An alternative form of Egoroff's theorem

by

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Given a sequence of measurable, real-valued functions $\{f_n(x)\}$ defined for $x \in E$, a Lebesgue measurable subset of finite measure of Euclidean m-space ($m \geqslant 1$), and such that

$$(1) \qquad f_n(x) \to f(x) \quad \text{as} \quad n \to \infty,$$

for all x in E; the well-known theorem due to Egoroff states that the convergence is almost uniform in the following sense. For any given $\varepsilon > 0$, there exists a measurable subset $S \subset E$ such that $|E-S| < \varepsilon$ (1) and, for all $x \in S$,

$$f_n(x) \to f(x) \quad \textit{uniformly} \text{ as } \quad n \to \infty.$$

In the present note it is shown that the difference $|f_n(x)-f(x)|$ satisfies some order condition in a subset $T \subset E$ such that $|E-T| = 0$. That is, under the condition (1), there exists a subset T and a decreasing sequence $\{\delta_n\}$ of positive numbers such that $\delta_n \to 0$ as $n \to \infty$, and for $x \in T$,

$$(2) \qquad \frac{|f_n(x)-f(x)|}{\delta_n} \to 0 \quad \text{as} \quad n \to \infty.$$

Further, in the new form of the theorem, it is not necessary to assume that $|E|$ is finite: the finiteness of $|E|$ is essential for the validity of the usual form.

The new form of the theorem clearly has many applications; in fact, many of the standard proofs which use Egoroff's theorem become clearer when this version is used. In [4], I showed that the Lebesgue density theorem could be strengthened and the proof given did not involve the use of the usual form of the density theorem. However, the direct methods of [4] did not give the strongest form of the density theorem in m-space ($m \geqslant 2$). If we assume the usual forms of the density theorem, stronger forms can be obtained by applying the new form of Egoroff's theorem. This solves the problem stated in [4].

(1) If E is a set in Euclidean m-space, $|E|$ will denote the m-dimensional Lebesgue outer measure of E.

The other application which I consider briefly formed the motivation for the present note. This shows that an absolutely continuous function is almost uniformly differentiable in the sense that the second order error term satisfies some order condition almost everywhere. In 1-dimension, if $f(x)$ is Lebesgue integrable over $[a, b]$, and

$$F(x) = \int_a^x f(t)\,dt,$$

we can say that there exists a positive function $\psi(h)$ monotonic for $h > 0$, with $\lim_{h\to 0+} \psi(h) = 0$, such that

$$\frac{1}{\psi(|h|)}\left|\frac{F(x+h)-F(x)}{h} - f(x)\right| \to 0 \quad \text{as} \quad h \to 0 \tag{3}$$

for almost all x in $[a, b]$. The problem as to whether or not (3) is true was suggested to me by Professor H. E. Daniels.

THEOREM 1. *Suppose that E is a Lebesgue measurable subset of Euclidean m-space, and $\{f_n(x)\}$ $(n = 1, 2, \ldots)$ is a sequence of measurable functions such that*

$$f_n(x) \to f(x) \quad \textit{as} \quad n \to \infty$$

for all x in E. Then there exists a monotonic sequence $\{\delta_n\}$ of positive numbers, with $\delta_n \to 0$ as $n \to \infty$, and a subset $S \subset E$ with $|E-S| = 0$ such that

$$\frac{|f_n(x)-f(x)|}{\delta_n} \to 0 \quad \textit{as} \quad n \to \infty$$

for all x in S.

Proof. Express $E = \bigcup_{k=1}^{\infty} E_k$, where each E_k is measurable and has finite measure. Then $f_n(x) \to f(x)$ for all x in E_k. Apply Egoroff's Theorem (see Saks [3], p. 18), to the sequence $\{f_n(x)\}$ defined on E_k. For $k = 1, 2, \ldots$, $\mu = 1, 2, \ldots$, let $E_{k,\mu}$ be a measurable subset of E_k such that

$$|E_k - E_{k,\mu}| < 2^{-\mu}, \tag{4}$$

and $f_n(x) \to f(x)$ uniformly for all x in $E_{k,\mu}$. Write $S = \bigcup_{k=1}^{\infty}\bigcup_{\mu=1}^{\infty} E_{k,\mu}$. Then it follows from (4) that $|E_k - \bigcup_{\mu=1}^{\infty} E_{k,\mu}| = 0$ and therefore $|E-S| = 0$. For each fixed k, μ, choose an increasing sequence $\{n_{k,\mu,r}\}$ $(r = 1, 2, \ldots)$ of positive integers such that, for all $x \in E_{k,\mu}$,

$$|f_n(x) - f(x)| < \frac{1}{r+1} \quad \text{when} \quad n \geqslant n_{k,\mu,r}. \tag{5}$$

Let $n_1, n_2, \ldots, n_t, \ldots$ be an increasing sequence of positive integers such that

$$\frac{n_t}{n_{k,\mu,t}} \to \infty \quad \text{as} \quad t \to \infty \tag{6}$$

for each fixed k, μ. We now define $\{\delta_n\}$ by

$$\delta_n = \begin{cases} 1 & \text{for} \quad 1 \leqslant n \leqslant n_1, \\ t^{-1/2} & \text{for} \quad n_{t-1}+1 \leqslant n \leqslant n_t \quad (t = 2, 3, \ldots). \end{cases} \tag{7}$$

Clearly $\{\delta_n\}$ is a monotonic decreasing sequence of positive numbers and $\delta_n \to 0$ as $n \to \infty$. If x is any point of S, then it is in $E_{k,\mu}$ for some integers k, μ. By (6) we can find t_0 such that $n_t \geqslant n_{k,\mu,t}$ for $t \geqslant t_0$. It follows from (5) and (7) that

$$|f_n(x) - f(x)| < \delta_n^2 \quad \text{for} \quad n \geqslant n_{t_0}.$$

A fortori, we have found a set S, and a sequence $\{\delta_n\}$ such that

$$\frac{|f_n(x)-f(x)|}{\delta_n} \to 0 \quad \text{as} \quad n \to \infty,$$

for every x in S.

COROLLARY. *Under the conditions of Theorem 1, in the case where E has finite measure, there exists a monotonic sequence $\{\delta_n\}$ of positive numbers, with $\delta_n \to 0$ as $n \to \infty$ such that, given any $\varepsilon > 0$, there is a subset $E_\varepsilon \subset E$ with $|E - E_\varepsilon| < \varepsilon$ and*

$$\frac{|f_n(x)-f(x)|}{\delta_n} \to 0 \quad \textit{uniformly for } x \textit{ in } E_\varepsilon.$$

This follows immediately on applying Egoroff's theorem to the functions

$$\varphi_n(x) = \frac{|f_n(x)-f(x)|}{\delta_n} \quad \text{for} \quad x \in S$$

where $S \subset E$, and $\{\delta_n\}$ satisfy the conditions of theorem 1. Thus this Corollary is an apparently stronger form of the standard Egoroff Theorem.

APPLICATION 1. Suppose that E is a measurable set in m-space. For x in E, $n = 1, 2, \ldots$, define

$$f_n(x) = \sup_{\substack{x \in I \\ d(I) \leqslant 1/n}} \left|\frac{|I \cap E|}{|I|} - 1\right| \tag{8}$$

the supremum being taken for all closed m-dimensional rectangles I containing x, having sides parallel to fixed coordinate axes in R_m, and having diameter $d(I)$, not greater than $1/n$.

