}^*_\varrho$ if and only if condition* B.2 *holds for any sequence of elements of X_w.*

Indeed, let the modular convergence be equivalent to the norm convergence and let us assume $\varrho(x_n) \to 0$. Then $\|x_n\| \to 0$; hence $\|ax_n\| \to 0$ and $\varrho(ax_n) \to 0$. Conversely, let B.2 hold and let $\varrho(x_n) \to 0$. Assuming $\|x_n\| > g > 0$, we obtain $\varrho(x_n/g) \to 0$. On the other hand, the definition of the norm yields $\varrho(x_n/g) > g$, in contradiction to the above convergence.

1.32. We now make some remarks concerning condition B.2.

(a) *If* B.2 *holds for any sequence of elements of $X_w \subset \bar{X}^*_\varrho$, and X'_w is the closure of X_w with respect to the norm, then* B.2 *holds also for any sequence of elements of X'_w. Moreover, we obviously have $X'_w \subset \bar{X}^*_\varrho$.*

(b) *Assuming X_w to be linear,* B.2 *in X_w is equivalent to the following condition: if $\varrho(x_n) \to 0$, then $\varrho(ax_n) \to 0$ for any $x_n \in X_w$, a being any fixed number larger than* 1.

(c) *X_w being a linear space, let us assume there the existence of positive numbers a and $\varkappa$ such that the condition*

$$(\Delta_2) \qquad \varrho(2x) \leqslant \varkappa\varrho(x)$$

holds for every $x \in X_w$ satisfying the inequality $\varrho(x) \leqslant a$. Then B.2 *holds for any sequence of elements of X_w.*

We first prove (a). Let $\{x_n\} \subset X'_w$ and $\varrho(x_n) \to 0$. We choose a sequence of elements $y_n \in X_w$ such that $\|y_n - x_n\| \to 0$. Then $\varrho(y_n - x_n) \to 0$ and the inequality $\varrho(\frac{1}{2}y_n) \leqslant \varrho(y_n - x_n) + \varrho(x_n) \to 0$ and B.2 imply $\varrho(2ay_n) \to 0$ for any real a. Hence $\varrho(ax_n) \leqslant \varrho[2a(y_n - x_n)] + \varrho(2ay_n) \to 0$ and $\varrho(ax_n) \to 0$. (b) and (c) being obvious, we note only that condition (Δ_2)

is not necessary for B.2. A counter example is provided by the case: X = the space of reals, $\varrho(x) = $ a continuous monotone function not satisfying (Δ_2).

2. In the present section a number of examples of spaces with a modular will be considered. As special cases, the well-known examples of spaces l^M, L^M and V_M come under consideration.

2.1. Let $X^1, X^2, \ldots$ be linear spaces with modulars $\varrho_1(x_1), \varrho_2(x_2), \ldots$ respectively. The question arises of defining in a natural sense a modular $\varrho(x)$ in the Cartesian product $X = X^1 \times X^2 \times \ldots$ by means of the modulars $\varrho_1(x_1), \varrho_2(x_2), \ldots$ This question may be put in various ways. When $x = (x_1, x_2, \ldots)$, the following definitions will be introduced:

$$1^\circ \quad \varrho^1(x) = \sum_{i=1}^{\infty} \varrho_i(x_i), \qquad 2^\circ \quad \varrho^2(x) = \sup_i \varrho_i(x_i),$$

$$3^\circ \quad \varrho^3(x) = \sup_n \frac{1}{n} \sum_{i=1}^{n} \varrho_i(x_i).$$

It is easily seen that, *if $\varrho_i(x_i)$ are modulars on X^i, respectively, then $\varrho^1(x)$, $\varrho^2(x)$, and $\varrho^3(x)$ are modulars on X.*

2.11. *The following inclusion holds:*

$$X^*_{\varrho^1} \cap (\overline{X}^*_{\varrho_1} \times \overline{X}^*_{\varrho_2} \times \ldots) \subset \overline{X}^*_{\varrho^1}.$$

Indeed, if $x = (x_1, x_2, \ldots)$ belongs to the left side of this inclusion, i. e. $kx \in X_{\varrho^1}$, then there exists for every $\varepsilon > 0$ a number N such that $\sum_{N+1}^{\infty} \varrho_i(kx_i) < \frac{1}{2}\varepsilon$; hence $\sum_{N+1}^{\infty} \varrho_i(a_n kx_i) < \frac{1}{2}\varepsilon$ for any $0 \leqslant a_n \leqslant 1$. If $a_n \to 0$, then $\sum_{1}^{N} \varrho_i(a_n kx_i) < \frac{1}{2}\varepsilon$ for sufficiently large n, and 2.11 follows. It will be noted that the inclusion 2.11 does not hold in general either for $\varrho^2(x)$ or for $\varrho^3(x)$.

