ANNALES
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On a problem of P. Erdös concerning the distribution of the zeros of polynomials

by A. Szybiak (Kraków)

Let C be a compact set in the z-plane. We assume that C is a sum of continua none of which reduces to a one-point set. Let $\{\alpha_n\}$ be an increasing sequence of indices. Let $\{p_i^{\alpha_n}\}$ and $\{q_i^{\alpha_n}\}$ $(i=1,\dots,\alpha_n)$ be two sequences of points of the set C. We consider two sequences of polynomials,

$$f_n(z)=(z-p_1^{\alpha_n})\dots(z-p_{\alpha_n}^{\alpha_n}),\qquad g_n(z)=(z-q_1^{\alpha_1})\dots(z-q_{\alpha_n}^{\alpha_n})$$

and two sequences of measures $\{\mu_n\}$ and $\{\nu_n\}$ defined by the formulae:

$$\int\varphi d\mu_n=\sum_{i=1}^{\alpha_n}\varphi(p_i^{\alpha_n})/\alpha_n,\qquad \int\varphi d\nu_n=\sum_{i=1}^{\alpha_n}\varphi(q_i^{\alpha_n})/\alpha_n$$

for every continuous function $\varphi(z)$. We know that $\{\mu_n\}$ contains a weakly convergent subsequence. For the sake of simplicity *we assume that* $\{\mu_n\}$ *converges*. We shall denote its limit by μ. We restrict our considerations to the set C and all the topological notions will relate to the topology on C which is induced by the common topology of the plane. We shall prove

THEOREM. *If for every closed neighbourhood V of every point $z\in C$ we have*

$$\lim_{n\to\infty}\left[\frac{\max_{z\in V}|f_n(z)|}{\max_{z\in V}|g_n(z)|}\right]^{1/\alpha_n}=1 \tag{1}$$

and the set of discontinuity points of the potential[1]

$$\int\log|z-\zeta|^{-1}d\mu(\zeta)$$

consists of isolated points then ν_n also converges as μ_n does to the measure μ.

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[1] The points at which the potential is (of course positively) infinite we treat as the continuity points.

To begin with we shall prove some lemmas.

LEMMA 1. *For every point $z_0 \in C$ and every neighbourhood V of it there exists a positive measure σ such that the carrier of σ contains z_0 and is contained in V and the potential*

$$\int \log|z-\zeta|^{-1} d\sigma(\zeta)$$

is continuous.

Proof. We consider two cases:

I. z_0 is an interior point of C with respect to the usual topology of the plane. Then there exists a circle which contains z_0 and we take as σ the plane Lebesgue measure of that circle. The continuity of $\int \log|z-\zeta|^{-1} d\sigma(\zeta)$ is here obvious.

II. z_0 is a boundary point of C. Then there exists a continuum $V_0 \subset V \subset C$ which contains z_0 and is not degenerated to a one-point set. Then we take as σ the equilibrium measure of V. The continuity of the potential $\int \log|z-\zeta|^{-1} d\sigma(\zeta)$ follows by a theorem of F. Leja which states that every compact continuum is a boundary of a domain with a continuous Green function (cf. [1]).

LEMMA 2. *Let μ_n be a sequence of measures on C which converges to a measure μ. Let V be a closed subset of C and x_n be a sequence of points such that*

$$\int \log|x_n-\zeta|^{-1} d\mu_n(\zeta) = \min_{z \in V} \int \log|z-\zeta|^{-1} d\mu_n(\zeta). \tag{2}$$

If $\{x_n\}$ converges to some point x_0 and $\int \log|z-\zeta|^{-1} d\mu(\zeta)$ is continuous at x_0, then we have

$$\lim \int \log|x_n-\zeta|^{-1} d\mu_n(\zeta) = \int \log|x_0-\zeta|^{-1} d\mu(\zeta).$$

Proof. First we shall show that

$$\underline{\lim} \int \log|x_n-\zeta|^{-1} d\mu_n(\zeta) \geqslant \int \log|x_0-\zeta|^{-1} d\mu(\zeta).$$

M being a positive number, we put

$$L_M(z, \zeta) = \min\{\log|z-\zeta|^{-1}, M\}.$$

Suppose that $\int \log|x_0-\zeta|^{-1} d\mu(\zeta) < \infty$. Let ε be any positive number. Then we choose such a large M that

$$\int L_M(x_0, \zeta)\, d\mu(\zeta) \geqslant \int \log|x_0-\zeta|^{-1} d\mu(\zeta) - \varepsilon.$$

Then we have

$$\underline{\lim} \int \log|x_n-\zeta|^{-1} d\mu_n(\zeta) \geqslant \lim \int L_M(x_n-\zeta)\, d\mu_n(\zeta)$$
$$= \int L_M(x_0, \zeta)\, d\mu(\zeta) \geqslant \int \log|x_0-\zeta|^{-1} d\mu(\zeta) - \varepsilon.$$

In view of ε being arbitrarily small we obtain inequality (2). The case $\int \log|x_0-\zeta|^{-1} d\mu(\zeta) = \infty$ we treat similarly.

