ANNALES
POLONICI MATHEMATICI
VIII (1960)

Parabolic differential inequalities and Chaplighin's method

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The purpose of the present paper is to discuss the extension of Chaplighin's method (see [1], [3]) to non-linear parabolic equations. Some results concerning such extension have been presented in [11]. An important factor in our investigations is the theorem of Nagumo-Westphal (see for instance [6]). The first section of the paper concerns differential inequalities of parabolic type. Next we discuss Chaplighin's scheme applied to parabolic equations. The last section deals with the limitation of the difference between the exact solution and the approximate one.

1.1. Let R be the rectangle $a \leqslant x \leqslant b$, $0 \leqslant t \leqslant T$. The function $u(x,t)$ is said to be regular in R if it is continuous in R and if it possesses the derivative $\partial u/\partial t$ and a continuous derivative $\partial^2 u/\partial x^2$ for $a < x < b$, $0 < t \leqslant T$. Let us denote by Γ the plane set composed of points $(x, 0)$ with $a \leqslant x \leqslant b$ and (a,t), (b,t) with $0 \leqslant t \leqslant T$. By $\mathrm{F}R$ we denote the boundary of R: by R^0 the interior of R. We now present the Nagumo-Westphal lemma:(1)

Lemma 1. *Let the functions $u(x,t)$, $v(x,t)$ be regular in R and let the inequalities*

$$\frac{\partial u}{\partial t} \leqslant \frac{\partial^2 u}{\partial x^2} + f\big(x, t, u(x,t), u_x(x,t)\big),$$

$$\frac{\partial v}{\partial t} > \frac{\partial^2 v}{\partial x^2} + f\big(x, t, v(x,t), v_x(x,t)\big)$$

be satisfied in $R^0 + (\mathrm{F}R - \Gamma)$. Suppose that $u(x,t) < v(x,t)$ for $(x,t) \in \Gamma$. Then the inequality $u(x,t) < v(x,t)$ holds for $(x,t) \in R$.

(1) An extensive review of papers dealing with parabolic differential inequalities will be found in [6].

1.2. Let the functions $u(x,t)$, $g(x,t,z)$, $\varphi(x,t)$ be continuous in R, $Q = \underset{(x,t,z)}{E}\{(x,t)\in R, -\infty < z < +\infty\}$ and Γ respectively. Define the function $r(x,t)$ by means of the formula

$$r(x,t) = \frac{1}{2\sqrt{\pi}}\int_0^t\int_a^b \frac{\exp\left(-(x-\xi)^2/4(t-\tau)\right)}{\sqrt{t-\tau}}\, g\left(\xi,\tau,u(\xi,\tau)\right)d\xi\,d\tau$$

and let $q(x,t)$ be the solution in R^0 of the equation

$$\partial z/\partial t = \partial^2 z/\partial x^2$$

such that $q(x,t) = \varphi(x,t) - r(x,t)$ for $(x,t)\in\Gamma$. We put $v(x,t) = q(x,t) + r(x,t)$. Denote by $T(u\colon g,\varphi)$ the transformation $u\to v$:

$$v = T(u\colon g,\varphi).$$

It is known that $T(u\colon g,\varphi)$ is completely continuous with respect to u in the Banach space C of all continuous functions([2]). The norm in C is defined as usual.

Suppose that $f(x,t,z)$ is Hölder continuous with respect to x and z. Let $f(x,t,z)$ be continuous in Q. Then each solution $v(x,t)$ of the equation $z = T(z\colon f,\varphi)$ is a regular solution of the equation

$$\partial v/\partial t = \partial^2 v/\partial x^2 + f(x,t,v) \tag{1}$$

and

$$v(x,t) = \varphi(x,t) \quad \text{for} \quad (x,t)\in\Gamma. \tag{2}$$

As in [2], [7], [8], [9] we introduce the following condition:

(H) The functions $u_0(x,t)$, $v_0(x,t)$ are regular in R and satisfy the inequalities

$$\partial u_0/\partial t < \partial^2 u_0/\partial x^2 + f(x,t,u_0) \quad \text{in} \quad R^0 + (FR-\Gamma), \tag{3}$$

$$\partial v_0/\partial t > \partial^2 v_0/\partial x^2 + f(x,t,v_0) \quad \text{in} \quad R^0 + (FR-\Gamma), \tag{4}$$

$$u_0(x,t) < \varphi(x,t) < v_0(x,t) \quad \text{for} \quad (x,t)\in\Gamma. \tag{5}$$

Following Prodi [8] (see also [2]) we formulate

THEOREM 1. *Let the assumption* (H) *be satisfied. Suppose that the functions $\varphi(x,t)$, $f(x,t,z)$ are continuous in Γ and Q respectively and let $f(x,t,z)$ be Hölder continuous with regard to x and z. Then there exists at least one regular solution of the boundary value problem* (1), (2).

1.3. The solution $u(x,t)$ of (1), (2) is called a *maximum* (*minimum*) *solution* if for every solution $v(x,t)$ of (1), (2) the inequality $v(x,t) \leqslant u(x,t)$ ($v(x,t) \geqslant u(x,t)$) holds in R. The existence of a maximum solution has been proved in [9] for generalized elliptic equations. For some generalized parabolic equations an analogous result has been obtained in [7]. In both papers appears the assumption that f is monotonic with regard to z. We shall prove that the boundary problem (1), (2) has the maximum solution when the condition (H) is satisfied. We do not assume that f is monotonic.

([2]) See [5], lemma 3.

THEOREM 2. *Let the assumptions of theorem* 1 *be satisfied. Then the boundary value problem* (1), (2) *has the maximum solution.*

Proof. As in [2], p. 805 we define the function $f^*(x,t,z)$ as follows:

$$f^*(x,t,z) = \begin{cases} f\left(x,t,u_0(x,t)\right) & \text{if} \quad z < u_0(x,t), \\ f(x,t,z) & \text{if} \quad u_0(x,t) \leqslant z \leqslant v_0(x,t), \\ f\left(x,t,v_0(x,t)\right) & \text{if} \quad z > v_0(x,t). \end{cases}$$

The function f^* is bounded. Let $\sup|f^*| < M$ and $\sup|\varphi| < K$. The assumption (H) holds for f^* if $u_0 = -Mt-K$ and $v_0 = Mt+K$. The function f^* is Hölder continuous. We can now apply theorem 1 to the equation

$$\partial z/\partial t = \partial^2 z/\partial x^2 + f^*(x,t,z) + 1/n \tag{6}$$

and thus conclude that (6) has a solution $z_n(x,t)$ such that

$$z_n(x,t) = \varphi(x,t) + 1/n \quad \text{for} \quad (x,t)\in\Gamma \tag{7}$$

for n sufficiently large. It follows from lemma 1 that

$$z_{n+1}(x,t) < z_n(x,t) \quad \text{in} \quad R. \tag{8}$$

It is easy to verify that $z_n = T(z_n\colon f^*+1/n, \varphi+1/n)$. Using the results of [5] we infer that z_n is compact in C. By (8) we therefore find that $z_n \rightrightarrows u$ in R. Obviously $u = T(u\colon f^*,\varphi)$ and consequently $u(x,t)$ satisfies the equation

$$\partial u/\partial t = \partial^2 u/\partial x^2 + f^*(x,t,u) \quad \text{in} \quad R^0 + (FR-\Gamma) \tag{9}$$

and $u(x,t) = \varphi(x,t)$ on Γ. On the other hand,

$$f^*\left(x,t,u_0(x,t)\right) = f\left(x,t,u_0(x,t)\right), \qquad f^*\left(x,t,v_0(x,t)\right) = f\left(x,t,v_0(x,t)\right).$$

Hence by (H) we have

$$\partial u_0/\partial t < \partial^2 u_0/\partial x^2 + f^*(x,t,u_0), \qquad \partial v_0/\partial t > \partial^2 v_0/\partial x^2 + f(x,t,v_0).$$

Applying lemma 1 and (9) we find that

$$u_0(x,t) < u(x,t) < v_0(x,t) \quad \text{in} \quad R. \tag{10}$$

Using the definition of f^* and (10) we conclude that u is a solution of (1), (2).

Let $v(x, t)$ be an arbitrary solution of (1), (2). From lemma 1 it follows that

$$u_0(x, t) < v(x, t) < v_0(x, t)$$

and consequently

$$\partial v/\partial t = \partial^2 v/\partial x^2 + f^*(x, t, v). \tag{11}$$

It follows from (7), (9), (11) and lemma 1 that $v(x, t) < z_n(x, t)$ in R. Therefore $v(x, t) \leqslant u(x, t)$ in R. We conclude that u is the maximum solution.

One easily proves the following theorem:

THEOREM 3. *Let the assumptions of theorem 1 be satisfied. Then the boundary value problem* (1), (2) *possesses the minimum solution.*

We shall prove the following theorem:

THEOREM 4. *Suppose that the assumptions of theorem 2 hold. Let the regular function $z(x, t)$ satisfy the inequalities*

$$\partial z/\partial t \leqslant \partial^2 z/\partial x^2 + f(x, t, z) \quad \text{in} \quad R^0 + (FR - \Gamma), \tag{12}$$

$$z(x, t) \leqslant \varphi(x, t) \quad \text{on} \quad \Gamma. \tag{13}$$

Then $z(x, t) \leqslant u(x, t)$ on R, where $u(x, t)$ is the maximum solution of (1), (2).

Proof. Using (12), (13) and lemma 1 one concludes that

$$z(x, t) < v_0(x, t) \quad \text{for} \quad (x, t) \in R.$$

In order to prove our theorem we shall prove that $z(x, t) < z_n(x, t)$ in R, for n sufficiently large: z_n is the sequence constructed in the proof of theorem 2. Suppose that there is n_0 such that the set

$$Z = \underset{(x,t)}{E}\{(x, t) \in R,\ z_{n_0}(x, t) \leqslant z(x, t)\}$$

is non-empty. Note that $z(x, t) \leqslant \varphi(x, t) < \varphi(x, t) + 1/n_0 = z_{n_0}(x, t)$ for $(x, t) \in \Gamma$. Let Z_t be the projection of Z on the t-ax and let $\xi = \inf Z_t$. Obviously $0 < \xi$ and $z(x, t) < z_{n_0}(x, t)$ for $a \leqslant x \leqslant b$ and $0 \leqslant t < \xi$. Moreover there is $\bar{x} \in (a, b)$ such that $z(\bar{x}, \xi) = z_{n_0}(\bar{x}, \xi)$. We conclude therefore that

$$\left(\frac{\partial z}{\partial t}\right)_{(\bar{x}, \xi)} \geqslant \left(\frac{\partial z_{n_0}}{\partial t}\right)_{(\bar{x}, \xi)}, \qquad \left(\frac{\partial^2 z_{n_0}}{\partial x^2}\right)_{(\bar{x}, \xi)} \geqslant \left(\frac{\partial^2 z}{\partial x^2}\right)_{(\bar{x}, \xi)}.$$

Furthermore

$$u_0(\bar{x}, \xi) < u(\bar{x}, \xi) < z_{n_0}(\bar{x}, \xi) = z(\bar{x}, \xi) < v_0(\bar{x}, \xi)$$

and consequently

$$f(\bar{x}, \xi, z(\bar{x}, \xi)) = f^*(\bar{x}, \xi, z(\bar{x}, \xi)) = f^*(\bar{x}, \xi, z_{n_0}(\bar{x}, \xi))$$
$$= \left(\frac{\partial z_{n_0}}{\partial t}\right)_{(\bar{x}, \xi)} - \left(\frac{\partial^2 z_n}{\partial x^2}\right)_{(\bar{x}, \xi)} - \frac{1}{n_0} < \left(\frac{\partial z}{\partial t}\right)_{(\bar{x}, \xi)} - \left(\frac{\partial^2 z}{\partial x^2}\right)_{(\bar{x}, \xi)}.$$

This contradicts (12).

We have just proved that $z(x, t) < z_n(x, t)$ in R for n sufficiently large. We conclude therefore that $z(x, t) \leqslant u(x, t) = \lim_{n\to\infty} z_n(x, t)$, q. e. d.

An analogous theorem for a minimum solution is the following one:

THEOREM 5. *Let the assumptions of theorem 3 be satisfied. Suppose that the regular function $z(x, t)$ satisfies the inequalities*

$$\partial z/\partial t \geqslant \partial^2 z/\partial x^2 + f(x, t, z) \quad \text{in} \quad R^0 + (FR - \Gamma),$$
$$z(x, t) \geqslant \varphi(x, t) \quad \text{for} \quad (x, t) \in \Gamma.$$

Then $z(x, t) \geqslant v(x, t)$ in R, where $v(x, t)$ is the minimum solution of (1), (2).

Remark 1. The theorems given above may be extended to the case of general parabolic equations and inequalities. The operator $\partial/\partial x^2$ is replaced by

$$\sum_{i,k=1}^{s} a_{ik}(x, t)\frac{\partial^2}{\partial x_i \partial x_k} + \sum_{j=1}^{s} b_j(x, t)\frac{\partial}{\partial x_j}.$$

One must assume that the coefficients are sufficiently regular as to guarantee the use of the potential theory. If f depends on the first spatial derivative of the solution, then the assumptions concerning f must be stable in the sense that they hold for $f \pm \varepsilon$ for $\varepsilon \geqslant 0$ and sufficiently small. Such stable assumptions are given in [2], [8]. There is no difficulty in extending our method to the case of other boundary value problems such as those of Neumann type, etc.

2.1. This section deals with the method of Chaplighin applied to the first boundary value problem for the equation

$$\partial z/\partial t = \partial^2 z/\partial x^2 + f(x, t, z). \tag{14}$$

In [11] appeared the assumption $f_z > 0$. We do not need such an assumption. Let us introduce the following condition:

(C_1) Suppose that the condition (H) holds. It is assumed that to every regular function $z(x, t)$ such that

$$\partial z/\partial t < \partial^2 z/\partial x^2 + f(x, t, z) \quad \text{in} \quad R^0 + (FR - \Gamma), \tag{15}$$

$$u_0(x, t) \leqslant z(x, t) \leqslant v_0(x, t) \tag{16}$$

there corresponds a function $\underline{\omega}(x, t: \xi, \eta: z)$, Hölder continuous in x, ξ and η, such that the following conditions hold:

(α) $\underline{\omega}(x, t: \xi, \xi: z) \equiv 0$ for each ξ,

(β) $\underline{\omega}(x, t: \xi, \eta: z) < f(x, t, \xi) - f(x, t, \eta)$ if $z(x, t) \leqslant \eta < \xi \leqslant v_0(x, t)$ and $(x, t) \in R$,

(γ) there is a $K > 0$ such that $|\underline{\omega}(x, t: \xi, \eta: z)| \leqslant K$ whenever $u_0(x, t) \leqslant z(x, t) \leqslant \eta < \xi \leqslant v_0(x, t)$ and $(x, t) \in R$,

(δ) $\underline{\omega}\big(x, t: z_{n+1}(x, t), z_n(x, t): z_n\big) \Rightarrow 0$ in R if $u_0(x, t) \leqslant z_n(x, t) \leqslant v_0(x, t)$ on R and $z_n(x, t)$ is uniformly convergent on R.

We shall prove the following theorem:

THEOREM 6. *Assume that the condition* (C_1) *holds. Suppose that*

$$\varphi_n(x, t) \Rightarrow \varphi(x, t) \quad \textit{and} \quad \varphi_n(x, t) < \varphi_{n+1}(x, t) \quad \textit{on} \quad \Gamma, \tag{17}$$

$$u_0(x, t) < \varphi(x, t) < v_0(x, t) \quad \textit{on} \quad \Gamma. \tag{18}$$

Then there exists a sequence $\{u_n\}$ *of regular functions which satisfy the following conditions:*

(19) $u_n(x, t)$ *is a regular solution of the equation* $\partial z/\partial t = \partial^2 z/\partial x^2 + \underline{\omega}\big(x, t: z, u_{n-1}(x, t): u_{n-1}\big) + f\big(x, t, u_{n-1}(x, t)\big)$,

$$u_n(x, t) = \varphi_n(x, t) \quad \textit{on} \quad \Gamma, \tag{20}$$

$$u_{n-1}(x, t) < u_n(x, t) \quad \textit{in} \quad R, \tag{21}$$

$$u_0(x, t) < u_n(x, t) < v_0(x, t) \quad \textit{in} \quad R, \tag{22}$$

$$\partial u_n/\partial t < \partial^2 u_n/\partial x^2 + f(x, t, u_n) \quad \textit{in} \quad R^0 + (\mathrm{F}R - \Gamma), \tag{23}$$

(24) $u_n(x, t)$ *converges uniformly in* R *to the minimum solution of the boundary problem* (1), (2).