2.12. In the sequel in 2.12-2.15 we shall always take $X^1 = X^2 = \ldots$ $\ldots = R^1$ = the space of reals, $\varrho_i(u) = M_i(u)$ = an even continuous function, non-decreasing for $u \geqslant 0$, $M_i(0) = 0$, $M_i(u) > 0$ for $u > 0$, where $i = 1, 2, \ldots$ Then

(a) $\overline{X}^*_{\varrho^1} = X^*_{\varrho^1}$,

(b) $X^*_{\varrho^1}$ *is strongly* ϱ^1*-complete,*

(c) *if $M_i(u)$ satisfy the condition*

$$(\Delta_2) \qquad M_i(2u) \leqslant \varkappa M_i(u) \quad \textit{for} \quad 0 \leqslant M_i(u) \leqslant a,$$

where $a > 0$ and $\varkappa$ are independent of i, then B.2 *holds for any sequence of elements of $X^*_{\varrho^1}$.*

The easy proofs will be omitted. Let us note that, when $M_i(u) = M(u)$ for $i = 1, 2, \ldots$, we obtain the spaces l^M, considered in many papers, with various additional assumptions (see e. g. [4], [5]). An F-norm of the form considered here was introduced for l^M in the case of $M(u)$ satisfying the condition (Δ_2) for small u in paper [2].

2.13. *Let us take $X^1 = X^2 = \ldots = R^1$, $M_i(u) = M(u)$, where $M(u) \to \infty$ as $u \to \infty$. Then the statements* (a), (b), (c) *of* 2.12 *hold if we put ϱ^2 in place of ϱ^1.*

2.14. *Assuming $M_1(u) = M_2(u) = \ldots = M(u)$, let us denote by X_m the set of all $x = \{a_i\}$ such that there exists a number a with the following property:*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} M[k(a_i - a)] = 0 \quad \textit{for every real} \quad k.$$

Then $X^*_{\varrho^3}$ is strongly ϱ^3-complete, X_m is a linear space contained in $\overline{X}^*_{\varrho^3}$, complete with respect to the norm.

In order to prove the completeness of $X^*_{\varrho^3}$, choose a sequence of elements $x_m = \{a_i^m\}$ of $X^*_{\varrho^3}$ satisfying the modular Cauchy condition, i. e.

$$\frac{1}{n} \sum_{i=1}^{n} M(a_i^p - a_i^q) \to 0 \quad \text{as} \quad p, q \to \infty$$

uniformly in n. Hence, the numerical sequence $\{a_i^m\}$ satisfies the Cauchy condition for each fixed i; hence $a_i^m \to a_i$ as $m \to \infty$ for each i and there results the strong ϱ^3-completeness of $X^*_{\varrho^3}$. We shall now prove the inclusion $X_m \subset \overline{X}^*_{\varrho^3}$. Let $x = \{a_i\} \in X_m$ and let $0 \leqslant \alpha \leqslant \frac{1}{2}$. Then

$$\frac{1}{n} \sum_{i=1}^{n} M(\alpha a_i) \leqslant \frac{1}{n} \sum_{i=1}^{n} M(a_i - a) + M(2\alpha a).$$

Since the first term on the right side of the above inequality tends to zero as $n \to \infty$ and the second tends to zero as $\alpha \to 0$, the expression on the left side of this inequality is small for n sufficiently large and α sufficiently (but independently of n) small. Now, $\varrho^3(\alpha x) \to 0$ as $\alpha \to 0$ results easily. Since $\overline{X}^*_{\varrho^3}$ is strongly ϱ^3-complete, it is complete in the norm. Then, to obtain the norm-completeness of X_m, it is sufficient to prove that X_m is closed with respect to the norm. Assuming $\{x_m\} \subset X_m$ and $\|x_m - x\| \to 0$, where $x_m = \{a_i^m\}$, $x = \{a_i^m\}$, we have, for every $k > 0$, $\varrho[k(x_m - x)] \to 0$; thus

$$\frac{1}{n} \sum_{i=1}^{n} M[k(a_i^m - a_i)] \to 0 \quad \text{as} \quad m \to \infty \text{ uniformly in } n,$$

$$\frac{1}{n} \sum_{i=1}^{n} M[k(a_i^m - a^m)] \to 0 \quad \text{as} \quad n \to \infty \quad \text{for} \quad m = 1, 2, \ldots,$$

k being arbitrary.

Let us choose a number $\varepsilon > 0$; since

$$M\left(k\frac{a^p-a^q}{4}\right) \leqslant \frac{1}{n}\sum_{i=1}^{n} M[k(a^p-a_i^p)]+\frac{1}{n}\sum_{i=1}^{n} M[k(a_i^p-a_i)]+ +\frac{1}{n}\sum_{i=1}^{n} M[k(a_i-a_i^q)]+\frac{1}{n}\sum_{i=1}^{n} M[k(a_i^q-a^q)] < M\left(k\frac{\varepsilon}{4}\right)$$

for all sufficiently large p, q and $n = n(p, q)$, the sequence $a^m \to a$ as $m \to \infty$. Thus the inequality

$$\frac{1}{n}\sum_{i=1}^{n} M[k(a_i-a)] \leqslant \frac{1}{n}\sum_{i=1}^{n} M[3k(a_i-a_i^m)]+\frac{1}{n}\sum_{i=1}^{n} M[3k(a_i^m-a^m)]+M[3k(a^m-a)]$$

implies that $x = \{a_i\} \epsilon X_m$; hence X_m is closed with respect to the norm.

