Bei der Abschätzung von $\sum_{\varrho}\left(L^{\varrho-\varrho_0}\frac{\varrho_0}{\varrho}\right)^{k+1}$ wir wenden den Satz von P. Turán an ([6], Satz X) und verfahren so wie in [4], Seite 193. In unserem Fall ist die Bedingung [4], 5.28, auch erfüllt, weil

$$(4.19)\qquad T_L \geqslant L > \log^{1/6}T > \exp(\tfrac{1}{3}\log\log\log\log T) > |\varrho_0| > |I\varrho_0|.$$

Nach dem Muster von [4] (Seite 193-195), kommen wir einfach zum Schluß des Beweises.

Durch Vergleich der beiden Abschätzungen (3.3) und (4.1) bekommt man (1.4).

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On the representation of $1, 2, \ldots, n$ by sums

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If $A\colon 0 = a_1 < a_2 < \ldots < a_k$ is a set of integers such that every positive integer not exceeding n is the sum of two elements of A then A is called a *2-basis for n*. In what follows we let $k = k(n)$ be the smallest integer for which a 2-basis for n with k elements exists, and let A be such a minimal 2-basis. The problem of estimating $k(n)$ was first proposed by I. Schur. Since we can form only $k(k-1)/2$ pairs of distinct elements of A (disregarding order) and k sums of the form $2a_r$, we have $(k^2+k)/2 \geqslant n+1$. On the other hand, the numbers $0, 1, 2, \ldots, [\sqrt{n}-1], [\sqrt{n}], 2[\sqrt{n}], \ldots, [\sqrt{n}+1][\sqrt{n}]$ are easily seen to form a 2-basis with $2[\sqrt{n}]+1$ elements. The only improvements on these trivial estimates seem to be those of Rohrbach [1] who proved that for n sufficiently large

$$(1)\qquad \frac{k^2}{2}(1-.0016) > n.$$

Although Rohrbach conjectured that $.25k^2 \sim n$ his proof of the much weaker result (1) is rather complicated. The object of this note is to prove the better estimate

$$(2)\qquad \frac{k^2}{2}(1-.0197) > n.$$

To the set A we make correspond the generating function

$$(3)\qquad f(x) = \sum_{j=1}^{n} x^{a_j}$$

and let

$$(4)\qquad g(x) = \big(f^2(x)+f(x^2)\big)/2.$$

The coefficient of x^j in $g(x)$ will be the number of representations of j in the form $a_r + a_s$ where order is not taken into account. We now define $\delta(j)$ by

$$(5)\qquad g(x) = 1+x+x^2+\ldots+x^n+\sum_{j=0}^{2n}\delta(j)x^j.$$

Since A is a 2-basis for n it follows that $\delta(j) \geqslant 0$ for all j. Setting $x=1$ we may use (3), (4), and (5) to regain the trivial estimate $(k^2+k)/2 \geqslant n+1$. We will show however that $\sum \delta(j)$ is large and thus improve on this estimate.

Note that for $x = e^{2\pi i t/(n+1)}$ (t a positive integer) the term $1+x+\ldots+x^n$ in (5) vanishes. We will use this value of x in (5) with $t=1$ and later with $t=2$. Thus from (5) we have

$$g(e^{2\pi i/(n+1)}) = \sum \delta(j) e^{2\pi i j/(n+1)}, \tag{6}$$

and using $\delta(j) \geqslant 0$

$$|g(e^{2\pi i/(n+1)})| \leqslant \sum \delta(j). \tag{7}$$

Now from (4), (7) and the triangle inequality,

$$\sum \delta(j) \geqslant \tfrac{1}{2}\Big[\Big|\sum e^{2\pi i a_j/(n+1)}\Big|^2 - k\Big] \geqslant \tfrac{1}{2}\Big[\Big(\sum \sin 2\pi a_j/(n+1)\Big)^2 - k\Big]. \tag{8}$$

Similarly, from $x = e^{4\pi i/(n+1)}$ we obtain

$$\tfrac{1}{4}\sum \delta(j) \geqslant \tfrac{1}{8}\Big[\Big(\sum \cos 4\pi a_j/(n+1)\Big)^2 - k\Big]. \tag{9}$$

Combining (8) and (9) yields

$$\sum \delta(j) \geqslant \tfrac{2}{5}\Big[\Big(\sum \sin 2\pi a_j/(n+1)\Big)^2 + \Big(\sum \tfrac{1}{2}\cos 4\pi a_j/(n+1)\Big)^2\Big] - k/2, \tag{10}$$

and since $a^2+b^2 \geqslant \frac{1}{2}(a+b)^2$,

$$\sum \delta(j) \geqslant \tfrac{1}{5}\Big(\sum \big(\sin 2\pi a_j/(n+1) + \tfrac{1}{2}\cos 4\pi a_j/(n+1)\big)\Big)^2 - k/2. \tag{11}$$

We next obtain a lower bound for the right hand side of (11) using the easily established fact that

$$\sin \pi\theta + \tfrac{1}{2}\cos 2\pi\theta \geqslant \begin{cases} \tfrac{1}{2} & \text{for} \quad 0 < \theta \leqslant 1, \\ -\tfrac{3}{2} & \text{for} \quad 1 < \theta \leqslant 2. \end{cases} \tag{12}$$

Indeed if we let l be the number of elements of A which exceed $n/2$ then (11) and (12) yield

$$\sum \delta(j) \geqslant \tfrac{1}{5}\Big(\frac{k-l}{2} - \frac{3l}{2}\Big)^2 - \frac{k}{2} = \frac{(k-4l)^2}{20} - \frac{k}{2}. \tag{13}$$

If l is small we will use the estimate (13). On the other hand, since l elements of A exceed $n/2$, at least $l^2/2$ pairs of a's (again not taking order into account) will exceed n. Hence (5) yields

$$\sum \delta(j) \geqslant l^2/2. \tag{14}$$

Now we consider the cases (i) $l < k(4-\sqrt{10})/6$ and (ii) $l \geqslant k(4-\sqrt{10})/6$. In case (i) (13) yields

$$\sum \delta(j) \geqslant k^2\Big(\frac{13-4\sqrt{10}}{36}\Big) - \frac{k}{2} \tag{15}$$

while in case (ii) (15) follows from (14). Finally combining (15) and (5) at $x=1$ gives

$$\frac{k^2+k}{2} \geqslant n+1+k^2\Big(\frac{13-4\sqrt{10}}{36}\Big) - \frac{k}{2}. \tag{16}$$

For all n (16) can be put in the form

$$k^2\Big(\frac{5+4\sqrt{10}}{36}\Big) + k \geqslant n+1. \tag{17}$$

For n sufficiently large (17) implies (2).

We remark that since we have made no use of the fact that the a's are integers our method actually proves that for k sufficiently large, the sums in pairs of k non negative numbers cannot represent 0 and the first $[.4902\, k^2]$ multiples of a fixed number.

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