Proof. The function $u_0(x, t)$ satisfies (15) and (16). We conclude therefore that there exists an $\underline{\omega}(x, t: \xi, \eta: u_0)$ satisfying (α), (β) and (γ). Consider the equation

$$\partial z/\partial t = \partial^2 z/\partial x^2 + \underline{\omega}\big(x, t: z, u_0(x, t): u_0\big) + f\big(x, t, u_0(x, t)\big) \tag{25}$$

and the boundary data

$$z(x, t) = \varphi_1(x, t) \quad \text{on} \quad \Gamma. \tag{26}$$

Using (α) we get

$$\partial u_0/\partial t < \partial^2 u_0/\partial x^2 + \underline{\omega}\big(x, t: u_0, u_0(x, t): u_0\big) + f(x, t, u_0). \tag{27}$$

It follows from the inequality $u_0(x, t) < v_0(x, t)$ and from (4) and (β) that

$$\partial v_0/\partial t > \partial^2 v_0/\partial x^2 + \underline{\omega}\big(x, t: v_0, u_0(x, t): u_0\big) + f\big(x, t, u_0(x, t)\big). \tag{28}$$

Using (18), (27), (28) and theorem 1 we conclude that there exists a solution $u_1(x, t)$ of the boundary problem (25), (26). Moreover, $u_0(x, t) < u_1(x, t) < v_0(x, t)$ in R. By (β) we therefore get

$$\partial u_1/\partial t < \partial^2 u_1/\partial x^2 + f(x, t, u_1) \quad \text{in} \quad R^0 + (\mathrm{F}R - \Gamma).$$

We have just proved that (19)-(23) hold for $n = 1$.

Let these conditions be satisfied for $n = k$. By (C_1) and (22) and (23) there is a function $\underline{\omega}(x, t: \xi, \eta: u_k)$ satisfying (α), (β) and (γ) for $z = u_k$. We consider the boundary problem (19), (20) for $n = k+1$. It is easy to verify that (H) holds for that problem if u_0 is replaced by u_k, v_0 being unchanged. By theorem 1 we find that there exists a solution u_{k+1} of (19), (20) for $n = k+1$. Moreover, $u_0(x, t) < u_k(x, t) < u_{k+1}(x, t) < v_0(x, t)$ in R. Using (β) we conclude therefore that

$$\partial u_{k+1}/\partial t < \partial^2 u_{k+1}/\partial x^2 + f(x, t, u_{k+1}).$$

It is thus seen that u_{k+1} satisfies (19)-(23) for $n = k+1$. It remains to prove (24). Using (22) and (γ) we conclude that the sequence $\underline{\omega}\big(x, t: u_{n+1}(x, t), u_n(x, t): u_n\big) + f\big(x, t, u_n(x, t)\big) = h_{n+1}(x, t)$ is equibounded in R. Obviously $u_n = T(u_n: h_n, \varphi_n)$. Applying arguments similar to that used in [5] one easily proves that u_n is compact in C. This fact together with (21) implies that $u_n(x, t)$ is uniformly convergent $u_n(x, t) \Rightarrow u(x, t)$ in R. We have $h_n \Rightarrow f\big(x, t, u(x, t)\big)$ and consequently by (17) $u = T(u: f, \varphi) = \lim\limits_{n\to\infty} T(u_n: h_n, \varphi_n)$. This implies that $u(x, t)$ is a solution of (1), (2). Let $z(x, t)$ be an arbitrary solution of the boundary value problem (1), (2). It follows from (17), (23) and from lemma 1 that $u_n(x, t) < z(x, t)$ in R. Hence $u(x, t)$ is the minimum solution of (1), (2), which was to be proved.

We introduce the following condition:

(C_2) Suppose that the condition (H) holds. We assume that to every regular function $z(x, t)$ satisfying the inequalities

$$\partial z/\partial t > \partial^2 z/\partial x^2 + f(x, t, z) \quad \text{in} \quad R^0 + (\mathrm{F}R - \Gamma),$$

$$v_0(x, t) \geqslant z(x, t) \geqslant u_0(x, t)$$

there corresponds a Hölder continuous function $\bar{\omega}(x,t:\xi,\eta:z)$ such that the following conditions hold:

(α') $\bar{\omega}(x,t,\xi,\xi:z)\equiv 0$ for each ξ,

(β') $f(x,t,\eta)-f(x,t,\xi)<\bar{\omega}(x,t:\xi,\eta:z)$ for $u_0(x,t)\leqslant\eta<\xi\leqslant z(x,t)$,

(γ') there is a $K>0$ such that $|\omega(x,t:\xi,\eta:z)|\leqslant K$ whenever $u_0(x,t)\leqslant\eta<\xi\leqslant z(x,t)\leqslant v_0(x,t)$ and $(x,t)\in R$,

(δ') $\bar{\omega}\big(x,t:z_n(x,t),z_{n+1}(x,t):z_n\big)\Rightarrow 0$ if z_n is uniformly convergent in R.