Suppose now that the assumptions of our lemma hold and that we have

$$\overline{\lim} \int \log|x_n-\zeta|^{-1} d\mu(\zeta) > \int \log|x_0-\zeta|^{-1} d\mu(\zeta).$$

Then there exist a subsequence $\{x_{k_n}\} \in \{x_n\}$ and a number $\eta > 0$ such that we have

$$\log|x_{k_n}-\zeta|^{-1} d\mu_{k_n}(\zeta) \geqslant \int \log|x_0-\zeta|^{-1} d\mu(\zeta) + 2\eta.$$

By the continuity at x_0 there exists a neighbourhood V of x_0 in which we have

$$\int \log|x_0-\zeta|^{-1} d\mu(\zeta) - \eta \leqslant \int \log|z-\zeta|^{-1} d\mu(\zeta).$$

Then we take the measure σ which has been described in lemma 1. In view of the fact that the points x_{k_n} realize $\min\limits_{z \in V} \int \log|z-\zeta|\, d\mu_{k_n}(\zeta)$ we have

$$\int d\sigma(z) \int \log|z-\zeta|^{-1} d\mu_{k_n}(\zeta) \geqslant \int d\sigma(z) \int \log|x_{k_n}-\zeta|^{-1} d\mu_{k_n}(\zeta)$$
$$\geqslant \int d\sigma(z) \Big[\int \log|x_0-\zeta|^{-1} d\mu(\zeta) + 2\eta\Big] \geqslant \int d\sigma(z) \int \log|z-\zeta|^{-1} d\mu(z) + \eta\Big].$$

Hence we obtain in the limit

$$\underline{\lim} \int d\sigma(z) \int \log|z-\zeta|^{-1} d\mu_{k_n}(\zeta) \geqslant \int d\sigma(z) \int \log|z-\zeta|^{-1} d\mu(\zeta) + \eta\sigma(V). \tag{3}$$

Using the theorem of Fubini we have

$$\lim \int d\sigma(z) \int \log|z-\zeta|^{-1} d\mu_{k_n}(\zeta) = \lim \int d\mu_{k_n}(\zeta) \int \log|z-\zeta|^{-1} d\sigma(z)$$
$$= \int d\mu(\zeta) \int \log|z-\zeta|^{-1} d\sigma(z) = \int d\sigma(z) \int \log|z-\zeta|^{-1} d\mu(\zeta),$$

which contradicts (3).

Proof of the theorem. We choose from $\{\nu_n\}$ a convergent subsequence and we denote its limit by ν. By assumption (1) and lemma 2 we conclude that every closed subset V of C which does not contain the discontinuity points of $\int \log|z-\zeta|^{-1} d\mu(\zeta)$ contains two points, x' and x'', such that

$$\int \log|x'-\zeta|^{-1} d\mu(\zeta) = \int \log|x''-\zeta|^{-1} d\nu(\zeta).$$

x' (resp. x'') is of course an accumulation point of the points realizing $\min\limits_{V} \int \log|z-\zeta|^{-1} d\mu_n(\zeta)$ (resp. $\min\limits_{V} \int \log|z-\zeta|^{-1} d\nu_n(\zeta)$). From this fact follows the equality of $\int \log|z-\zeta|^{-1}d\mu(\zeta)$ and $\int \log|z-\zeta|^{-1}d(\zeta)$ outside at most a set of isolated points which is of capacity 0. Hence

$$\int \log|z-\zeta|^{-1}d\mu(\zeta) \equiv \int \log|z-\zeta|^{-1}d\nu(\zeta)$$

and in consequence $\mu = \nu$ (cf. [2]). Since the above treatment may be applied to the arbitrary convergent subsequence of the original sequence $\{\nu_n\}$, then $\{\nu_n\}$ converges to ν.

The theorem gives a positive answer to the hypothesis which P. Erdös has put in a slightly less general form.

References

[1] F. Leja, *Une condition de la régularité et irrégularité des points frontières dans le problème de Dirichlet*, Ann. Soc. Polon. Math. 20 (1947), p. 223-228.

[2] H. Cartan, *Théorie du potentiel Newtonien: énergie, capacité, suites de potentiels*, Bull. Soc. Math. France 69 (1954), p. 74-106.

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Sur les solutions périodiques et presque-périodiques de l'équation différentielle $x''+kf(x)x'+g(x) = kp(t)$

par Z Opial (Kraków)

1. Considérons l'équation différentielle non linéaire du second ordre

$$x''+kf(x)x'+g(x) = kp(t), \tag{1}$$

où k est un paramètre. Admettons que les fonctions $f(x)$, $g(x)$ et $p(t)$ soient continues $(-\infty < x < +\infty,\ -\infty < t < +\infty)$ et posons

$$F(x) = \int_0^x f(u)\,du, \quad G(x) = \int_0^x g(u)\,du \quad \text{et} \quad P(t) = \int_0^t p(u)\,du.$$

L'équation (1) est équivalente au système d'équations du premier ordre

$$x' = y-kF(x), \quad y' = -g(x)+kp(t). \tag{2}$$

Pour toute solution $x(t)$ de l'équation (1) nous désignerons par $(x(t), y(t))$ la solution correspondante du système (2), de sorte que l'on aura

$$y(t) = x'(t)+kF(x(t)).$$

Supposons que les fonctions $f(x)$, $g(x)$ et $p(t)$ satisfassent aux conditions

$$f(x) > 0 \quad \text{pour tout } x, \quad \lim_{|x|\to\infty} F(x)\operatorname{sgn} x = +\infty, \tag{3}$$

$$xg(x) > 0 \quad \text{pour } x \neq 0, \quad \lim_{|x|\to\infty} G(x) = +\infty, \tag{4}$$

$$|p(t)| \leqslant P, \quad |P(t)| \leqslant P \quad (-\infty < t < +\infty), \tag{5}$$

P étant une constante positive.

G. E. H. Reuter [5] a démontré que, dans ces hypothèses, pour tout $k > 0$ il existe dans le plan (x, y) un ensemble K limité par une courbe régulière (de classe C^1 sauf en un nombre fini de points), simple et jouissant des propriétés suivantes: