[9] A. Kolmogoroff, *Grundbegriffe der Wahrscheinlichkeitsrechnung*, Springer, Berlin, 1933.

[10] A. Rényi, *Valószinüségszámitás* (Textbook of Probability theory), Budapest, Tankönyvkiadó, 1954.

[11] P. Erdös and P. Turán, *On a problem of Sidon in additive number theory and on some related problems*, Journal London Math. Soc. 16 (1941), p. 212-215.

[12] S. Kakutani, *On equivalence of infinite product measures*, Annals of Math. 49 (1948), p. 214-224.

[13] A. Stöhr, *Gelöste und ungelöste Fragen über Basen der natürlichen Zahlen*, Journal für reine und angew. Math. 194 (1955), p. 40-65, 111-140.

[14] P. Erdös, *On some problems of Bellmann and a theorem of Romanoff*, Journal of the Chinese Math. Soc. (1951), p. 409-421.

[15] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, 3rd. edition, Oxford, 1954.

[16] P. Erdös and A. Rényi, *On Cantor's series with convergent $\sum 1/q_n$*, Annales Univ. R. Eötvös. Budapest, Sect. Math. 2 (1959) (in print).

[17] J. Karamata, *Neuer Beweis und Verallgemeinerung der Tauberschen Sätze, welche die Laplacesche und Stieltjessche Transformation betreffen*, J. reine u. angew. Math. 164 (1931), p. 27-39.

[18] N. G. de Bruijn and P. Erdös, *On a recursion formula and on some Tauberian theorems*, Journal of Research of the Nat. Bureau of Stand. 50 (1953), p. 161-164.

Reçu par la Rédaction le 28. 10. 1959

ACTA ARITHMETICA
VI (1960)

A sum involving the function of Möbius

by

D. H. Lehmer and S. Selberg (Berkeley, Cal., and Trondheim, Norway)

Let $\mu(n)$ be the Möbius function. The sum

$$g(x) = \sum_{n\leqslant x} \mu(n)/n$$

may itself be summed to give

$$G(x) = \sum_{n\leqslant x} g(n).$$

In this note we show that $G(x)-2$ changes sign infinitely often. Some numerical calculations of the first 56 sign changes are described. These show that these "zeros" of $G(x)-2$ are remarkably close to being in geometric progression with two exceptions. An heuristic explanation of this phenomenon is given.

It is equally easy to show that for any real K, the function $G(x)-K$ changes sign infinitely often. For this purpose we may treat $G(x)$ as a continuous, piecewise linear, function defined for $n\leqslant x<n+1$ by

$$G(x) = G(n)+(x-n)g(n+1). \tag{1}$$

Let $s=\sigma+it$ and suppose there is a u_0 such that $G(u)-K$ is of fixed sign for all $u\geqslant u_0$. For $\sigma>1$ we may write

$$\int_{u_0}^{\infty} \frac{G(u)-K}{u^{s+1}}\,du = -\int_{1}^{u_0} \frac{G(u)-K}{u^{s+1}}\,du + \int_{1}^{\infty} \frac{G(u)-K}{u^{s+1}}\,du$$
$$= \int_{1}^{\infty} \frac{G(u)}{u^{s+1}}\,du + \frac{1}{s}f(s)$$

where $f(s)$ is regular for $\sigma>0$. Using (1) we have, with $\sigma>1$,

$$\int_{u_0}^{\infty} \frac{G(u)-K}{u^{s+1}}\,du = \frac{1}{s}f(s)+\frac{1}{s}\sum_{n=1}^{\infty}\frac{g(n)}{n^s} + \sum_{n=1}^{\infty} g(n+1)\int_{0}^{1}\frac{\Theta\,d\Theta}{(n+\Theta)^{s+1}}. \tag{2}$$