Applying a reasoning similar to that applied in the proof of theorem 6 one easily proves the following theorem:

THEOREM 7. *Suppose that the condition* (C_2) *holds. We assume that* $\psi_n(x,t)$ *are continuous and* $\psi_n(x,t)\Rightarrow\varphi(x,t)$, $\psi_{n+1}(x,t)<\psi_n(x,t)$ *on* Γ. *Then there exists a sequence* $\{v_n\}$ *of regular functions and the following conditions hold:* $v_n(x,t)$ *is a regular solution of the equation*

$$\partial z/\partial t=\partial^2 z/\partial x^2+\bar{\omega}\big(x,t:v_{n-1}(x,t),z:v_{n-1}\big)+f\big(x,t,v_{n-1}(x,t)\big)$$

and $v_n(x,t)=\psi_n(x,t)$ *on* Γ. *Moreover,* $u_0(x,t)<v_n(x,t)<v_{n-1}(x,t)<v_0(x,t)$ in R *and*

$$\partial v_n/\partial t>\partial^2 v_n/\partial x^2+f(x,t,v_n).$$

The sequence $v_n(x,t)$ *converges uniformly in* R *to the maximum solution of* (1), (2).

2.2. We say that the boundary value problem

$$\partial z/\partial t=\partial^2 z/\partial x^2+g(x,t,z)\quad\text{in}\quad R^0+(\mathrm{F}R-\Gamma),$$

$$z(x,t)=\sigma(x,t)\quad\text{on}\quad\Gamma$$

has the property (S) if it possesses a unique solution $w(x,t)$ and the following condition holds: for every regular function $r(x,t)$ the inequalities

$$\partial r/\partial t\leqslant\partial^2 r/\partial x^2+g(x,t,r)\quad\text{in}\quad R^0+(\mathrm{F}R-\Gamma),$$

$$r(x,t)\leqslant\sigma(x,t)\quad\text{on}\quad\Gamma$$

imply that $r(x,t)\leqslant w(x,t)$ in R.

Suppose for example that $|g(x,t,\bar{z})-g(x,t,\bar{\bar{z}})|\leqslant M|\bar{z}-\bar{\bar{z}}|$ and let g be Hölder continuous. Define $g_\varepsilon(x,t,z)=g(x,t,z)+\varepsilon$ for $\varepsilon>0$. Then $|g_\varepsilon(x,t,\bar{z})-g_\varepsilon(x,t,\bar{\bar{z}})|\leqslant M|\bar{z}-\bar{\bar{z}}|$ and consequently there exists (see for instance [11]) a unique solution $z_\varepsilon(x,t)$ of the equation

$$\partial z/\partial t=\partial^2 z/\partial x^2+g_\varepsilon(t,t,z)$$

such that $z_\varepsilon(x,t)=\sigma(x,t)+\varepsilon$ $(\varepsilon>0)$, on Γ. It follows from lemma 1 that $z_{\varepsilon_1}(x,t)<z_{\varepsilon_2}(x,t)$ in R whenever $\varepsilon_1<\varepsilon_2$. From theorem 2.2 of [10] one concludes that $z_\varepsilon(x,t)\underset{\varepsilon\to 0+}{\Rightarrow} z_0(x,t)$ in R. Suppose that the regular function $z(x,t)$ satisfies the inequalities

$$\partial z/\partial t\leqslant\partial^2 z/\partial x^2+g(x,t,z)\quad\text{in}\quad R^0+(\mathrm{F}R-\Gamma)$$

and $z(x,t)\leqslant\sigma(x,t)$ on Γ. Using lemma 1 we find that $z(x,t)<z_\varepsilon(x,t)$ in R for $\varepsilon>0$ and consequently $z(x,t)\leqslant z_0(x,t)$ in R. Hence the property (S) holds. In particular if $g=\alpha(x,t)z+\beta(x,t)$ then property (S) holds whenever $\alpha(x,t)$ is Hölder continuous.

If we replace in (C_1) (resp. (C_2)) the sign $<$ in (15) and (β) (resp. (β')) by $\leqslant$ we get a new condition, which we denote by (C_1^*) (resp. (C_2^*)). All the remaining conditions of (C_1) (resp. (C_2)) are not changed. One easily proves the following theorem:

THEOREM 8. *Suppose that* (C_1^*) *holds. Let the problem* (1), (2) *have the property* (S). *We assume that for every regular function* $w(x,t)$ *the problem*

$$\partial z/\partial t=\partial^2 z/\partial x^2+\underline{\omega}\big(x,t:z,w(x,t):w\big)+f\big(x,t,w(x,t)\big),$$

$$z(x,t)=\sigma(x,t)\quad\text{on}\quad\Gamma$$

where $u_0(x,t)\leqslant\sigma(x,t)\leqslant v_0(x,t)$ *on* Γ *has the property* (S). *Under our assumptions there exists a sequence* $u_n(x,t)$ *of regular functions converging uniformly in* R *to the solution of* (1), (2) *and the following condition hold:* $u_n(x,t)$ *is the solution of the equation*

$$\partial z/\partial t=\partial^2 z/\partial x^2+\underline{\omega}\big(x,t:z,u_{n-1}(x,t):u_{n-1}\big)+f\big(x,t,u_{n-1}(x,t)\big).$$

Moreover $u_0(x,t)\leqslant u_n(x,t)\leqslant u_{n+1}(x,t)<v_0(x,t)$ *in* R *and*

$$\partial u_n/\partial t\leqslant\partial^2 u_n/\partial x^2+f(x,t,u_n).$$

A similar theorem may be formulated when (C_2^*) holds.