2.15. *Assuming $M_1(u) = M_2(u) = \ldots = M(u)$, we have the equality $\bar{X}^*_{\varrho 3} = X^*_{\varrho 3}$ if and only if for any $\varepsilon > 0$ numbers $A_\varepsilon > 0$ and $a_\varepsilon > 0$ exist such that $M(au) < \varepsilon M(u)$ for every $0 \leqslant a \leqslant a_\varepsilon$, $u \geqslant A_\varepsilon$.*

In order to prove the sufficiency, let us choose a number $\varepsilon > 0$ and take an $x \epsilon X_{\varrho^3}$. Then

$$\frac{1}{n}\sum_{i=1}^{n} M(aa_i) \leqslant M(aA_\varepsilon)+\varepsilon\varrho^3(x) \quad \text{for} \quad 0 \leqslant a \leqslant a_\varepsilon;$$

hence $\varrho^3(ax) \leqslant M(aA_\varepsilon)+\varepsilon\varrho^3(x)$ and the sufficiency results. Now, we shall prove the necessity. Let us suppose that there exists an $\varepsilon > 0$ and two sequences of positive numbers: u_n increasing to infinity and a_n decreasing to zero, satisfying the inequality $M(a_n u_n) > \varepsilon M(u_n)$ for $n = 1, 2, \ldots$ It may be assumed that $M(u_1) > \frac{3}{2}$. Now, we shall define two sequences of indices $n_1', n_2', \ldots; n_1, n_2, \ldots,$ and a sequence of numbers $a_1, a_2, \ldots$ by induction. Let n_1' be the least positive integer such that $M(u_1)/n_1' < \frac{1}{4}$. Moreover, let n_1 be the largest positive integer satisfying the inequality $(n_1-n_1')M(u_1)/n_1 < \frac{5}{4}$. We define the first n_1 terms of the sequence $a_1, a_2, \ldots$ as follows: $a_n = 0$ for $n \leqslant n_1'$, $a_n = u_1$ for $n_1' < n \leqslant n_1$. Now, let us suppose that the sequences n_i', n_i, and a_j are defined for $i < k$ and $j \leqslant n_{k-1}$ in such a way that

$$(+) \qquad \frac{1}{n}\sum_{i=1}^{n} M(a_i) < \tfrac{3}{2} \quad \text{for any} \quad n \leqslant n_{k-1},$$

$$(++) \qquad \frac{(n_{k-1}-n_{k-1}')M(u_{k-1})}{n_{k-1}} > \tfrac{3}{4}.$$

Let n_k' be the least positive integer, larger than n_{k-1} and satisfying the inequalities

$$\frac{M(u_k)}{n_k'} < \frac{1}{4}, \qquad \frac{1}{n_k'}\sum_{i=1}^{n_{k-1}} M(a_i) < \frac{1}{4}.$$

Moreover, let $n_k > n_k'$ be the largest positive integer stisafying the inequality

$$\frac{(n_k-n_k')M(u_k)}{n_k} < \frac{5}{4}.$$

Obviously we have, for any $n_k' \leqslant n \leqslant n_k$,

$$\frac{(n-n_k')M(u_k)}{n} < \frac{5}{4} \quad \text{and} \quad \frac{3}{4} < \frac{(n_k-n_k')M(u_k)}{n_k} < \frac{5}{4}.$$

Now, we put $a_n = 0$ for $n_{k-1} < n \leqslant n_k'$, $a_n = u_k$ for $n_k' < n \leqslant n_k$. It is easily seen that

$$\frac{1}{n}\sum_{i=1}^{n} M(a_i) < \frac{3}{2} \quad \text{for any} \quad n \leqslant n_k.$$

Indeed, it is sufficient to prove the above inequality for $n_k' < n \leqslant n_k$; however, for such n,

$$\frac{1}{n}\sum_{i=1}^{n} M(a_i) \leqslant \frac{1}{n}\sum_{i=1}^{n_{k-1}} M(a_i)+\frac{(n-n_k')M(u_k)}{n} < \frac{1}{4}+\frac{5}{4} = \frac{3}{2}.$$

Thus, (+) and (++) are satisfied if we put n_k instead of n_{k-1}. We shall now prove that $x = \{a_i\}$ belongs to X_{ϱ^3} but does not belong to $\bar{X}_{\varrho^3}$. Evidently, since $\varrho^3(x) \leqslant \frac{3}{2}$, it is sufficient to prove that $\varrho^3(a_k x) > \frac{3}{4}\varepsilon$ for $k = 1, 2, \ldots$ However,

$$\varrho^3(a_k x) \geqslant \frac{1}{n_k}\sum_{i=n_k'+1}^{n_k} M(a_k a_i) = \frac{n_k-n_k'}{n_k} M(a_k u_k) \geqslant \varepsilon\frac{(n_k-n_k')M(u_k)}{n_k} > \frac{3}{4}\varepsilon \quad \text{for} \quad k = 1, 2, \ldots$$

Finally, x cannot belong to $\bar{X}^*_{\varrho^3}$, since it would belong to $\bar{X}_{\varrho^3}$, x belonging to X_{ϱ^3}.

