Note that in (4.1) the modulus is m^r while in (4.2) it is only m^{r_1}. As remarked at the end of § 3, the hypothesis (3.13) implies in particular

$$U_2^{(r)} \equiv 0 \pmod{m^r},$$

but the converse is apparently not true.

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On the average number of direct factors of a finite abelian group

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1. Introduction. For positive integers n, let $\tau(n)$ denote the number of divisors of n, and let $t(n)$ denote the number of decompositions of n into two relatively prime factors. In this paper we prove analogues for the finite abelian groups of the classical results of Dirichlet and Mertens on the average order of $\tau(n)$ and $t(n)$. We recall Dirichlet's formula [4], with $x \geqslant 2$,

$$D(x) \equiv \sum_{n \leqslant x} \tau(n) = x(\log x + 2\gamma - 1) + O(\sqrt{x}), \tag{1.1}$$

γ denoting Euler's constant, and Mertens's estimate [8],

$$D^*(x) \equiv \sum_{n \leqslant x} t(n) = ax(\log x + 2\gamma - 1) + 2bx + O(\sqrt{x}\log x), \tag{1.2}$$

where $a = \eta(2)$, $b = \eta'(2)$, $\eta(s) = 1/\zeta(s)$, $\zeta(s)$ denoting the Riemann zeta-function, $s > 1$. For proofs and discussions of (1.1) and (1.2) we mention [1], §§ 13.2, 13.5, 13.9, [3], p. 282-283, 289, [7], p. 665-666.

The functions $\tau(n)$ and $t(n)$ can be generalized from the (multiplicative) semigroup J^* of the integers n to the semigroup X of the finite abelian groups with respect to the direct product. A general discussion of functions defined in X appears in [2]. For groups G, H contained in X, denote by (G, H) the group of maximal order in X which is simultaneously a direct factor of G and H. Denoting by E_0 the identity of X, we say that G and H are *relatively prime* if $(G, H) = E_0$. A direct factor D of G will be called *unitary* if $D \times E = G$, $(D, E) = E_0$.

For groups G in X, let $\tau(G)$ denote the number of direct factors of G in X, or equivalently, the total number of decompositions, $G = D \times E$, in X. Analogously, let $t(G)$ denote the number of *unitary* factors of G in X, that is, the total number of direct decompositions of G into two relatively prime factors of X. In view of the isomorphism [2] of J^* with the sub-semigroup J of the completely reducible groups of X, it follows that the functions $\tau(G)$ and $t(G)$ reduce to $\tau(n)$ and $t(n)$, respectively, when G is restricted to J.

The *level function* $l_f(n)$ of a function $f(G)$ over X is defined by

$$l_f(n) = \sum_{\varrho(G)=n} f(G),$$

where the summation is over all G in X of order $\varrho(G) = n$; the *summatory function* of $f(G)$ is defined by

$$S_f(x) = \sum_{\varrho(G)\leqslant x} f(G) = \sum_{n\leqslant x} l_f(n).$$

In this paper we obtain estimates analogous to (1.1) and (1.2) for the summatory functions $T(x)$ and $T^*(x)$ of $\tau(G)$ and $t(G)$, respectively. The remainder of this section is devoted to an account of the results of the paper and the methods of proof.

It is recalled that Dirichlet's proof of (1.1) is based on an ingenious though natural transformation of $D(x)$, while Mertens's proof of (1.2) is derived from Dirichlet's result by means of a simple identity expressing $t(n)$ in terms of $\tau(n)$. Using an analogous approach in § 3, we obtain elementary estimates for $T(x)$ and $T^*(x)$ (Theorems 3.1 and 3.2). An important tool in this approach is the Erdös-Szekeres estimate [5] for the number $A(x)$ of abelian groups of order $\leqslant x$ (see Lemma 2.6).

While the proofs of § 3 are quite short, the remainder terms obtained are relatively crude in comparison to those of (1.1) and (1.2). The two final sections (§§ 4, 5) are devoted to a more penetrating treatment of $T(x)$ and $T^*(x)$, the specific goal being to reduce significantly the error terms of § 2 without resorting to non-elementary methods. In accord with this aim, the remainder terms for $T(x)$ and $T^*(x)$ are decreased from $O(x^{3/4})$ to $O(\sqrt{x}\log^2 x)$ in the case of $T(x)$, (Theorem 5.1) and to $O(\sqrt{x}\log x)$ in the case of $T^*(x)$ (Theorem 4.1). The principal devices employed are generating functions defined by (real) Dirichlet series and the classical estimate (1.1) for $D(x)$.

