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ABOUT AN ESTIMATION PROBLEM OF ZAHORSKI

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Z. Zahorski [4] has asked for the best possible estimation from above of the integral

$$\int_0^{2\pi} |\cos n_1 x + \cos n_2 x + \ldots + \cos n_k x|\, dx,$$

where $0 < n_1 < n_2 < \ldots < n_k$ are integers. He observes that the estimation of $c\sqrt{k}$ is trivial, but he conjectures that $c \log n_k$ is also valid. We shall refute this question twice.

I. *We find a sequence n_i for which*

$$\int_0^{2\pi} \Big| \sum_{i=1}^{k} \cos n_i x \Big|\, dx > ck^{\frac{1}{2}-\varepsilon}.$$

II. *We find a sequence n_i for which*

$$\int_0^{2\pi} \Big| \sum_{i=1}^{k} \cos n_i x \Big|\, dx = \sqrt{\pi}\sqrt{n_k} + o(\sqrt{n_k}),$$

which proves that $O(\sqrt{n_k})$ is the best estimation.

Since the proof of I is much more elementary than the proof of II, we also include it.

The problem remains whether for every sequence $n_1 < n_2 < \ldots < n_k < \ldots$ and for every $\varepsilon > 0$ we have for $k > k_0(\varepsilon)$

$$\int_0^{2\pi} \Big| \sum_{i=1}^{k} \cos n_i x \Big|\, dx < (\sqrt{\pi} + \varepsilon)\sqrt{n_k}.$$

Proof of I. Let us put $n_i = i^2$; $1 \leqslant i \leqslant k$. We are going to prove that

$$\int_0^{2\pi} \Big| \sum_{i=1}^{k} \cos i^2 x \Big|\, dx > ck^{\frac{1}{2}-\varepsilon}. \tag{1}$$

To check this observe that clearly

$$\int_0^{2\pi}\Big(\sum_{i=1}^{k}\cos i^2x\Big)^2dx = \pi k, \tag{2}$$

and it is not difficult to see that for every $\eta>0$ and $k>k_0(\eta)$

$$\int_0^{2\pi}\Big(\sum_{i=1}^{k}\cos i^2x\Big)^4dx < k^{2+\eta}. \tag{3}$$

Namely, in order to prove (3), observe that

$$\int_0^{2\pi}\Big(\sum_{i=1}^{k}\cos i^2x\Big)^4dx < c_1\sum_{\substack{i_1^2\pm i_2^2\pm i_3^2\pm i_4^2=0\\ 1\leqslant i_1,i_2,i_3,i_4\leqslant k}} 1 < k^{2+\eta}. \tag{4}$$

Indeed, at least two terms in the sum $i_1^2\pm i_2^2\pm i_3^2\pm i_4^2$ have the same sign. If these terms are i_1^2 and i_2^2, we can write $2\leqslant i_1^2+i_2^2 = \pm i_3^2\pm i_4^2\leqslant 2k^2$. The inequalities $2\leqslant \pm i_3^2\pm i_4^2\leqslant 2k^2$, $1\leqslant i_3, i_4<k$ have $O(k^2)$ solutions. We denote by $\lambda(x)$ the number of solutions of the equation $i_1^2+i_2^2=x$. It is well known that $\lambda(x)=o(x^\varepsilon)$ (1). Hence the number of solutions of the equation $i_1^2\pm i_2^2\pm i_3^2\pm i_4^2=0$ is

$$k^2\max_{x=\pm i_3^2\pm i_4^2}\lambda(x) = o(k^{2+\varepsilon}).$$

From (3) we observe that the set in x for which

$$\Big|\sum_i \cos i^2x\Big| > tk^{1/2}$$

has a measure less than k^η/t^4. Thus, a simple computation shows that

$$\int_I\Big(\sum_{i=1}^{k}\cos i^2x)^2dx = \sum_{u=0}^{\infty}\int_{I_u}\Big(\sum_{i=1}^{k}\cos i^2x\Big)^2dx = o(k), \tag{5}$$

where I is the set in which

$$\Big|\sum_{i=1}^{k}\cos i^2x\Big| > k^{\frac12+\eta},$$

and the sets I_u are those in which

$$2^uk^{\frac12+\eta} < \Big|\sum_{i=1}^{k}\cos i^2x\Big| \leqslant 2^{u+1}k^{\frac12+\eta}.$$

(1) Indeed, $\lambda(x)\leqslant\tau(x)$, where $\tau(x)$ is the number of the divisors of x (see e. g. [2], p. 398), and $\tau(x)=o(x^\varepsilon)$ (see e. g. [3], p. 26). *(Remark of the Editors).*

Formulae (2) and (5) imply

$$\int_{I'}\Big(\sum\cos i^2x\Big)^2dx = \pi k+o(k), \tag{6}$$

where I' is the complement of I, i. e. for $x\in I'$ we have

$$\Big|\sum_{i=1}^{k}\cos i^2x\Big| \leqslant k^{\frac12+\eta}.$$

Thus

$$\int_0^{2\pi}\Big|\sum_{i=1}^{k}\cos i^2x\Big|dx \geqslant \int_{I'}\Big|\sum_{k=1}^{k}\cos i^2x\Big|dx \geqslant \frac{1}{k^{\frac12+\eta}}\int_{I'}\Big(\sum_{i=1}^{k}\cos i^2x\Big)^2dx$$

$$= \frac{\pi k+o(k)}{k^{\frac12+\eta}} > ck^{\frac12-\eta},$$

which completes the proof of I.

