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An Abelian theorem for number-theoretic sums

by

L. A. RUBEL (Urbana, Ill.)

Wintner ([2], § 8-§ 12) has studied pairs f, f^* of functions on the positive integers related by

$$f(n) = \sum_{d|n} f^*(d), \tag{1}$$

or, equivalently, from the Möbius inversion formula, by

$$f^*(n) = \sum_{d|n} f(d)\mu(n/d). \tag{2}$$

He proved, among other negative results, that unless one imposes certain Tauberian restrictions on f or f^*, the convergence of $\sum_{n=1}^{\infty} f^*(n)/n$ does not follow from the existence of $\lim_{n\to\infty} n^{-1}\{f(1)+f(2)+\ldots+f(n)\}$, even though the sum and the limit must be equal if they both exist. We show here that despite this negative result, there is a purely Abelian theorem connecting the behaviour of $f(n)$ as $n\to\infty$ with the convergence of $\sum f^*(n)/n$.

The proof of our theorem applies the Silverman-Toeplitz conditions for the regularity of a matrix summation method. Like the proof ([1], Appendix IV) of the Abelian theorem, "L implies A", for Lambert and Abel summability, ours requires the assertion (5), which is somewhat stronger than the prime number theorem. Because the Silverman-Toeplitz conditions are both necessary and sufficient, the results from prime number theory used to prove our result are in turn immediately recoverable from it.

THEOREM. *If* $\lim_{n\to\infty} f(n) = L$ *exists, then* $\sum_{n=1}^{\infty} f^*(n)/n$ *converges to* L.

Proof. If we put $S_m = \sum_{n=1}^{m} f^*(n)/n$ and use (2), we get

$$S_m = \sum_{n=1}^{\infty} n^{-1} \sum_{d|n} f(d)\mu(n/d).$$

Changing the order of summation, we write

$$S_m = \sum_{n=1}^{\infty} C_{mn} f(n), \tag{3}$$

where

$$C_{mn} = \frac{1}{n} N\left(\frac{m}{n}\right) \quad \text{and} \quad N(x) = \sum_{k \leqslant x} \mu(k)/k.$$

In the usual way, (3) expresses the sequence-to-sequence transformation $\{f(n)\} \to \{S_m\}$ by means of the matrix $((C_{mn}))$, and our theorem is precisely the assertion that the matrix is regular. The threefold conditions for regularity are customarily written ([1], p. 43):

(i) For some $H < \infty$, $\sum_{n=1}^{\infty} |C_{mn}| < H$, for each $m = 1, 2, \ldots$

(ii) $\lim_{m\to\infty} \sum_{n=1}^{\infty} C_{mn} = 1$.

(iii) $\lim_{m\to\infty} C_{mn} = 0$ for each $n = 1, 2, \ldots$

The assertion (ii) is merely a restatement of the assertion that for the function $f(n) \equiv 1$, $\lim_{m\to\infty} S(m) = 1$. But in this case, $S_m = 1+0+ +0+\ldots+0$. The assertion (iii) is equivalent to

$$\sum_{k=1}^{\infty} \mu(k)/k = 0, \tag{4}$$

which is known to be "equivalent" to the prime number theorem.

For assertion (i), we write

$$\sum_{n=1}^{\infty} |C_{mn}| = \sum_{n=1}^{m} \frac{1}{n} \left| N\left(\frac{m}{n}\right) \right|$$

$$= \sum_{k=1}^{m} \sum_{\frac{m}{k+1} < n \leqslant \frac{m}{k}} \frac{1}{n} \left| N\left(\frac{m}{n}\right) \right| = \sum_{k=1}^{m} |N(k)| \sum_{\frac{m}{k+1} < n \leqslant \frac{m}{k}} \frac{1}{n}.$$

Since $\sum_{\frac{m}{k+1} < n \leqslant \frac{m}{k}} n^{-1} \leqslant k^{-1}$, we see that (i) is implied by the known result

$$\sum_{k=1}^{\infty} k^{-1} |N(k)| < \infty. \tag{5}$$

On the other hand, (i) implies (5), since for $k \leqslant \sqrt{m}-1$,

$$\sum_{\frac{m}{k+1} < n \leqslant \frac{m}{k}} n^{-1} \geqslant \tfrac{1}{2} k^{-1},$$

so that

$$\sum_{n=1}^{\infty} |C_{mn}| \geqslant \tfrac{1}{2} \sum_{k \leqslant \sqrt{m}-1} k^{-1} |N(k)|.$$

For a brief discussion of the relations between (4) and (5) and the prime number theorem, see [1], Appendix IV.

References

[1] G. H. Hardy, *Divergent series*, Oxford 1949.
[2] A. Wintner, *Eratosthenian averages*, Baltimore 1943.

UNIVERSITY OF ILLINOIS

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