Oddly enough, the rôles performed by $\tau(G)$ and $t(G)$ in the Dirichlet-Mertens procedure followed in § 3, are reversed in the more thorough treatment of §§ 4 and 5. More precisely, $T^*(x)$ is investigated independently, while the evaluation of $T(x)$ is made to depend upon that obtained for $T^*(x)$ (cf. Remark 5.1). The reverse procedure is applicable, but is less effective, yielding an error term of $O(\sqrt{x}\log^3 x)$ in the estimate for $T^*(x)$. Moreover, a self-contained treatment of $T(x)$ leads to an estimate no better than the one obtained in § 5; consequently, the alternative procedure will not be developed in this paper.

A second item merits a brief reference. In his study of $D^*(x)$, Mertens found it convenient to replace $t(n)$ by an equivalent function $r(n)$, defined to be the number of square-free divisors of n. For the sake of historical analogy, a similar device is employed in § 3. Such an artifice is not, however, essential to the paper, and is in fact abandoned in the independent discussion of § 4.

An unexpected by-product of the paper is a simple evaluation of the abelian analogue of the Euler constant (Corollary 5.1, cf. (2.7)). This result arises on comparing the estimate for $T(x)$ obtained in § 3 with that obtained in § 5.

A number of the details of the paper are absorbed in the preliminary lemmas of § 2. Some of these results were proved in [2]; others are well known and a few are new. Finally, as in [2], we make consistent use of the Basis Theorem for finite abelian groups ([10], III, § 4).

Remark 1.1. In this paper x will usually be assumed $\geqslant 2$. All asymptotic results are valid for $1\leqslant x\leqslant 2$ with the O-term replaced by $O(1)$.

2. Lemmas. We first review some terminology and notation introduced in [2]. A direct product, $G\times\ldots\times G$ to k factors, will be denoted G^k and called a k-th power group. If G possesses no square direct factor other than E_0, that is, if its indecomposable factors are distinct, then G will be termed *separable* (or square-free). The function $I(G)$ is defined to be 1 for all G; the level function of $I(G)$ is denoted $a(n)$ and its summatory function $A(x)$. We define $\gamma(G)$ to be 1 or 0 according as G is or is not separable, and $\gamma'(G)$ to be 1 or 0 according as G is or is not a square. The level function of $\gamma(G)$ is denoted $q(n)$ and its summatory function $Q(x)$. The inversion function $\mu(G)$ of X is defined, for separable groups G, to be $(-1)^r$ if G possesses exactly r indecomposable direct factors; otherwise $\mu(G)$ is defined to be 0. The level function of $\mu(G)$ is denoted $\nu(n)$, while the level functions of $\tau(G)$ and $t(G)$ are designated $\tau_1(n)$ and $t_1(n)$, respectively.

If e is a non-negative integer, then $q_e(n)$ will denote the number of abelian groups in X of order n whose indecomposable factors G_i are of order $p_i^{r_i}$, p_i prime, r_i odd and $\geqslant t = 2e+1$, for all i. For positive integers k, $a_k(n)$ is used to denote the number of groups in X whose indecomposable direct factors G_i are of order $p_i^{r_i}$, p_i prime, $r_i\geqslant k$ for all i. It is noted that $a(n) = a_1(n)$ is the total number of abelian groups of order n.

If $f(G)$ and $g(G)$ are two (complex-valued) functions defined in X, then the direct convolution of $f(G)$ and $g(G)$ is a function $h(G)$ defined by

$$h(G) = \sum_{D\times E=G} f(D)g(E), \tag{2.1}$$

where the summation is over all D, E in X such that $D\times E = G$. Symbolically, (2.1) will be written $h = f\cdot g$. Let R denote the binary system of functions defined in X with products determined by (2.1) and sums by ordinary function addition.

LEMMA 2.1 ([2], Theorem 2.1). *The system R is a commutative ring with identity element, $\varepsilon(G)\equiv 1$ or 0 according as $G = E_0$ or $G\neq E_0$.*

LEMMA 2.2 ([2], Lemma 3.1). *If $k\geqslant 1$, then*

$$Z_k(s)\equiv\sum_{n=1}^{\infty}\frac{a_k(n)}{n^s}=\prod_{i=k}^{\infty}\zeta(is),\qquad s>1/k, \tag{2.2}$$

where the series and product are absolutely convergent for $s>1/k$.

LEMMA 2.3 ([2], Lemma 3.2). *For $s>1$,*

$$Z(s)\equiv Z_1(s)=\prod_{i=1}^{\infty}\zeta(is),\qquad \sum_{G\epsilon X}\frac{\mu(G)}{\varrho^s(G)}=\prod_{i=1}^{\infty}\zeta^{-1}(is). \tag{2.3}$$

The second relation in (2.3) arises from the relation,

LEMMA 2.4 ([2], Theorem 2.5).

$$\sum_{G\epsilon X}\frac{\mu(G)}{\varrho^s(G)}=\sum_{n=1}^{\infty}\frac{\nu(n)}{n^s}=\frac{1}{Z(s)}\equiv Y(s),\qquad s>1, \tag{2.4}$$

which is itself a consequence of

LEMMA 2.5 ([2], Theorem 2.2).

$$\sum_{D\times E=G}\mu(D)=\varepsilon(G). \tag{2.5}$$

The latter result states that $\mu(G)$ and $I(G)$ are multiplicative inverses in the ring R.

LEMMA 2.6 ([2], Lemma 3.3, also cf. [5], [6]). *If $x\geqslant 1$, then*

$$A(x)=ax+O(\sqrt{x}),\qquad a=Z_2(1). \tag{2.6}$$

Lemmas 2.2 and 2.6 are due to Erdös and Szekeres.

LEMMA 2.7 ([2], (2.12)). *The function $\nu(n)$ is bounded.*

LEMMA 2.8 ([2], Theorem 5.2). *There exists a constant γ^* such that*

$$\sum_{\varrho(G)\leqslant x}\frac{1}{\varrho(G)}=\sum_{n\leqslant x}\frac{a(n)}{n}=a\log x+\gamma^*+O\left(\frac{1}{\sqrt{x}}\right),\qquad x\geqslant 1. \tag{2.7}$$

The constant γ^*, which is the analogue in X of the Euler-Mascheroni constant is evaluated in § 5. Corresponding to (2.2) we have ([2], Lemma 6.2),

LEMMA 2.9. *For e and t as defined above,*

$$F_e(s)\equiv\sum_{n=1}^{\infty}\frac{q_e(n)}{n^s}=\prod_{j=e}^{\infty}\zeta\big((2j+1)s\big),\qquad s>1/t, \tag{2.8}$$

the series and product being (absolutely) convergent for all $s>1/t$.

In case $e=0$ we have

LEMMA 2.10 ([2], Lemma 6.1, (6.3)).

$$q(n)=q_0(n),\qquad F(s)\equiv\sum_{n=1}^{\infty}\frac{q(n)}{n^s}=\prod_{j=0}^{\infty}\zeta\big((2j+1)s\big), \tag{2.9}$$

for all $s>1$.

Define $\mu_2(G)$ to be $\mu(H)$ if $G=H^2$ is a square and 0 otherwise.

LEMMA 2.11 ([2], Lemma 4.2, $k=2$).

$$\sum_{D\times E=G}\mu_2(D)=\gamma(G), \tag{2.10}$$

$$\sum_{D\times E=G}\mu_2(D)\gamma'(E)=\varepsilon(G). \tag{2.11}$$

Remark 2.1. The level function of $\mu_2(G)$ is the function $\nu_2(n)$ defined to be $\nu(m)$ if $n=m^2$ is a square, 0 otherwise.

Recalling the notation of § 1 for level functions, we prove now

LEMMA 2.12. *If the relation (2.1) holds for functions $f(G)$, $g(G)$, $h(G)$, then*

$$\sum_{de=n}l_f(d)l_g(e)=l_h(n). \tag{2.12}$$

Proof. We sum both sides of (2.1) over all $G\epsilon X$ of order n. The result obtained from the left member is clearly $l_h(n)$. On the right one obtains

$$\sum_{\substack{D\times E=G\\ \varrho(G)=n}}f(D)g(E)=\sum_{\varrho(D)\varrho(E)=n}f(D)g(E)=\sum_{de=n}\sum_{\varrho(D)=d}f(D)\sum_{\varrho(E)=e}g(E),$$

from which (2.12) follows.

LEMMA 2.13.

$$\sum_{n\leqslant x}\frac{a(n)}{n^s}=\begin{cases}O(x^{1-s}) & \textit{if}\quad s<1,\\ O(\log x) & \textit{if}\quad s=1,\\ O(1) & \textit{if}\quad s>1.\end{cases}\tag{2.13}$$

Proof. Since $a(n)>0$, one obtains by (2.6) and partial summation,

$$\sum_{n\leqslant x}\frac{a(n)}{n^s}=\sum_{n\leqslant x}A(n)\frac{\big((1+1/n)^s-1\big)}{(n+1)^s}+\frac{A(x)}{([x]+1)^s}$$
$$=O\Big(\sum_{n\leqslant x}\frac{A(n)}{n^{s+1}}\Big)+O\Big(\frac{1}{x^{s-1}}\Big)=O\Big(\sum_{n\leqslant x}\frac{1}{n^s}\Big)+O(x^{1-s}),$$

and (2.13) easily follows.