The proof of II is based on a theorem of Salem and Zygmund [1]. Let us write

$$S_N = \sum_1^N \varphi_k(t)(a_k\cos kx+b_k\sin kx),$$

where $\{\varphi_n(t)\}$ is the system of Rademacher functions,

$$c_k^2 = a_k^2+b_k^2;\qquad B_N^2 = \tfrac12\sum_1^N c_k^2,$$

and let $\omega(p)$ be a function of p increasing to $+\infty$ with p, such that $p/\omega(p)$ increases and that $\sum 1/p\,\omega(p)<\infty$. Then, under the assumptions $B_N^2\to\infty$, $c_N^2 = O\{B_N^2/\omega(B_N^2)\}$, the distribution function of S_N/B_N tends, for almost every t, to the Gaussian distribution with mean value zero and dispersion 1.

Let us set $a_k=1$, $b_k=0$ $(k=1,2,\ldots)$; then $c_N^2=1$, $B_N^2=\frac12 N$ where $N=1,2,\ldots$ Moreover, it is easy to verify that the function $\omega(p)=\sqrt{p}$ satisfies the conditions of the Salem-Zygmund theorem. Consequently, for almost all t, the distribution function of

$$\frac{S_N}{B_N} = \frac{\sqrt2}{\sqrt N}\sum_{k=1}^{N}\varphi_k(t)\cos kx$$

tends to the Gaussian distribution with mean value zero and dispersion 1. Furthermore, since the variance of S_N/B_N is equal to 1, we have for almost all t the convergence of the absolute moments of S_N/B_N to the absolute moment of the normalized Gaussian distribution. In other words, we have the relation

$$\lim_{N\to\infty}\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{\sqrt{2}}{\sqrt{N}}\sum_{k=1}^{N}\varphi_k(t)\cos kx\right|dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|x|e^{-x^2/2}dx=\sqrt{\frac{2}{\pi}}$$

for almost all t. Hence, using the well-known equality

$$\lim_{N\to\infty}\frac{1}{\sqrt{N}}\int_0^{2\pi}\left|\sum_{k=1}^{N}\cos kx\right|dx=0,$$

we obtain the relation

$$\lim_{N\to\infty}\frac{1}{\sqrt{N}}\int_0^{2\pi}\left|\sum_{k=1}^{N}(\varphi_k(t)+1)\cos kx\right|dx=2\sqrt{\pi} \tag{7}$$

for almost all t.

Let us fix an irrational number t_0 with this property. Let $n_1, n_2, \ldots$ denote the successive indices k for which $\varphi_k(t_0)=1$. Then

$$\sum_{k=1}^{n_N}(\varphi_k(t_0)+1)\cos kx=2\sum_{k=1}^{N}\cos n_k x$$

and, according to (7),

$$\int_0^{2\pi}\left|\sum_{k=1}^{N}\cos n_k x\right|dx=\sqrt{\pi}\sqrt{n_N}+o(\sqrt{n_N}),$$

which completes the proof of II.

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ON A PROBLEM OF MAZUR AND ULAM ABOUT IRREDUCIBLE GENERATING SYSTEMS IN GROUPS

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1. INTRODUCTION

In 1935 S.Mazur and S. Ulam have stated the following problem [1] (for the terminology see the end of this section):

Let a group possess an irreducible generating system. Does each of its subgroups also have this property?

The problem mentioned has been solved negatively in paper [1]; e. g. the abelian group

$$G(p^\infty)+\sum_{i=1}^{\infty}G_i(p),$$

where $G(p^\infty)$ is the Prüfer group of the type p^∞ and $G_i(p)$ cyclic groups of the prime order p, possesses an irreducible generating system but none of its non-reduced subgroup with a finite reduced component has this property (it is easy to see that the subgroups in question are all those having no irreducible generating system; see also [2]). It is the purpose of this note to give some more general constructions of groups G with

PROPERTY P. *The group G possesses an irreducible generating system, but there exists a subgroup $H\subset G$ every generating system of which is reducible.*

Throughout this article we consider predominantly non-abelian groups written multiplicatively; $\times$ and (in the abelian case) $+$ denote the direct product. $\sum$ denotes the weak direct sum of abelian groups. G^n for a fixed natural n is the subset (of the group G) of all elements g^n with $g\in G$. The power of a set $\mathfrak{M}$ will be denoted by $m(\mathfrak{M})$ and the order of an element $g\in G$ by $O(g)$.

For any non-void subset $\mathfrak{M}$ of G, $\{\mathfrak{M}\}$ denotes the subgroup of G generated by the elements of $\mathfrak{M}$; thus $\{\mathfrak{G}\}=G$ means that $\mathfrak{G}$ is a *genera-*

[1] The Scottish Book, Problem 63, p. 27. I am indebted to Jan Mycielski for calling my attention to this problem.