Then there is a subset $E' \subset E$, with $|E - E'| = 0$, such that, when $x \in E'$

$$f_n(x) \to 0 \quad \text{as} \quad n \to \infty. \tag{9}$$

This result is due to Riesz [2], and Buseman and Feller [1]: a simplified proof is given in Saks [3], p. 130. In R_1 the result is no more than the Lebesgue density theorem, while in R_m ($m \geqslant 2$), it is a more precise theorem. In the above definition of density it is essential that the intervals I have their sides parallel to given prescribed directions. Without this condition, (9) ceases to be true even for closed sets E: for a proof see [1], p. 243.

THEOREM 2. *Given any measurable set E in Euclidean m-space, there is a real function $\psi(t)$, monotonic increasing, and defined for $t>0$ with $\lim\limits_{t\to 0+}\psi(t)=0$, such that, for $x\in S$*

$$\frac{1}{\psi(d(I))}\left|\frac{|I\cap E|}{|I|}-1\right|\to 0 \quad as \quad d(I)\to 0$$

where I is any rectangle containing x with sides parallel to the coordinate axes of R_m, and $S\subset E$ is such that $|E-S|=0$.

Proof. For rectangles I with sides parallel to the coordinate axes of R_m, $|I\cap E|/|I|$ is a continuous function of the vertices of I. Hence

$$f_n(x)=\sup_{\substack{x\in I_r\\ d(I_r)\leqslant 1/n}}\left|\frac{|I\cap E|}{|I|}-1\right|$$

where the supremum is now taken over closed rectangles whose vertices have rational coordinates. Hence $f_n(x)$ is a measurable function of x and we can apply theorem 1 to the sequence $\{f_n(x)\}$ in the set E' for which (9) holds. Let $\{\delta_n\}$, $S\subset E'$ satisfy the conditions of theorem 1, so that for $x\in S$

$$\frac{f_n(x)}{\delta_n}\to 0 \quad \text{as} \quad n\to\infty. \tag{10}$$

Define a function $\psi(t)$ as follows:

$$\psi(0)=0,\quad \psi\left(\frac{1}{n}\right)=\delta_{n-1}\quad (n=2,3,\dots),$$

$$\psi(t) \text{ is linear for } \frac{1}{n+1}\leqslant t\leqslant\frac{1}{n}\ (n=2,3,\dots).$$

Clearly $\psi(t)$ is monotonic and $\lim\limits_{t\to 0+}\psi(t)=0$. If x is a point of S, I is any closed rectangle containing x and $1/(n+1)<d(I)\leqslant 1/n$, then $\psi(d(I))\geqslant\delta_n$, and therefore

$$\frac{1}{\psi(d(I))}\left|\frac{|I\cap E|}{|I|}-1\right|\leqslant\frac{f_n(x)}{\delta_n},\quad \text{by (8)}.$$

By (10), it follows that

$$\frac{1}{\psi(d(I))}\left|\frac{|I\cap E|}{|I|}-1\right|\to 0 \quad \text{as} \quad d(I)\to 0.$$

APPLICATION 2. One can think of the Lebesgue density theorem as a special case of the theorem about differentiating an indefinite integral. Thus if

$$c_E(x)=\begin{cases}1 & \text{for} \quad x\in E,\\ 0 & \text{for} \quad x\notin E,\end{cases}$$

and

$$F(x)=\int^x c_E(t)\,dt;$$

we know that $F'(x)$ exists and equals $c_E(x)$ for all values of x except for an exceptional set of zero measure. Theorem 2 in 1 dimension deals with the rate at which $(F(x+h)-F(x))/h$ approaches its limit when $h\to 0$. Clearly a similar method can be applied to any absolutely continuous function $F(x)$. This yields

THEOREM 3. *Suppose that $F(x)$ is an absolutely continuous function of the real variable x, with $F'(x)=f(x)$ for $x\in E$; then there is a monotonic increasing function $\psi(t)$, defined for $t>0$, with $\lim\limits_{t\to 0+}\psi(t)=0$, such that*

$$\frac{1}{\psi(|h|)}\left|\frac{F(x+h)-F(x)}{h}-f(x)\right|\to 0 \quad \text{as} \quad h\to 0$$

for any $x\in S\subset E$ where $|E-S|=0$.

