Some decomposition theorems for certain invariant continua and their minimal sets

by

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1.1. Let $\mathfrak{T}$ denote throughout this paper a $(1,1)$-continuous orientation preserving transformation of the $x = (\xi, \eta)$ plane into itself which leaves a certain bounded continuum I invariant. We suppose throughout that $\mathfrak{C}(I)$, the complement of I with respect to the plane, is a single simply connected domain if the point at infinity is adjoined to the plane, and we say that a continuum satisfying this condition is one which does not separate the plane. The properties of such an I with respect to its fixed and periodic points have been discussed elsewhere (1), in particular properties of the points of I considered as belonging to the prime ends of $\mathfrak{C}(I)$ and the rotation number of $\mathfrak{T}$ with respect to $\mathfrak{C}(I)$. The following theorem which we shall use repeatedly was proved by Cartwright and Littlewood (see [5] and [8]):

THEOREM A. *If $\mathfrak{T}$ leaves a bounded continuum I invariant, and if I does not separate the plane, then I contains a fixed point.*

This result is more general than the Brouwer fixed point theorem in one respect; for I need not be locally connected, but the conditions on $\mathfrak{T}$ are more restrictive.

1.2. A compact set M such that $\mathfrak{T}(M) = M$ and M is irreducible with respect to these properties is a *minimal set for* $\mathfrak{T}$, or simply a *minimal set*. If a minimal set is locally connected, it is a *continuous* minimal set, otherwise it is a *discontinuous* minimal set. A fixed point, and a set of periodic points which permute among themselves are finite continuous minimal sets. All other minimal sets are infinite sets. Birkhoff (see [1], p. 104, 105) has shown that an infinite continuous minimal set consists of "a finite number of closed two-sided curves all outside of one another" which undergo a permutation under $\mathfrak{T}$, each curve is invariant under $\mathfrak{T}^N$ for some N, and its interior domain contains a point fixed under $\mathfrak{T}^N$. Birkhoff's definition of a continuous minimal set is somewhat ambiguous,

(1) See [5], where further references will be found.

and it is not all together clear what he means by a curve. We shall in fact show (see theorem 3) that his main results hold for any minimal set consisting of N continua which permute among themselves under $\mathfrak{T}$. These are included among the *partially discontinuous minimal sets*. Minimal sets which do not contain any non-degenerate continuum are *totally discontinuous minimal sets*. (See Addendum.) We shall make repeated use of the well-known fact (see [7], p. 14), that each point x of an *infinite* minimal set M is a limit point of $\mathfrak{T}^n(y)$ as $n\to\infty$ and as $n\to-\infty$ for every point y in M.

1.3. The main object of this paper is to study the structure of I in relation to its minimal sets by considering continua in I containing a given minimal set M, and invariant under $\mathfrak{T}$ or under $\mathfrak{T}^N$ for some N. Keeping in mind the results mentioned in § 1.2 about periodic points in the interior domains of continuous minimal sets, we shall investigate whether any corresponding result holds for suitable continua invariant under $\mathfrak{T}^N$ and containing a discontinuous minimal set. On the other hand the simplest type of continuum I such that $I=\mathfrak{F}(I)$, the frontier of I, seems to be a star (see [10]) with the minimal set forming the end points of the rays, but an I with a branching structure (²) is also well known. We shall be concerned with the problem whether subcontinua of I can be found invariant under $\mathfrak{T}^N$ for some N which have properties similar to these simple structures, or whether for certain I any set of continua invariant under $\mathfrak{T}^N$ containing the given minimal set are necessarily of an extremely complicated type.

1.4. Throughout the paper we shall denote by M a minimal set in I. For any bounded continuum C, the complement $\mathfrak{C}(C)$ consists of an unbounded simply connected domain which we shall call $E(C)$ and possibly one or more bounded simply connected domains, the union of which we call $B(C)$. Then

$$I(C)=C\cup B(C)$$

is a continuum which does not separate the plane, and when C_N is a continuum such that $\mathfrak{T}^N(C_N)=C_N$, it follows from theorem A that there is a point p_N in $I(C_N)$ fixed for $\mathfrak{T}^N$. We first establish some general results about any continuum $C^a(M)$ in I containing M such that $\mathfrak{T}\big(C^a(M)\big)=C^a(M)$, and about sets of continua $\mathfrak{T}^\nu(C^a_N)$, $\nu=0,1,2,\ldots,N-1$, such that

$$\mathfrak{T}^N(C^a_N)=C^a_N\,, \tag{1}$$

$$\mathfrak{T}^\nu(C^a_N)\cap C^a_N=\emptyset\,,\qquad \nu=1,2,\ldots,N-1\,, \tag{2}$$

(²) See [10] and also § 2.4.

and about the periodic points in $I(C^a_N)$ when C^a_N satisfies (1) and (2). In particular we discuss the case in which there is a sequence of values of N and continua C^a_N for which (1) and (2) hold.