2.2. The example of a modular which we shall give now is the generalized variation of a function. Let $M(u)$ be an even continuous function, non-decreasing for $u \geqslant 0$, $M(0) = 0$, $M(u) > 0$ for $u > 0$. Given a real function $x(t)$, defined in a closed finite interval $\langle a, b\rangle$, the value

$$V_M(x) = \sup_{\Pi} \sum_{i=1}^{m} M[x(t_i) - x(t_{i-1})],$$

where Π: $a = t_0 < t_1 < \ldots < t_m = b$ is an arbitrary partition of the interval $\langle a, b\rangle$, is called the *M-th variation of $x(t)$ in $\langle a, b\rangle$* (see [3]). Denoting by X the class of all real functions in $\langle a, b\rangle$ vanishing at a, we define in X

$$\varrho(x) = V_M(x).$$

Obviously *$\varrho(x)$ is a modular and X_ϱ^* is strongly ϱ-complete.* The authors defined a B-norm in the space X_ϱ^* in the case when $M(u)$ is a convex function in [3]. Several results have been obtained for $M(u)$ convex and satisfying the condition $M(u) = o(u)$ as $u \to 0$. As an example of the modulars, an F-norm will be introduced in $\bar{X}_\varrho^*$ for the opposite case, namely $u = o[M(u)]$ as $u \to 0$. The following conditions will be of importance:

(A) $M(u)/u \to \infty$ as $u \to 0$,

(B) there exists a constant $\varkappa$ such that $M(u_1 + \ldots + u_n) \leqslant \varkappa[M(u_1) + \ldots + M(u_n)]$ for $u_1, \ldots, u_n \geqslant 0$.

Condition (B) is satisfied for example, if $M(u)$ is concave or sub-additive for $u \geqslant 0$. The following lemma will be useful:

2.21. *If $M(u)$ satisfies (A), $x \in X_\varrho$ and $x(t)$ is continuous in $\langle \alpha, \beta\rangle \subset \langle a, b\rangle$, then $x(t) = \text{const}$ in $\langle \alpha, \beta\rangle$.*

In order to prove 2.21, let us suppose $x(t)$ to be continuous in $\langle \alpha, \beta\rangle$, $x(\alpha) = c$, $x(\beta) = d$, where $c < d$, and let us take for each positive integer n a partition $\alpha = t_0 < t_1 < \ldots < t_{2^n} = \beta$ of $\langle \alpha, \beta\rangle$ such that $x(t_i) = 2^{-n}(d-c) + c$; hence

$$V_M(x) \geqslant \sum_{i=1}^{2^n} M[x(t_i) - x(t_{i-1})] = |d-c| \frac{M(2^{-n}|d-c|)}{2^{-n}|d-c|} \to \infty \quad \text{as} \quad n \to \infty,$$

and this contradicts the hypothesis $x \in X_\varrho$.

To formulate a further lemma, let us put for a function $x(t)$ of bounded variation (in the usual sense, i. e. with $M(u) = |u|$) in $\langle a, b\rangle$,

$$s_x(t) = x(a+0) - x(a) + \sum_{t_i < t} [x(t_i+0) - x(t_i-0)] + x(t) - x(t-0) \quad \text{for} \quad a < t \leqslant b,$$

$$s_x(a) = 0,$$

where $t_1, t_2, \ldots$ are all the points of discontinuity of $x(t)$. It is well-known

that, $x(t)$ being of bounded variation in the usual sense, $s_x(t)$ and $x(t) - s_x(t)$ are also of bounded variation in the usual sense; moreover, $x(t) - s_x(t)$ is continuous in $\langle a, b\rangle$. The following lemma holds:

2.22. *Assuming $M(u)$ to satisfy (A) and (B), we have $x(t) = s_x(t)$ in $\langle a, b\rangle$ for any $x \in X_\varrho$.*