Now

$$\sum_{n=1}^{\infty}\frac{g(n)}{n^s}=\int_1^{\infty}\frac{g(u)}{u^s}\,du+\sum_{n=1}^{\infty}g(n)\left\{\frac{1}{n^s}-\int_0^1\frac{d\Theta}{(n+\Theta)^s}\right\}$$

$$=\frac{1}{(s-1)\zeta(s)}+\sum_{n=1}^{\infty}g(n)\left\{\frac{1}{n^s}-\frac{1}{(n+1)^s}-s\int_0^1\frac{\Theta\,d\Theta}{(n+\Theta)^{s+1}}\right\}.$$

Substituting this into (2) gives, for $\sigma>1$,

$$\int_{u_0}^{\infty}\frac{G(u)-K}{u^{s+1}}\,du \tag{3}$$

$$=\frac{1}{s}f(s)+\frac{1}{s(s-1)\zeta(s)}+\frac{1}{s\zeta(s+1)}+\sum_{n=1}^{\infty}\frac{\mu(n+1)}{n+1}\int_0^1\frac{\Theta\,d\Theta}{(n+\Theta)^{s+1}}.$$

Now this last sum, being less than

$$\sum_{n=1}^{\infty}\frac{1}{n^{\sigma+2}}$$

in absolute value, represents a regular function for $\sigma>0$. By a theorem of E. Landau [1] the left member of (3) is regular for $\sigma>0$. But the right member of (3) is certainly not regular for $\sigma>0$. Thus $G(u)-K$ must change sign infinitely often.

Numerical calculations. A preliminary calculation of $G(x)$ for $x=1\ (1)\ 6017$ was made in 1954 at Norsk Regnesentral in Oslo. The 27 zeros x_k of $G(x)-2$ in this interval were seen to be nearly in geometric progression such that the ratio

$$r_k=x_k/x_{k-1}\approx\tfrac{5}{4}.$$

It was decided to test this phenomenon further by a more powerful computer, the IBM 701 at the University of California. The function $G(x)$ was explored as far as $x=2{,}125{,}000$, just beyond x_{56}. The following table gives first 56 zeros of $G(x)-2$, that is, integers x_k such that $G(x)-2$ is of different sign from $G(x-1)-2$, together with the ratio r_k.

The larger values of x_k given in the table may contain small errors due to accumulated round-off errors in $g(x)$. However the values of r_k are correct as given.

It is seen that zeros x_{47} and x_{48} do not conform. Leaving them out we find $x_{49}/x_{46}=1.2792$. Averaging this with the other values of r_k $(k\neq 47,48)$ we have a mean of

$$\bar{r}=1.2495. \tag{4}$$

k	x_k	r_k	k	x_k	r_k	k	x_k	r_k	k	x_k	r_k
1	16	—	15	353	1.2172	29	8378	1.2635	43	179979	1.2285
2	19	1.1875	16	464	1.3144	30	10108	1.2065	44	230512	1.2808
3	26	1.3684	17	572	1.2328	31	12263	1.2132	45	297509	1.2906
4	31	1.1923	18	684	1.1958	32	16299	1.3291	46	352766	1.1857
5	39	1.2581	19	919	1.3436	33	19995	1.2268	47	430018	1.2190
6	48	1.2308	20	1116	1.2144	34	24283	1.2145	48	437595	1.0176
7	62	1.2917	21	1400	1.2545	35	31359	1.2914	49	451257	1.0312
8	79	1.2742	22	1701	1.2150	36	38843	1.2387	50	588047	1.3031
9	96	1.2152	23	2178	1.2804	37	48709	1.2540	51	696967	1.1852
10	115	1.1979	24	2821	1.2952	38	60078	1.2334	52	881576	1.2649
11	150	1.3043	25	3351	1.1879	39	73075	1.2163	53	1067178	1.2105
12	193	1.2867	26	4137	1.2346	40	96510	1.3207	54	1381082	1.2941
13	227	1.1762	27	5381	1.3007	41	118603	1.2289	55	1751318	1.2681
14	290	1.2775	28	6631	1.2323	42	146503	1.2352	56	2119483	1.2102

By partial summation

$$G(x-1)=xg(x)-M(x) \tag{5}$$

where, as usual

$$M(x)=\sum_{n\leqslant x}\mu(n).$$

This identity was used to test the amount of round-off error in G and g. As $g(x)$ converges to zero with increasing x it becomes more difficult to decide just when $G(x-1)$ and

$$G(x)=G(x-1)+g(x)$$

are of different sign.