For the sake of simplicity we assume in what follows that the equations considered have the property (S). Now we introduce the following condition:

(C) Let the assumption (H) be satisfied. It is supposed that to every couple $\underline{z}$, $\bar{z}$ of regular functions such that $u_0(x,t)\leqslant\underline{z}(x,t)\leqslant\bar{z}(x,t)\leqslant v_0(x,t)$ there correspond two Hölder continuous functions $\underline{\omega}(x,t:\xi,\eta:\underline{z},\bar{z})$, $\bar{\omega}(x,t:\xi,\eta:\underline{z},\bar{z})$ and the following relations hold:

(α'') $\underline{\omega}(x,t:\xi,\xi:\underline{z},\bar{z})\equiv\bar{\omega}(x,t:\xi,\xi:\underline{z},\bar{z})\equiv 0$ for each ξ,

(β'') $\underline{\omega}(x,t:\xi,\eta:\underline{z},\bar{z})\leqslant f(x,t,\xi)-f(x,t,\eta)$ and $f(x,t,\eta)-f(x,t,\xi)\leqslant\bar{\omega}(x,t:\xi,\eta:\underline{z},\bar{z})$ whenever $\underline{z}(x,t)\leqslant\eta<\xi\leqslant\bar{z}(x,t)$,

(γ'') the functions $\underline{\omega}$, $\bar{\omega}$ are bounded,

(δ'') $\lim\limits_{n\to\infty}\underline{\omega}(x,t:\underline{z}_n,\underline{z}_{n-1}:\underline{z}_{n-1},\bar{z}_{n-1})=\lim\limits_{n\to\infty}\bar{\omega}(x,t:\bar{z}_{n-1},\bar{z}_n:\underline{z}_{n-1},\bar{z}_{n-1})=0$ uniformly in R if $\lim\limits_{n\to\infty}\bar{z}_n=\lim\limits_{n\to\infty}\underline{z}_n=z$ uniformly in R.

One easily proves the following theorem:

THEOREM 9. *Let the condition* (C) *be satisfied. It is supposed that* $u_0(x,t) \leqslant \varphi_n(x,t) \leqslant \varphi_{n+1}(x,t) \leqslant \varphi(x,t) \leqslant \psi_{n+1}(x,t) \leqslant \psi_n(x,t) \leqslant v_0(x,t)$ *for* $(x,t)\in\Gamma$ *and* $\lim\limits_{n\to\infty}\psi_n = \lim\limits_{n\to\infty}\varphi_n = \varphi$ *uniformly on* Γ. *Then there exist two sequences* u_n, v_n *of regular functions and the following relations hold:*

$$\partial u_n/\partial t \leqslant \partial^2 u_n/\partial x^2 + f(x,t,u_n), \qquad \partial v_n/\partial t \geqslant \partial^2 v_n/\partial x^2 + f(x,t,v_n), \tag{29}$$

$$\begin{aligned} u_0(x,t) \leqslant u_n(x,t) \leqslant u_{n+1}(x,t) &\leqslant v_{n+1}(x,t) \leqslant v_n(x,t) \\ &\leqslant v_0(x,t) \quad \text{for} \quad (x,t)\in R, \end{aligned} \tag{30}$$

$$\begin{aligned} \partial u_n/\partial t &= \partial^2 u_n/\partial x^2 + \underline{\omega}(x,t: u_n, u_{n-1}: u_{n-1}, v_{n-1}) + f(x,t,u_{n-1}), \\ \partial v_n/\partial t &= \partial^2 v_n/\partial x^2 + \bar{\omega}(x,t: v_{n-1}, v_n: u_{n-1}, v_{n-1}) + f(x,t,v_{n-1}) \end{aligned} \tag{31}$$

and $u_n(x,t) = \varphi_n(x,t)$, $v_n(x,t) = \psi_n(x,t)$ *for* $(x,t)\in\Gamma$,

(32) $u_n \Rightarrow u$, $v_n \Rightarrow u$ *uniformly in* R *where* u *is the solution of* (1), (2).

We shall present some examples.

EXAMPLE 1. Suppose that

$$a(x,t)(\xi-\eta) \leqslant f(x,t,\xi) - f(x,t,\eta)$$

for $\xi > \eta$. We can put $\underline{\omega} = a(x,t)(\xi-\eta)$ in theorem 6.

EXAMPLE 2. Assume that f_z exists (see [11]) and

$$m = \inf_{\substack{(x,t)\in R \\ u_0(x,t)\leqslant z\leqslant v_0(x,t)}} f_z(x,t,z).$$

We can put $\underline{\omega} = (m-\varepsilon)(\xi-\eta)$ in theorem 6. If $|f_z| < M$ then we put $\underline{\omega} = -M(\xi-\eta)$ in theorem 6 and $\bar{\omega} = M(\xi-\eta)$ in theorem 7. One can also take in theorem 8

$$\underline{\omega}(x,t;\xi,\eta:w) = \inf_{\substack{(x,t)\in R \\ w(x,t)\leqslant z\leqslant v_0(x,t)}} f_z(x,t,z)(\xi-\eta).$$

EXAMPLE 3. This example deals with the original method of Chaplighin (see [1] and [3]). It is supposed that $f_z(x,t,z)$ is Hölder continuous and increases in z. Then $f(x,t,z)$ is convex with respect to z. The assumptions of theorem 9 are satisfied if

$$\underline{\omega}(x,t:\xi,\eta:\underline{z},\bar{z}) = f_z(x,t,\underline{z})(\xi-\eta)$$

and

$$\begin{aligned} \bar{\omega}(x,t:\xi,\eta:\underline{z},\bar{z}) = \frac{f(x,t,\underline{z}(x,t)) - f(x,t,\xi)}{\underline{z}(x,t)-\xi}\big(\eta-\underline{z}(x,t)\big) + \\ + f(x,t,\underline{z}(x,t)) - f(x,t,\xi). \end{aligned}$$

If $\xi = \underline{z}(x,t)$, then the coefficient at $(\eta - \underline{z}(x,t))$ is replaced by $f_z(x,t,\underline{z}(x,t))$.