If we choose an arbitrary partition $a = \tau_0 < \tau_1 < \ldots < \tau_m = b$ of the interval $\langle a, b\rangle$, we obtain

$$\sum_{j=1}^{m} M[s_x(\tau_j) - s_x(\tau_{j-1})] = \sum_{j=1}^{m} M\Big\{\sum_{\tau_{j-1} \leqslant t_i < \tau_j} [x(t_i+0) - x(t_i-0)] + [x(\tau_j) - x(\tau_{j-1})] + [x(\tau_{j-1}-0) - x(\tau_j-0)]\Big\}$$

$$\leqslant \varkappa\Big\{\sum_i M[x(t_i+0) - x(t_i-0)] + \sum_{j=1}^{m} M[x(\tau_j) - x(\tau_{j-1})] + \sum_{j=1}^{m} M[x(\tau_j-0) - x(\tau_{j-1}-0)]\Big\} \leqslant 3\varkappa V_M(x),$$

where we write by convention $x(a-0) = x(a+0)$; hence $V_M(s_x) \leqslant 3\varkappa V_M(x) < \infty$ and $\frac{1}{2}(x - s_x) \in X_\varrho$. Thus, Lemma 2.21 implies, by the continuity of $x(t) - s_x(t)$, the equality $x(t) - s_x(t) = \text{const}$ in $\langle a, b\rangle$. This yields $x(t) = s_x(t)$ for every $t \in \langle a, b\rangle$.

2.23. *If $M(u)$ satisfies the conditions (A) and (B), then $\bar{X}_\varrho^* = X_\varrho^*$.*

Assuming $x \in X_\varrho$ and defining $\bar{x}(t) = x(t-0)$ for $a < t \leqslant b$, $\bar{x}(a) = x(a) = 0$ and $y(t) = x(t) - \bar{x}(t)$, we easily obtain $V_M(\bar{x}) \leqslant V_M(x)$. Therefore $\bar{x} \in X_\varrho$ and $z = \frac{1}{2}y \in X_\varrho$. The following inequalities hold:

$$(+) \qquad 2\sum_i M[z(t_i)] - M[z(b)] \leqslant V_M(z) \leqslant 2\varkappa \sum_i M[z(t_i)].$$

Since $V_M(z) < \infty$, the series $\sum_i M[z(t_i)]$ is convergent. Let us fix the arrangement of the sequence $t_1, t_2, \ldots$ and let us choose a number $\varepsilon > 0$; then there exists an integer N such that

$$\sum_{i=N+1}^{\infty} M[\alpha z(t_i)] \leqslant \sum_{i=N+1}^{\infty} M[z(t_i)] < \varepsilon/4\varkappa$$

for any $0 \leqslant \alpha \leqslant 1$. Moreover, $M[\alpha z(t_i)] < \varepsilon/4\varkappa N$ for sufficiently small α; hence

$$\sum_i M[\alpha z(t_i)] < \varepsilon/2\varkappa$$

and the second of the inequalities (+) implies $V_M(\alpha z) < \varepsilon$ for sufficiently small α. The relation $\bar{x}\in\bar{X}_\varrho^*$ may be deduced from the inequalities

$$(++)\qquad \frac{1}{\varkappa}\sum_i M[\bar{x}(t_i+0)-\bar{x}(t_i)] \leqslant V_M(\bar{x})$$
$$\leqslant \varkappa\Big\{M[\bar{x}(a+0)]+\sum_i M[\bar{x}(t+0)-\bar{x}(t_i)]\Big\}$$

in the same way as the relation $x\in\bar{X}_\varrho^*$ from the inequalities (+). We still have to prove (++). The left-hand inequality being easily obtained, we prove only the right-hand one. Lemma 2.22 implies

$$\bar{x}(t) = s_{\bar{x}}(t) = \bar{x}(a+0)+\sum_{t_i<t}[\bar{x}(t_i+0)-\bar{x}(t_i)] \quad \text{for} \quad a<t\leqslant b,$$
$$\bar{x}(a) = 0,$$

whence, given an arbitrary partition $a = \tau_0 < \tau_1 < \ldots < \tau_m = b$ of the interval $\langle a, b\rangle$, we have

$$\sum_{j=1}^m M[\bar{x}(\tau_j)-\bar{x}(\tau_{j-1})] = M\Big\{\bar{x}(a+0)+\sum_{\tau_0\leqslant t_i<\tau_1}[\bar{x}(t_i+0)-\bar{x}(t_i)]\Big\}+$$
$$+\sum_{j=2}^m M\Big\{\sum_{\tau_{j-1}\leqslant t_i<\tau_j}[\bar{x}(t_i+0)-\bar{x}(t_i)]\Big\}$$
$$\leqslant \varkappa\Big\{M[\bar{x}(a+0)]+\sum_i M[\bar{x}(t_i+0)-\bar{x}(t_i)]\Big\}.$$

2.24. *Let $M(u)$ satisfy the following condition: there exists a $u_0>0$ such that $\sup_{u<u_0} M(\alpha u)/M(u)\to 0$ as $\alpha\to 0$. Then $\bar{X}_\varrho^* = X_\varrho^*$.*

It will be noted that all convex functions $M(u)$ satisfy 2.24.