The following additional estimate results on comparing sums to integrals.

LEMMA 2.14. *If $s>1$, then*

$$\sum_{n>x}\frac{\log n}{n^s}=O\Big(\frac{\log x}{x^{s-1}}\Big).\tag{2.14}$$

In the next two lemmas, the term, summatory function, has the significance of ordinary number theory.

LEMMA 2.15 (cf. [9], Satz 2). *Let $f(n)$, $g(n)$ denote arithmetical functions and $G(x)$ the summatory function of $g(n)$; then*

$$H(x)\equiv\sum_{n\leqslant x}h(n)=\sum_{n\leqslant x}f(n)G(x/n),\qquad \textit{where}\qquad h(n)=\sum_{de=n}f(d)g(e).$$

LEMMA 2.16 (cf. [3], p. 317-318). *In the notation of the preceding lemma, with $F(x)$ representing the summatory function of $f(n)$,*

$$H(x)=\sum_{n\leqslant\sqrt{x}}f(n)G(x/n)+\sum_{n\leqslant\sqrt{x}}g(n)F(x/n)-F(\sqrt{x})G(\sqrt{x}).$$

Finally, we obtain a representation for the level function of $\tau(G)$ in terms of $a(n)$.

LEMMA 2.17.

$$\tau_1(n)=\sum_{de=n}a(d)a(e).\tag{2.15}$$

Proof. By definition

$$\tau(G)=\sum_{D\times E=G}1=\sum_{D\times E=G}I(D)I(E),\tag{2.16}$$

and (2.16) results by Lemma 2.12.

3. Initial estimates for $T(x)$ and $T^*(x)$. Corresponding to Dirichlet's formula (1.1), we now prove

THEOREM 3.1. *If a and γ^* are defined as in § 2, then for $x\geqslant 1$,*

$$T(x)=ax(a\log x+2\gamma^*-a)+O(x^{3/4}).\tag{3.1}$$

Proof. We apply Lemma 2.16 to (2.15), with $f(n)=g(n)=a(n)$, $h(n)=\tau_1(n)$, to obtain

$$T(x)=2\sum_{n\leqslant\sqrt{x}}a(n)A(x/n)-A^2(\sqrt{x})\equiv 2R_1-R_2.\tag{3.2}$$

Application of (2.6) yields

$$R_1=ax\sum_{n\leqslant\sqrt{x}}\frac{a(n)}{n}+O\Big(\sqrt{x}\sum_{n\leqslant\sqrt{x}}\frac{a(n)}{n^{1/2}}\Big);$$

which becomes, on the basis of Lemma 2.8 and Lemma 2.13 ($s=1/2$), with x replaced by $\sqrt{x}$,

$$R_1=\tfrac{1}{2}a^2x\log x+a\gamma^*x+O(x^{3/4}).\tag{3.3}$$

Also, by (2.6),

$$R_2=\big(ax^{1/2}+O(x^{1/4})\big)^2=a^2x+O(x^{3/4}).\tag{3.4}$$

The Theorem results on combining (3.2), (3.3), and (3.4).

We denote by $r(G)$ the number of separable direct factors of G and let $r_1(n)$ and $R^*(x)$ denote the level and summatory functions, respectively, of $r(G)$.

LEMMA 3.1. *For all G in X, $r(G)=t(G)$.*

Proof. Corresponding to each separable direct factor $D\neq E_0$ of G, there is determined a u n i q u e unitary factor D' of G, such that D is the product of the distinct indecomposable factors of D'. Since E_0 is both separable and unitary, there is a one-to-one correspondence between the unitary and separable factors in X of G. This proves the Lemma.

LEMMA 3.2.

$$r(G)=\sum_{D\times E=G}\mu_2(D)\tau(E),\qquad r_1(n)=\sum_{de=n}\nu_2(d)\tau_1(e).\tag{3.5}$$

Proof. By definition

$$r(G) = \sum_{D\times E} \gamma(D) = \sum_{D\times E} \gamma(D) I(E);$$

in the abbreviated notation of § 2, this may be written, by (2.10) and (2.16), $r = \gamma\cdot I = (\mu_2\cdot I)\cdot I = \mu_2\cdot(I\cdot I) = \mu_2\cdot\tau$, using the fact that the convolution is associative (Lemma 2.1). This proves the first formula of (3.5). The second follows from the first by Lemma 2.12.