Proof. Put

$$q_n(x)=\sup_{0<|h|\leqslant 1/n}\left|\frac{F(x+h)-F(x)}{h}-f(x)\right|.$$

Then $q_n(x)$ is measurable because the same supremum is obtained by restricting h to rational values. Further $q_n(x)\to 0$ for $x\in E$, and E is a measurable set. We can now proceed exactly as in theorem 2 to obtain the function $\psi(t)$, and the set S.

In an obvious way theorem 3 extends the idea that if $f(x)$ is continuous as a function of x in $a\leqslant x\leqslant b$, and $F(x)=\int_a^x f(t)\,dt$, then there is some order function $\psi(h)$, again monotonic with $\lim\limits_{t\to 0+}\psi(t)=0$, such that

$$\left|\frac{F(x+h)-F(x)}{h}-f(x)\right|<\psi(h)$$

for all x, h such that $a \leqslant x < x+h \leqslant b$. This last result is quite elementary, and an immediate corollary of the uniform continuity of $f(x)$.

Remark. When first proving theorems 2 and 3, I based them on a version of theorem 1 for a continuous parameter family of functions

$$\{f_h(x)\} \quad \text{with} \quad f_h(x) \to f(x) \quad \text{as} \quad h \to 0$$

for $x \in E$. Unfortunately this continuous parameter version of theorem 1 is untrue since it is not valid even for the standard version of Egoroff's theorem. A simple counterexample has recently been given by Weston [5]; this shows how the theorem can break down in this case.

References

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On the existence of conjugate functions of higher order*

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1. In this paper we investigate properties of certain extensions of the notion of conjugate function. Before we formulate these extensions we recall some known facts about generalized derivatives. Proofs and bibliographic references can be found e. g. in [5], vol. II, Chapter XI, §§ 1-5 (see the References at the end of the paper).

A function $f(x)$ defined in the neighbourhood of a point x_0 is said to have at x_0 a generalized derivative of order r $(r = 1, 2, \ldots)$ if

$$f(x_0+t) = a_0 + a_1 t + \ldots + \frac{a_r}{r!} t^r + o(t^r)$$

for $t \to 0$, the a_j denoting constants. The number a_r is called the rth *generalized derivative* of f at x_0 and will be denoted by $f_{(r)}(x_0)$. Clearly, if an ordinary derivative $f^{(r)}(x_0)$ exists so does $f_{(r)}(x_0)$ and both have the same value $(r = 1, 2, \ldots)$; the existence of $f_{(r)}(x_0)$ implies that of $f_{(r-1)}(x_0)$; finally, if $f_{(r)}(x_0)$ exists and F is the indefinite integral of f, then $F_{(r+1)}(x_0)$ exists and equals $f_{(r)}(x_0)$ (F is defined near x_0 since the hypothesis implies that f is bounded near x_0).

Suppose that f, defined in the neighborhood of x_0, has a generalized derivative $f_{(r-1)}(x_0)$. Writing a_j for $f_{(j)}(x_0)$ we define the function $\delta_r(x_0, t)$ by the formula

$$\text{(1.1)} \quad \frac{1}{2}[f(x_0+t)+f(x_0-t)] = a_0 + \frac{a_2}{2!}t^2 + \ldots + \frac{a_{r-1}}{(r-1)!}t^{r-1} + \frac{1}{2}\delta_r(x_0, t)\frac{t^r}{r!}$$

if r is odd, and by

$$\text{(1.2)} \quad \frac{1}{2}[f(x_0+t)-f(x_0-t)] = a_1 t + \frac{a_3}{3!}t^3 + \ldots + \frac{a_{r-1}}{(r-1)!}t^{r-1} + \frac{1}{2}\delta_r(x_0, t)\frac{t^r}{r!}$$

if r is even. If

$$\delta_r(x_0) = \lim_{t\to+0} \delta_r(x_0, t)$$

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