Since

$$\sum_{n=1}^{\infty}\frac{\mu(n)\,n^{-s}}{n}=1/\zeta(s+1)\qquad(\sigma\geqslant 0)$$

we have

$$2\pi i\,g(x)=\int_{0-\infty i}^{0+\infty i}\frac{1}{s}\cdot\frac{1}{\zeta(s+1)}\,x^s\,ds=\int_{1-\infty i}^{1+\infty i}\frac{x^{s-1}}{s-1}\cdot\frac{ds}{\zeta(s)}.$$

Also

$$2\pi i\,M(x)=\int_{1-\infty i}^{1+\infty i}\frac{x^s\,ds}{s\,\zeta(s)}.$$

Hence by (5)

$$2\pi i\,G(x-1)=\int_{1-\infty i}^{1+\infty i}\frac{x^s\,ds}{s(s-1)\zeta(s)}.$$

Assuming that the zeros of $\zeta(s)$ are all simple and noting that

$$\zeta(0) = -\tfrac{1}{2},$$

we have, by the residue theorem,

$$G(x-1)-2 = -\sum \frac{x^{\varrho}}{\varrho(1-\varrho)\zeta'(\varrho)} \tag{6}$$

where the sum extends over the zeros ϱ of $\zeta(s)$. The contribution made by the trivial zeros ($\varrho = -2, -4, -6, \ldots$) is of order x^{-2} and is negligible in comparison to that made by the complex zeros

$$\varrho_\nu = \tfrac{1}{2}+\gamma_\nu i \quad \text{and} \quad \bar{\varrho} = \tfrac{1}{2}-\gamma_\nu i$$

where $\gamma_\nu > 0$, on the assumption of the Riemann hypothesis.

With this notation (6) becomes

$$G(x-1)-2 = 8x^{1/2}\sum_{\nu=1}^{\infty} \frac{\cos\{\gamma_\nu \log x - a_\nu\}}{(1+4\gamma_\nu^2)|\zeta'(\tfrac{1}{2}+\gamma_\nu i)|} \qquad (a_\nu = \arg\zeta'(\varrho_\nu)). \tag{7}$$

If we neglect all terms but the first, we obtain a crude approximation to $G(x-1)-2$ which is periodic in $\log x$ of halfperiod

$$\pi/\gamma_1 = .22226061.$$

Thus we can expect that, on the average, the roots of $G(x)-2$ will be in geometric progression of ratio

$$e^{\pi/\gamma_1} = 1.2488968$$

which agrees fairly well with the evidence (4).

The unexpected zeros x_{47} and x_{48} result from a conspiracy between the other zeros of $\zeta(s)$ to upset the usual dominating first zeros $\frac{1}{2}\pm\gamma_1 i$.

The series (7) can be used to calculate isolated values of $G(x)$ with limited accuracy. Thus the first 25 terms give 12.204 for the value of $G(323000)$ whereas the true value is close to 12.360. Values of γ_ν, a_ν, $|\zeta'(\frac{1}{2}+\gamma_\nu i)|$ were kindly supplied independently by Drs. C. B. Haselgrove and R. S. Lehman from as yet unpublished tables.

Reference

[1] E. Landau, *Vorlesungen über Zahlentheorie*, Leipzig 1927, v. 2, p. 126, theorem 454.

Reçu par la Rédaction le 3. 11. 1959

ACTA ARITHMETICA
VI (1960)

ERRATA

In the line immediately following formula (7) on page 78 replace

$$\tfrac{1}{2}[(u_2+u_3)] = [\tfrac{1}{2}(k-u_1-1)]-1$$

by

$$[\tfrac{1}{2}(u_2+u_3)] = \tfrac{1}{2}[(k-u_1-1)]-1.$$