THEOREM 3.2.

$$T^*(x) = a\alpha^*(a\log x + 2\gamma^* - a) + 2a^2\beta^* x + O(x^{3/4}), \tag{3.6}$$

where $\alpha^* = Y(2)$, $\beta^* = Y'(2)$, $Y(s) = 1/Z(s)$.

Proof. By (3.5), Lemma 3.1, and Lemma 2.15,

$$T^*(x) = R^*(x) = \sum_{n\leqslant x} \nu_2(n) T(x/n) = \sum_{n\leqslant\sqrt{x}} \nu(n) T(x/n^2). \tag{3.7}$$

Application of (3.1) to (3.7) yields, with $c = 2\gamma^* - a$, since $\nu(n)$ is bounded (Lemma 2.7),

$$T^*(x) = ax(a\log x + c)\sum_{n\leqslant\sqrt{x}} \frac{\nu(n)}{n^2} - 2a^2x \sum_{n\leqslant\sqrt{x}} \frac{\nu(n)\log n}{n^2} + O\Big(x^{3/4}\sum_{n\leqslant\sqrt{x}} \frac{1}{n^{3/2}}\Big),$$

which will be denoted, for convenience,

$$T^*(x) = ax(a\log x + c)S_1 - 2a^2xS_2 + O(x^{3/4}). \tag{3.8}$$

As for S_1, we have, by Lemma 2.7,

$$S_1 = \sum_{n\leqslant\sqrt{x}} \frac{\nu(n)}{n^2} = \sum_{n=1}^{\infty} \frac{\nu(n)}{n^2} + O\Big(\sum_{n>\sqrt{x}} \frac{1}{n^2}\Big),$$

so that by (2.4),

$$S_1 = \alpha^* + O(1/\sqrt{x}). \tag{3.9}$$

Again by Lemma 2.7,

$$S_2 = \sum_{n\leqslant\sqrt{x}} \frac{\nu(n)\log n}{n^2} = \sum_{n=1}^{\infty} \frac{\nu(n)\log n}{n^2} + O\Big(\sum_{n>\sqrt{x}} \frac{\log n}{n^2}\Big),$$

and hence by (2.4) and (2.14),

$$S_2 = -\beta^* + O\Big(\frac{\log x}{\sqrt{x}}\Big). \tag{3.10}$$

Thus the Theorem is a consequence of (3.8), (3.9), and (3.10).

Historical remark. The formula (3.2) is the analogue in $\mathfrak{X}$ of Dirichlet's transformation ([3], Chap. 10, (11)), while (3.5) is the abelian analogue of the Mertens identity ([3], p. 289).

4. Refined estimate for $T^*(x)$. We give a new proof of Lemma 3.2 in terms of $t(G)$.

LEMMA 4.1.

$$t(G) = \sum_{D\times E=G} \mu_2(D)\tau(E), \qquad t_1(n) = \sum_{d\delta=n} \nu_2(d)\tau_1(\delta). \tag{4.1}$$

Proof. By (2.5), the definition of $\varepsilon(G)$, and the Basis Theorem,

$$T(G) = \sum_{\substack{D\times E=G\\(D,E)=E_0}} 1 = \sum_{D\times E=G} \varepsilon\big((D,E)\big) = \sum_{D\times E=G}\ \sum_{H\times H_1=(D,E)} \mu(H)$$

$$= \sum_{D\times E=G}\ \sum_{\substack{H\times D_1=D\\H\times E_1=E}} \mu(H) = \sum_{H^2\times G_1=G} \mu(H) \sum_{D_1\times E_1=G_1} 1$$

$$= \sum_{H^2\times G_1=G} \mu(H)\tau(G_1),$$

which yields the first relation of (4.1). The second follows by Lemma 2.12.

LEMMA 4.2. *If* $s>1$, *then*

$$f(s) \equiv \sum_{n=1}^{\infty} \frac{t_1(n)}{n^s} = \frac{Z^2(s)}{Z(2s)} = \prod_{i=1}^{\infty} \zeta^2\big((2i-1)s\big)\zeta(2is). \tag{4.2}$$

Proof. Using $*$ to denote convolution of ordinary arithmetical functions, it follows by (2.15) and (4.1) that $t_1(n) = \nu_2(n)*\tau_1(n) = \nu_2(n)*$ $*\big(a(n)*a(n)\big)$. Thus (4.2) follows by Dirichlet multiplication, on the basis of (2.3) and (2.4).

The function $f(s)$ in (4.2) is called the *generating function* of $t(G)$.