3.1. In this section we derive some estimates for the difference $u-u_n$ where u is the solution of (1), (2) and u_n are solutions of suitable equations considered in § 2. To begin with we formulate a lemma of J. Szarski [10]:

LEMMA 2. *Suppose that the function* $\sigma(t,y) \geqslant 0$ *is continuous for* $t\in\langle 0, T+\varepsilon)$ $(\varepsilon > 0)$ *and* $y \geqslant 0$. *Let* $\tau(t,\eta)$ *be the right-hand maximum solution of the equation* $y' = \sigma(t,y)$ *such that* $\tau(0,\eta) = \eta$. *It is supposed that for each* $\eta \geqslant 0$ $\tau(t,\eta)$ *exists in* $\langle 0, T+\varepsilon)$. *We are given two regular solutions* $u(x,t)$, $v(x,t)$ *of equations*

$$\partial u/\partial t = \partial^2 u/\partial x^2 + g(x,t,u)$$

and

$$\partial v/\partial t = \partial^2 v/\partial x^2 + h(x,t,v)$$

respectively. Assume that

$$\big|g(x,t,u(x,t)) - h(x,t,v(x,t))\big| \leqslant \sigma\big(t, |u(x,t)-v(x,t)|\big) \quad \text{in} \quad R^0.$$

Let $\eta \geqslant \max\limits_{\Gamma}|u(x,t)-v(x,t)|$. *Then*

$$|u(x,t)-v(x,t)| \leqslant \tau(t,\eta) \quad \text{for} \quad (x,t)\in R.$$

Now let the derivative f_z be Hölder continuous and increasing in z. Suppose that (H) holds and

$$u_0(x,t) < \varphi(x,t) < v_0(x,t) \quad \text{on} \quad \Gamma. \tag{33}$$

Using theorem 8 we obtain a sequence u_n such that

$$\begin{aligned} \partial u_n/\partial t = \partial^2 u_n/\partial x^2 + f_z(x,t,u_{n-1}(x,t))\big(u_n(x,t)-u_{n-1}(x,t)\big) + \\ + f(x,t,u_{n-1}(x,t)). \end{aligned} \tag{34}$$

Moreover,

$$u_0(x,t) \leqslant u_n(x,t) \leqslant u_{n+1}(x,t) \leqslant u(x,t) < v_0(x,t), \tag{35}$$

where $u(x,t)$ is the solution of (1), (2). The sequence u_n converges uniformly in R to u.

3.2. We shall prove a theorem concerning the limitation of difference $u_n - u$.

THEOREM 10. *Suppose that* $f_z(x,t,z)$ *is Hölder continuous and increasing in* z. *Let* (H) *be satisfied and suppose that* (33) *holds. Let* u *be the* *solution of* (1), (2). *Suppose that the function* $\omega(t, y)$ *is continuous for* $y \geqslant 0$ *and increases in* y. *We assume that*

$$(36)\qquad |f_z(x, t, \bar{z}) - f_z(x, t, \bar{\bar{z}})| \leqslant \omega(t, |\bar{z} - \bar{\bar{z}}|)$$

whenever $u_0(x, t) \leqslant \bar{z}, \bar{\bar{z}} \leqslant v_0(x, t)$. *Let* $\tau_0(t) \geqslant \max_{a \leqslant x \leqslant b} \{v_0(x, t) - u_0(x, t)\}$ *and define*

$$(37)\qquad \tau_{n+1}(t) = \varepsilon_{n+1} e^{Kt} + \int_0^t e^{K(t-s)} \omega\big(s, \tau_n(s)\big) \tau_n(s)\, ds,$$

where

$$K = \sup_{\substack{(x,t) \epsilon R \\ u_0(x,t) \leqslant z \leqslant v_0(x,t)}} |f_z(x, t, z)|.$$

Suppose that

$$(38)\qquad |u_n(x, t) - u(x, t)| \leqslant \varepsilon_n \quad \text{on} \quad \Gamma.$$

Our assumptions imply that

$$(39)\qquad |u(x, t) - u_n(x, t)| \leqslant \tau_n(t), \quad (x, t) \epsilon R, \quad n = 0, 1, 2, \ldots$$

Proof. Obviously (39) holds for $n = 0$. Suppose that (39) holds for $n = k$. Remark that

$$\partial u/\partial t = \partial^2 u/\partial x^2 + f\big(x, t, u(x, t)\big)$$

and

$$\frac{\partial u_{n+1}}{\partial t} = \frac{\partial^2 u_{n+1}}{\partial x^2} + f_z\big(x, t, u_n(x, t)\big)\big(u_{n+1}(x, t) - u_n(x, t)\big) + f\big(x, t, u_n(x, t)\big).$$

It follows from the mean value theorem that

$$(40)\qquad \begin{aligned} &f\big(x, t, u(x, t)\big) - f\big(x, t, u_n(x, t)\big) \\ &\quad = f_z\big(x, t, u_n(x, t)\big) + \vartheta_n\big(u(x, t) - u_n(x, t)\big)\big(u(x, t) - u_n(x, t)\big), \end{aligned}$$

where $0 \leqslant \vartheta_n \leqslant 1$. On the other hand

$$\begin{aligned} &f_z\big(x, t, u_n(x, t)\big) + \vartheta_n\big(u(x, t) - u_n(x, t)\big)\big(u(x, t) - u_n(x, t)\big) - \\ &\quad - f_z\big(x, t, u_n(x, t)\big)\big(u_{n+1}(x, t) - u_n(x, t)\big) \\ &= \big[f_z\big(x, t, u_n(x, t)\big) + \vartheta_n\big(u(x, t) - u_n(x, t)\big) - f_z\big(x, t, u_n(x, t)\big)\big]\big(u(x, t) - \\ &\quad - u_n(x, t)\big) + f_z\big(x, t, u_n(x, t)\big)\big(u(x, t) - u_{n+1}(x, t)\big). \end{aligned}$$