2.25. *If $M(u)$ satisfies the condition*

$$(\Delta_2)\qquad M(2u)\leqslant \varkappa M(u) \text{ for small } u,$$

then $\varrho(x)$ also satisfies the condition (Δ_2) (see 1.32 (c)); *hence* B.2 *holds for any sequence of elements of $\bar{X}_\varrho^*$.*

2.3. In many known examples the space X consists of M-integrable functions. These examples will be generalized as follows. Given a set E and a σ-additive and σ-finite measure μ defined on a σ-algebra $\mathcal{F}$ of subsets of the set E, we take a real function $M(u, v)$, defined in $E\times R^1$, R^1 being the space of reals, satisfying the following conditions:

(a) $M(u, v)\geqslant 0$; $M(u, v) = 0$ if and only if $v = 0$,

(b) $M(u, v)$ is an even, continuous and non-decreasing (for $v\geqslant 0$) function of v, for every $u\in E$,

(c) $M(u, v)$ is measurable as a function of u for every $v\in R^1$.

We take as X_1 the set of all real-valued μ-measurable functions $x(t)$ in E, finite almost everywhere on E. It is easily seen that $M[t, x(t)]$ is measurable on E for any $x\in X_1$; denoting by X the quotient space $X = X_1/X_0$, X_0 being the set of all $x(t) = 0$ almost everywhere, we may define a modular $\varrho(x)$ on X by

$$\varrho(x) = \int_E M[t, x(t)]\,d\mu,$$

the uniqueness of this definition being granted by (a).

2.31. *The modular $\varrho(x)$ satisfies the condition $\bar{X}_\varrho^* = X_\varrho^*$; moreover, X_ϱ^* is strongly ϱ-complete.*

The equality $\bar{X}_\varrho^* = X_\varrho^*$ being implied by the Lebesgue bounded-convergence theorem, we have only to prove the strong ϱ-completeness. First, let us assume $\mu E<\infty$. We apply the following lemma:

(*) *If $f(t)>0$ is measurable in a set of finite measure E, then for every $\varepsilon>0$ there exists a number $\eta>0$ such that $\int_A f(t)\,d\mu<\eta$ implies $\mu(A)<\varepsilon$ for any $A\subset E$, η being independent of A.*

Choose a number $\varepsilon>0$. Writing $f(t) = M(t, \varepsilon)$ we apply the above lemma. We find an $\eta>0$ such that for every $A\subset E$, $\int_A M(t, \varepsilon)\,d\mu<\eta$ implies $\mu(A)<\varepsilon$. If a sequence $\{x_n\}$ satisfies the modular Cauchy condition, then we may find a number N such that

$$\int_E M[t, x_m(t)-x_n(t)]\,d\mu<\eta \quad \text{for} \quad m, n>N.$$

Writing $A_{m,n} = \{t\in E: |x_m(t)-x_n(t)|\geqslant\varepsilon\}$, we obtain

$$\int_{A_{m,n}} M(t, \varepsilon)\,d\mu<\eta;$$

hence $\mu(A_{m,n})<\varepsilon$ for $m, n>N$. Thus the sequence $x_n(t)$ is convergent in measure to a function $x(t)$ in E. Taking an arbitrary subsequence x_{m_k} we may extract from x_{m_k} a sequence $x_{m_{k_l}}(t)\to x(t)$ almost everywhere. Then $M[t, x_n(t)-x_{m_{k_l}}(t)]\to M[t, x_n(t)-x(t)]$ almost everywhere, n being fixed, and Fatou's lemma yields

$$(+)\qquad \varrho(x_n-x)\leqslant \varliminf_{l\to\infty}\varrho(x_n-x_{m_{k_l}})<\varepsilon$$

for sufficiently large n.

Now, let $\mu(E) = \infty$, $E = \bigcup_1^\infty E_k$, where $\mu E_k<\infty$ for $k = 1, 2, \ldots$ and the sets E_k are ascending. Applying the above results we obtain a function $x(t)$ such that $x_n(t)$ converges in measure to $x(t)$ in each E_k and that

$$\int_{E_k} M[t, x_n(t)-x(t)]d\mu \to 0 \quad \text{as} \quad n\to\infty$$

for each k. Let us write for a fixed n, $y_k(t) = M[t, x_n(t)-x(t)]\chi_{E_k}(t)$, where $\chi_{E_k}(t)$ denotes the characteristic function of the set E_k, $y(t) = M[t, x_n(t)-x(t)]$. Applying once more the inequality (+) with E_k instead of E we obtain

$$\int_E y_k(t)d\mu = \int_{E_k} M[t, x_n(t)-x(t)]d\mu$$
$$\leqslant \varliminf_{l\to\infty}\int_{E_k} M[t, x_n(t)-x_{m_{k_l}}(t)]d\mu \leqslant \varliminf_{l\to\infty}\varrho(x_n-x_{m_{k_l}}) \leqslant \varepsilon$$

for sufficiently large (independently of k) n. However, $y_k(t)$ is convergent to $y(t)$ everywhere; hence Fatou's lemma yields

$$\varrho(x_n-x) = \int_E y(t)d\mu \leqslant \varliminf_{k\to\infty}\int_E y_k(t)d\mu \leqslant \varepsilon$$

for sufficiently large n.