Remark 4.1. The following relation, which will not be used, may be deduced from (4.2), by virtue of (2.3) and (2.9):

$$t_1(n) = \sum_{d\delta=n} q(d)a(\delta). \tag{4.3}$$

This relation also follows directly from Lemma 3.1, by Lemma 2.12, and could be used, in place of (4.1) as an alternative basis of discussion for this section.

Next we define

$$t_2(n) = \sum_{d\delta=n} a_2(d)q_1(\delta), \qquad t_3(n) = \sum_{d\delta=n} a_3(d)q_1(\delta), \tag{4.4}$$

using the notation of § 2. Application of Lemmas 2.2 and 2.9 to (4.4) gives

LEMMA 4.3.

$$H(s) \equiv \sum_{n=1}^{\infty} \frac{t_2(n)}{n^s} = \sum_{i=1}^{\infty} \zeta(2is)\zeta^2((2i+1)s), \quad s > \frac{1}{2}, \tag{4.5}$$

$$H_1(s) \equiv \sum_{n=1}^{\infty} \frac{t_3(n)}{n^s} = \prod_{i=2}^{\infty} \zeta^2((2i-1)s)\zeta(2is), \quad s > \frac{1}{3}, \tag{4.6}$$

the series and products being (absolutely) convergent for the indicated values of s.

We note that (4.5) can be written

$$H(s) = \frac{Z_2^2(s)}{Z(2s)}, \quad s > \frac{1}{2}. \tag{4.7}$$

LEMMA 4.4.

$$t_1(n) = \sum_{d\delta=n} \tau(d) t_2(\delta), \tag{4.8}$$

$$t_2(n) = \sum_{d^2\delta=n} t_3(\delta). \tag{4.9}$$

Proof. By Lemmas 4.2 and 4.3, $f(s) = \zeta^2(s)H(s)$ and $H(s) = \zeta(2s)H_1(s)$ for $s > 1$. By Dirichlet products, these two analytical relations yield the arithmetical formulas (4.8) and (4.9), respectively.

LEMMA 4.5.

$$B(x) \equiv \sum_{n\leqslant x} t_2(n) = O(\sqrt{x}). \tag{4.10}$$

Proof. By definition, $t_3(n)$ is positive; hence, by (4.9),

$$\sum_{n\leqslant x} t_2(n) = \sum_{d^2\delta\leqslant x} t_3(\delta) = \sum_{\delta\leqslant x} t_3(\delta)\left[\frac{x^{1/2}}{\delta^{1/2}}\right] = O\left(\sqrt{x}\sum_{\delta\leqslant x}\frac{t_3(\delta)}{\delta^{1/2}}\right)$$

but the O-sum is bounded, because the series in (4.6) converges for $s > 1/3$.

We are now in a position to prove our principal result concerning $t(G)$.

THEOREM 4.1.

$$T^*(x) = c_1 x(\log x + 2\gamma - 1) + c_2 x + O(\sqrt{x}\log x), \tag{4.11}$$

where $c_1 = H(1)$, $c_2 = H'(1)$, *and* $H(s)$ *is defined by* (4.7).

Proof. By (4.8) and Lemma 2.15, one obtains

$$T^*(x) = \sum_{n\leqslant x} t_2(n) D(x/n). \tag{4.12}$$

Application of (1.1) yields, since $t_2(n) \geqslant 0$,

$$T^*(x) = (\log x + 2\gamma - 1)x \sum_{n\leqslant x}\frac{t_2(n)}{n} - x\sum_{n\leqslant x}\frac{t_2(n)\log n}{n} + O\left(\sqrt{x}\sum_{n\leqslant x}\frac{t_2(n)}{n^{1/2}}\right),$$

which for brevity is written

$$T^*(x) = (\log x + 2\gamma - 1)xP_1 - xP_2 + O(\sqrt{x}P_3). \tag{4.13}$$

The case $s = 1$ in (4.5) gives

$$P_1 = \sum_{n\leqslant x}\frac{t_2(n)}{n} = H(1) + \sum_{n>x}\frac{t_2(n)}{n}. \tag{4.14}$$

By (4.10), since $t_2(n) \geqslant 0$, it follows, on partial summation, that

$$\sum_{n>x}\frac{t_2(n)}{n} = \sum_{n>x}\frac{B(n)}{n(n+1)} - \frac{B(x)}{[x]+1} = O\left(\sum_{n>x}\frac{B(n)}{n^2}\right) + O\left(\frac{B(x)}{x}\right)$$

$$= O\left(\sum_{n>x}\frac{1}{n^{3/2}}\right) + O\left(\frac{1}{\sqrt{x}}\right) = O\left(\frac{1}{\sqrt{x}}\right);$$

hence (4.14) leads to

$$P_1 = c_1 + O(1/\sqrt{x}). \tag{4.15}$$

As for P_2, one obtains, by (4.5) with $s = 1$,

$$P_2 = \sum_{n\leqslant x}\frac{t_2(n)\log n}{n} = -H'(1) + \sum_{n>x}\frac{t_2(n)\log n}{n}. \tag{4.16}$$