Note that $0 \leqslant u(x, t) - u_n(x, t) \leqslant \tau_n(t)$ and $0 \leqslant \vartheta_n \leqslant 1$. The function $\omega(t, y)$ increases in y. Hence

$$\begin{aligned} &\big|\big[f_z\big(x, t, u_n(x, t)\big) + \vartheta_n\big(u(x, t) - u_n(x, t)\big) - f_z\big(x, t, u_n(x, t)\big)\big]\big(u(x, t) - \\ &\quad - u_n(x, t)\big)\big| \leqslant \omega\big(t, \vartheta_n\big(u(x, t) - u_n(x, t)\big)\big)\tau_n(t) \leqslant \omega\big(t, \tau_n(t)\big)\tau_n(t). \end{aligned}$$

Obviously

$$\big|f_z\big(x, t, u_n(x, t)\big)\big(u(x, t) - u_{n+1}(x, t)\big)\big| \leqslant K|u(x, t) - u_{n+1}(x, t)|.$$

Using the above relations and applying lemma 2 we get

$$|u(x, t) - u_{n+1}(x, t)| \leqslant \tau(t, \varepsilon_{n+1}), \qquad (x, t) \epsilon R,$$

where $\tau(t, \varepsilon_{n+1})$ is the solution of the equation

$$y' = Ky + \omega\big(t, \tau_n(t)\big)\tau_n(t)$$

such that $\tau(0, \varepsilon_{n+1}) = \varepsilon_{n+1}$. We conclude therefore that

$$|u(x, t) - u_{n+1}(x, t)| \leqslant \varepsilon_{n+1} e^{Kt} + \int_0^t e^{K(t-s)} \omega\big(s, \tau_n(s)\big)\tau_n(s)\, ds = \tau_{n+1}(t)$$

and our theorem is proved.

Remark 2. Theorem 10 gives some generalization of results of Lusin [3] obtained for ordinary differential equations. In [3] $\omega(t, y) = Hy$ where $H = \sup|\partial^2 f/\partial x^2|$. Following Lusin one immediately proves that if $\partial^2 f/\partial x^2 > 0$ and if

$$\tau_0(t) \leqslant 1/2HTe^{KT} = C$$

and $\varepsilon_n = 0$, then

$$|u(x, t) - u_n(x, t)| \leqslant 2C/2^{2^n}.$$

Remark 3. Suppose that $\varepsilon_n = 0$ and $e^{K(t-s)} \leqslant N$ for $0 \leqslant s \leqslant t \leqslant T$. Consider the ordinary differential equation

$$(40)\qquad y' = N\omega(t, y)y.$$

The unique solution of (40) passing through the point $(0, 0)$ is the solution identically equal to zero. Suppose that $\tau_0(t) \leqslant \psi_0(t)$ in $\langle 0, T\rangle$ and define

$$\psi_{n+1}(t) = \int_0^t N\omega\big(s, \psi_n(s)\big)\psi_n(s)\, ds.$$

$\psi_n(t)$ is a sequence of successive approximations for (40). It is easy to show that $\tau_n(t) \leqslant \psi_n(t)$. In general ψ_n converges to zero more strongly than $t^n/n!$.

Remark 4. The methods developed in § 2 and § 3 may be extended to equations of the following form:

$$(41)\qquad \frac{\partial u}{\partial t} = \sum_{i,k=1}^n a_{ik}(x, t)\frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{j=1}^n b_j(x, t)\frac{\partial u}{\partial x_j} + f(x, t, u, u_{x_1}, u_{x_2}, \ldots, u_{x_n}) \qquad (x = (x_1, \ldots, x_n)).$$

Our derivations given before may serve as a model to derive analogous results for (41). Such generalizations must be based on lemmas analogous to lemma 1 and lemma 2. Generalizations of lemma 1 and 2 are respectively proved in [4] and [10]. Following Zeragia [11] we can introduce the following assumption: the form

$$\sum \frac{\partial^2 f}{\partial p_i \partial p_k} \xi_i \xi_k \geqslant 0 \qquad (f = f(x, t, p_1, \ldots, p_{n+1})).$$

A new difficulty arises in that case. This is the problem of equi-boundedness of first spatial derivatives of approximate solutions. The approximate equations are of the following form:

$$\begin{aligned} \frac{\partial u_{\nu+1}}{\partial t} = & \sum_{i,k=1}^{n} a_{ik}(x, t) \frac{\partial^2 u_{\nu+1}}{\partial x_i \partial x_k} + \sum_{j=1}^{n} b_j(x, t) \frac{\partial u_{\nu+1}}{\partial x_j} + \\ & + f_u\left(x, t, u_\nu, \frac{\partial u_\nu}{\partial x_1}, \ldots, \frac{\partial u_\nu}{\partial x_n}\right)(u_{\nu+1} - u_\nu) + \\ & + \sum_{i=1}^{n} f_{q_i}\left(x, t, u_\nu, \frac{\partial u_\nu}{\partial x_1}, \ldots, \frac{\partial u_\nu}{\partial x_n}\right)\left(\frac{\partial u_{\nu+1}}{\partial x_i} - \frac{\partial u_\nu}{\partial x_i}\right) + \\ & + f\left(x, t, u_\nu, \frac{\partial u_\nu}{\partial x_1}, \ldots, \frac{\partial u_\nu}{\partial x_n}\right) \qquad (f = f(x, t, u, q_1, \ldots, q_n)). \end{aligned}$$

A suitable regularity assumptions for a_{ik}, b_j must be introduced.

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Reçu par la Rédaction le 7. 4. 1959