2.32. (a) *If there exists a constant $\varkappa > 0$ such that, E_1 denoting the set of all $u\in E$ which satisfy the inequality $M(u, 2v) \leqslant \varkappa M(u, v)$ for all v, $\mu(E-E_1) = 0$, then B.2 holds for any sequence of elements of X_ϱ^*.*

(b) *Let $\mu E < \infty$, and assume that $M(u, v)$ is integrable in E for each finite v and satisfies the following condition: there exist a $v_0 > 0$ and a $\varkappa > 0$ such that the set of all $u\in E$ satisfying the inequality $M(u, 2v) \leqslant \varkappa M(u, v)$ for any $v\geqslant v_0$ is of measure μE. Then B.2 holds for any sequence of elements of X_ϱ^*.*

The assumption of (a) easily implies the condition (Δ_2) for $\varrho(x)$; hence, (a) follows from 1.32 (c). To prove (b), let us write $E_1 = \{u: M(u, 2v)\leqslant \varkappa M(u, v)$ for all $v\geqslant v_0\}$, $E_{2,n} = \{t: |x_n(t)| < v_0\}$, where $x_n\in X_\varrho$. Then $\varrho(2x_n) \leqslant \int_{E_{2,n}} M[t, 2x_n(t)]d\mu + \varkappa\varrho(x_n)$. Now, let us assume $\varrho(x_n)\to 0$. It follows from lemma (*) that $x_n(t)$ is convergent in measure to zero in E, and $M[t, 2\min(x_n(t), v_0)]$ also converges in measure to zero in E. Hence

$$\int_{E_{2,n}} M[t, 2x_n(t)]d\mu \leqslant \int_E M[t, 2\min(x_n(t), v_0)]d\mu \to 0,$$

and $\varrho(2x_n)\to 0$ as $n\to\infty$; thus B.2 follows from 1.32 (b).

2.33. Finally, it will be noted that example 2.12 may be obtained from the present example if we put $E =$ the set of all positive integers, $\mu(A) =$ the number of elements of the set $A\subset E$, $M(u, v) = M_n(v)$

for $u = n$. Another special case of the function $M(u, v)$ may be obtained by putting

$$M(u, v) = [M(v)]^{p(u)},$$

where $M(v)$ is an even, continuous and non-decreasing (for $v\geqslant 0$) function, $M(0) = 0$, $M(v) > 0$ for $v > 0$, and $p(u)$ is a measurable positive function on E.

2.4. Given an even, continuous function $M(u)$, non-decreasing for $u\geqslant 0$, $M(0) = 0$, $M(u) > 0$ for $u > 0$. We define, in the class X of all real functions $x(t)$ measurable in $\langle 0, \infty)$, a pseudomodular $\varrho(x)$ as follows:

$$\varrho(x) = \varlimsup_{T\to\infty}\frac{1}{T}\int_0^T M[x(t)]dt.$$

2.41. *The class X_ϱ^* with the pseudomodular $\varrho(x)$ defined above is strongly ϱ-complete. Moreover, $\bar{X}_\varrho^* = X_\varrho^*$ if and only if for every $\varepsilon > 0$ there exist numbers $A_\varepsilon > 0$ and $\alpha_\varepsilon > 0$ such that $M(\alpha u) < \varepsilon M(u)$ for any $0\leqslant\alpha\leqslant\alpha_\varepsilon$, $u\geqslant A_\varepsilon$.*

First, we prove the strong ϱ-completeness of X_ϱ^* (see also [1]). Let the sequence $\{x_n\}\subset X_\varrho^*$ satisfy the modular Cauchy condition. Given an arbitrary sequence of numbers ε_k decreasing to zero, let us choose an increasing sequence of indices $n_1, n_2, \ldots$ satisfying the inequalities $\varrho(x_{n_k}-x_n) < \varepsilon_k$ for $n\geqslant n_k$. We now define a sequence $T_1, T_2, \ldots$ by induction. Put $T_1 = 0$; if $T_1, T_2, \ldots, T_{i-1}$ are defined, we choose T_i with the following properties:

1° if $T\geqslant T_i$, then

$$\frac{1}{T}\int_0^T M[x_{n_k}(t)-x_{n_i}(t)]dt < \varepsilon_k \quad \text{for} \quad k = 1, 2, \ldots, i-1,$$

$$\frac{1}{T}\int_0^T M[x_{n_i}(t)-x_{n_{i+1}}(t)]dt < \varepsilon_i;$$

2° $T_i > 2T_{i-1}$.

Now, we define the function $x(t)$ by the equalities $x(t) = x_{n_i}(t)$ for $T_{i-1}\leqslant t < T_i$. We shall prove that $\varrho[\frac{1}{4}(x_n-x)]\to 0$ as $n\to\infty$. Take an arbitrary index k, $m\geqslant k$ and a positive number T, where $T_m\leqslant T < T_{m+1}$. Then

$$\frac{1}{T}\int_0^T M\left(\frac{x_{n_k}(t)-x(t)}{2}\right)dt = \frac{1}{T}\sum_{i=1}^{k}\int_{T_{i-1}}^{T_i} M\left(\frac{x_{n_k}(t)-x_{n_i}(t)}{2}\right)dt +$$
$$+\frac{1}{T}\sum_{i=k+1}^{m}\int_{T_{i-1}}^{T_i} M\left(\frac{x_{n_k}(t)-x_{n_i}(t)}{2}\right)dt + \frac{1}{T}\int_{T_m}^{T} M\left(\frac{x_{n_k}(t)-x_{n_{m+1}}(t)}{2}\right)dt.$$

Since

$$\frac{1}{T}\sum_{i=k+1}^{m}\int_{T_{i-1}}^{T_i} M[x_{n_k}(t)-x_{n_i}(t)]dt \leqslant \frac{1}{T_m}\sum_{i=k+1}^{m} T_i \frac{1}{T_i}\int_0^{T_i} M[x_{n_k}(t)-x_{n_i}(t)]dt$$
$$\leqslant \frac{\varepsilon_k}{T_m}\sum_{i=1}^{m} T_i < 2\varepsilon_k$$

and

$$\frac{1}{T}\int_{T_m}^{T} M\left(\frac{x_{n_k}(t)-x_{n_{m+1}}(t)}{2}\right)dt \leqslant \frac{1}{T}\int_0^T M[x_{n_k}(t)-x_{n_m}(t)]dt+$$
$$+\frac{1}{T}\int_0^T M[x_{n_m}(t)-x_{n_{m+1}}(t)]dt < \varepsilon_k+\varepsilon_m,$$

we have $\varrho[\frac{1}{2}(x_{n_k}-x)] \leqslant 3\varepsilon_k \to 0$ as $k\to\infty$. Hence,

$$\varrho[\tfrac{1}{4}(x_n-x)] \leqslant \varrho[\tfrac{1}{2}(x_{n_k}-x_n)]+\varrho[\tfrac{1}{2}(x_{n_k}-x)] < 4\varepsilon_k \text{ for } n \geqslant n_k.$$

In order to prove the sufficiency in the second part of 2.41, let us take a sequence $a_n\to 0$. Writing $E_1=\{t\geqslant 0: |x(t)|\leqslant A_\varepsilon\}$ and $E_2=\langle 0,\infty)-E_1$, we obtain

$$\frac{1}{T}\int_0^T M[a_n x(t)]dt = \frac{1}{T}\int_{E_1\cap\langle 0,T\rangle} M(a_n x(t)]dt+\frac{1}{T}\int_{E_2\cap\langle 0,T\rangle} M[a_n x(t)]dt$$
$$\leqslant M(a_n A_\varepsilon)+\varepsilon\varrho(x)$$

for $0\leqslant a_n\leqslant a_\varepsilon$. In proving the necessity, we apply the method used in the proof of 2.15, with an arbitrary sequence v_n such that $M(v_n)>\frac{3}{2}$ for $n=1,2,\ldots$ instead of the sequence u_n. Defining the sequences n_k', n_k and a_k as in 2.15 but with v_n instead of u_n, we obtain

$$\frac{(n_k-n_k')M(v_k)}{n_k} > \frac{3}{4} \quad\text{and}\quad \frac{1}{n}\sum_{i=1}^{n} M(a_i) < \frac{3}{2}$$

for $k=1,2,\ldots$; $n=1,2,\ldots$ Then we choose $v_k=u_{i_k}$, u_k being defined as in the proof of 2.15, where i_k is the sequence $1,1,2,1,2,3,\ldots$ $\ldots,1,2,3,\ldots,n,\ldots$ and put $x(t)=a_i$ in $\langle i-1,i)$; thus, we obtain $\varrho(x)\leqslant\frac{3}{2}$ and $\varrho(a_k x)\geqslant\frac{3}{4}\varepsilon$, the sequence a_k and the number $\varepsilon>0$ being chosen as in 2.15.

2.42. *The class $\bar{X}_\varrho^*$ is an F-space with respect to the pseudonorm induced by the pseudomodular $\varrho(x)$, assuming $\varrho(x)$ such that $\varrho(x)=0$ implies $\varrho(2x)=0$* [1]. *Denote by X_0 the class of all $x\in X_\varrho^*$ such that the pseudonorm $\|x\|=0$; then the quotient space $\bar{X}_\varrho^*/X_0$ is an F-space complete with respect to the norm generated by the above pseudonorm.*

2.43. Let us note that the conditions sufficient for any sequence of elements of X_ϱ^* to satisfy B.2 may be formulated similarly to 2.32.

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[1] The theorem remains true without last hypothesis, Namely the norm-completeness of the $\bar{X}_\varrho^*$ may be proved directly by a slight modification of the argument of the proof on p. 63-64.