Again by (4.10) and partial summation, since $B(x) \geqslant 0$,

$$\sum_{n>x}\frac{t_2(n)\log n}{n} = \sum_{n>x} B(n)\left(\frac{\log n - n\log(1+1/n)}{n(n+1)}\right) - B(x)\left(\frac{\log([x]+1)}{[x]+1}\right)$$

$$= O\left(\sum_{n>x}\frac{B(n)\log n}{n^2}\right) + O\left(\frac{B(x)\log x}{x}\right)$$

$$= O\left(\sum_{n>x}\frac{\log n}{n^{3/2}}\right) + O\left(\frac{\log x}{\sqrt{x}}\right),$$

so that by (2.14) and (4.16),

$$P_2 = -c_2 + O\left(\frac{\log x}{\sqrt{x}}\right). \tag{4.17}$$

By partial summation and (4.10),

$$P_3 = \sum_{n\leqslant x} \frac{t_2(n)}{n^{1/2}} = \sum_{n\leqslant x} B(n)\left(\frac{(1+1/n)^{1/2}-1}{(n+1)^{1/2}}\right) + \frac{B(x)}{([x]+1)^{1/2}}$$

$$= O\left(\sum_{n\leqslant x} \frac{B(n)}{n^{3/2}}\right) + O\left(\frac{B(x)}{\sqrt{x}}\right) = O\left(\sum_{n\leqslant x} \frac{1}{n}\right) + O(1),$$

so that

$$P_3 = O(\log x). \tag{4.18}$$

The theorem results on collecting (4.13), (4.15), (4.17), and (4.18).

5. Refined estimate for $T(x)$. The function $\gamma'(G)$ defined in § 2 evidently has the level function $b(n)$, where $b(n) = a(m)$ in case n is a square ($n = m^2$) and is otherwise 0. We use this fact to prove

LEMMA 5.1.

$$\tau_1(n) = \sum_{d\delta=n} b(d) t_1(\delta). \tag{5.1}$$

Proof. Using the short-hand notation for direct convolution, it follows, by (2.11) and (4.1), that $t = \mu_2\cdot\tau$ and $\mu_2\cdot\gamma' = \varepsilon$. Since ε is the identity of R (see Lemma 2.1), we have then $\gamma'\cdot t = \gamma'\cdot(\mu_2\cdot\tau) = (\gamma'\cdot\mu_2)\cdot\tau = \tau$; that is,

$$\tau(G) = \sum_{D\times E=G} \gamma'(D) t(E). \tag{5.1a}$$

Application of Lemma 2.12 proves the Lemma.

We now prove our main result concerning $\tau(G)$.

THEOREM 5.1.

$$T(x) = a' x(\log x + 2\gamma - 1) + \beta' x + O(\sqrt{x}\log^2 x), \tag{5.2}$$

where $a' = L(1)$, $\beta' = L'(1)$, *and* $L(s) = Z_2^2(s)$.

Proof. Application of Lemma 2.15 to (5.1) gives

$$T(x) = \sum_{n\leqslant x} b(n) T^*(x/n) = \sum_{n\leqslant\sqrt{x}} a(n) T^*(x/n^2). \tag{5.3}$$

Applying Theorem 4.1, with $c' = c_1(2\gamma-1)+c_2$, one finds, since $a(n) \geqslant 0$ (cf. Remark 1.1),

$$T(x) = (c_1\log x + c')xQ_1 - 2c_1xQ_2 + O(\sqrt{x}Q_3) + O(A(\sqrt{x})), \tag{5.4}$$

where

$$Q_1 = \sum_{n\leqslant\sqrt{x}} \frac{a(n)}{n^2},\qquad Q_2(x) = \sum_{n\leqslant\sqrt{x}} \frac{a(n)\log n}{n^2},\qquad Q_3 = \sum_{n\leqslant\sqrt{x/2}} \frac{a(n)}{n}\log\frac{x}{n^2}.$$

We have, by (2.3),

$$Q_1 = Z(2) + \sum_{n>\sqrt{x}} \frac{a(n)}{n^2},$$

but by partial summation and (2.6),

$$\sum_{n>\sqrt{x}} \frac{a(n)}{n^2} = \sum_{n>\sqrt{x}} A(n)\frac{((1+1/n)^2-1)}{(n+1)^2} - \frac{A(\sqrt{x})}{([\sqrt{x}]+1)^2}$$

$$= O\left(\sum_{n>\sqrt{x}} \frac{A(n)}{n^3}\right) + O\left(\frac{A(\sqrt{x})}{x}\right) = O\left(\sum_{n>\sqrt{x}} \frac{1}{n^2}\right) + O\left(\frac{1}{\sqrt{x}}\right) = O\left(\frac{1}{\sqrt{x}}\right);$$

therefore,

$$Q_1 = \alpha_1 + O(1/\sqrt{x}),\qquad \alpha_1 = Z(2). \tag{5.5}$$

Similarly, for Q_2, one obtains

$$Q_2 = -Z'(2) + \sum_{n>\sqrt{x}} \frac{a(n)\log n}{n^2};$$

moreover, by (2.6) and partial summation,

$$\sum_{n>\sqrt{x}} \frac{a(n)\log n}{n^2} = \sum_{n>\sqrt{x}} A(n)\left(\frac{(2n+1)\log n - n^2\log\left(1+\frac{1}{n}\right)}{n^2(n+1)^2}\right) -$$

$$-\frac{A(\sqrt{x})\log([\sqrt{x}]+1)}{([\sqrt{x}]+1)^2}$$

$$= O\left(\sum_{n>\sqrt{x}} \frac{A(n)\log n}{n^3}\right) + O\left(\frac{A(\sqrt{x})\log x}{x}\right)$$

$$= O\left(\sum_{n>\sqrt{x}} \frac{\log n}{n^2}\right) + O\left(\frac{\log x}{\sqrt{x}}\right);$$

therefore, by (2.14),

$$Q_2 = -\beta_1 + O\left(\frac{\log x}{\sqrt{x}}\right), \quad \beta_1 = Z'(2). \tag{5.6}$$

As for Q_3, we have by Lemma 2.13,

$$Q_3 = O\left(\log x \sum_{n \leqslant \sqrt{x}} \frac{a(n)}{n}\right) = O(\log^2 x). \tag{5.7}$$

Combining (5.4), (5.5), (5.6), and (5.7) we have, since $A(\sqrt{x}) = O(\sqrt{x})$,

$$T(x) = a_1 c_1(\log x + 2\gamma - 1) + (a_1 c_2 + 2\beta_1 c_1)x + O(\sqrt{x}\log^2 x). \tag{5.8}$$

It remains to simplify the coefficients of (5.8). We have $a_1 c_1 = Z_2^2(1) = a'$. Note that $H(s) = L(s)/Z(2s)$, $s > \frac{1}{2}$, so that $H'(s) = \big(Z(2s)L'(s) - L(s)Z'(2s)\big)Z^{-2}(2s)$. Since $Z'(2s) = 2Z'(t)$ with $t = 2s$, it follows that $c_2 = H'(1) = (a_1\beta' - 2a'\beta_1)/a_1^2$. Therefore $a_1 c_2 + 2\beta_1 c_1 = \beta'$, and (5.8) becomes (5.2), which proves the Theorem.

As a curious consequence of Theorem 5.1, we obtain the following evaluation of the constant γ^*.

COROLLARY 5.1.

$$\gamma^* = \beta + a\gamma, \tag{5.9}$$

where $a = Z_2(1)$, $\beta = Z_2'(1)$.

Proof. Comparing the two estimates (3.1) and (5.2) of $T(x)$, we have, since $a' = a^2$, $\beta' = 2a\beta$,

$$ax(\gamma^* - a\gamma - \beta) = O(x^{3/4}),$$

which implies that the coefficient of x on the left is 0. Since $a > 0$, (5.9) results.

Remark 5.1. Theorem 5.1 could have been proved independently in the manner of Theorem 4.1. In this connection, the generating function of $\tau(G)$, by (2.15) and Lemma 2.3, is $\zeta^2(s)\zeta^2(2s)\zeta^2(3s)\ldots$ $(s > 1)$. In comparison, $t(G)$ has the generating function $\zeta^2(s)\zeta(2s)\zeta^2(3s)\zeta(3s)\ldots$ (Lemma 4.2), $s > 1$. The choice of $t(G)$ to play the leading rôle in the main discussion of the paper (§§ 4, 5) is justified by the simpler nature of its generating function. The reverse is true in the integral case, where $t(n)$ has the generating function $\zeta^2(s)\zeta^{-1}(2s)$ as compared with the simpler function $\zeta^2(s)$ generating $\tau(n)$.

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