The idea of a component of a continuous minimal set is generalised by considering an irreducible continuum $C^a\big(x,\mathfrak{T}^N(x)\big)$ containing a point x of M and $\mathfrak{T}^N(x)$, and obtaining from it a continuum $C^a_N(x)$ satisfying (1) and containing a subset of M minimal for I^N. The example of a star or branching type of continuum suggests the use of an irreducible continuum $C^a(x,p_N)$ containing a point x of M and some point p_N in I with period N to form an apparently star-shaped continuum

$$C^a_N(x:p_N)=\overline{\bigcup_{-\infty<n<\infty}\mathfrak{T}^{nN}\big(C^a(x,p_N)\big)} \tag{3}$$

invariant for $\mathfrak{T}^N$ and containing a subset of M minimal for $\mathfrak{T}^N$. The relations between the various continua containing M obtainable by these methods are discussed, and it is shown that if $I=\mathfrak{F}(I)$ the continua $C^a_N(x)$ and $C^a_N(x:p^N)$ obtained by all choices of $C^a\big(x,\mathfrak{T}^N(x)\big)$, $C^a(x,p_N)$ respectively are the same.

It is not necessarily the case that a suitable choice of N and $C^a\big(x,\mathfrak{T}^N(x)\big)$ or $C^a(x,p_N)$ will lead to a continuum C^a_N for which (2) holds (unless of course x has least period N and $C^a\big(x,\mathfrak{T}^N(x)\big)=x$), nor does the continuum (3) necessarily have any of the features usually associated with a star. As far as we know at present it is possible for the irreducible continuum $C^a\big(x,\mathfrak{T}^N(x)\big)$ or the irreducible continuum $C^a(x,p_N)$ to contain all the other points of M, and either of them may be an indecomposable continuum. The remainder of the paper is concerned with these and other pathological cases, and in particular with may occur when $I=\mathfrak{F}(I)$.

1.5. The original incentive in writing this paper was to investigate whether a certain type of second order non-linear differential equation with positive damping can have uniformly almost periodic solutions. The most usual type of almost periodic solution of second order differential equations is the type called biperiodic (³) which corresponds to a continuous minimal set consisting of a single closed Jordan curve. The hypothesis of positive damping corresponds to the hypothesis $I=\mathfrak{F}(I)$ which excludes such minimal sets.

In § 2.4 we show how to construct a continuum I with an infinite minimal set which is the limit set of sets of periodic points with arbitrarily

(³) See [3], p. 232 (1). There is a misprint, and it should be $x(\lambda t,\lambda t/\varrho)$. See also [1], p. 119.

large periods. This suggests the possibility of solutions which are limit periodic functions, but so far as I know no such solutions are known.

2.1. In the following result there is no restriction on the type of the minimal set M, and the only restrictions on I and $\mathfrak{T}$ are those laid down in § 1.1. For any set S we denote the frontier by $\mathfrak{F}(S)$ and the interior by $\mathfrak{I}(S)$.

THEOREM 1. *Suppose that $C^a(M)$ is a continuum in I containing M such that $\mathfrak{T}(C^a(M)) = C^a(M)$. Then* (i) $I(C^a(M))$ *contains a fixed point,* (ii) *either*

$$(1) \qquad M \cap \mathfrak{I}(C^a(M)) = M\,,$$

or

$$(2) \qquad M \cap \mathfrak{F}(E(C^a(M))) = M\,,$$

or

$$(3) \qquad M \cap \mathfrak{F}(B(C^a(M))) = M\,,$$

or both (2) *and* (3) *hold.*

COROLLARY 1. *If B_0 is a component of $B(C^a(M))$ such that $\mathfrak{T}(B_0) = B_0$, then either $M \cap \mathfrak{F}(B_0) = \emptyset$, or*

$$(4) \qquad M \cap \mathfrak{F}(B_0) = M\,.$$

In particular if B_0 contains a fixed point and $M \cap \mathfrak{F}(B_0) \neq \emptyset$ then (4) *holds.*

COROLLARY 2. *If $I = \mathfrak{F}(I)$, then* (2) *holds but not* (3), *and there is one and only one continuum $C^a(M)$ irreducible with respect to the properties stated in the hypothesis of the theorem.*

We shall see later that each of the three possibilities (1), (2), (3) can occur even when $C^a(M)$ is irreducible.

Since $\mathfrak{T}(C^a(M)) = C^a(M)$, the sets $I(C^a(M))$, $E(C^a(M))$, $B(C^a(M))$, and the frontier of each of these sets is invariant, the existence of the fixed point in $I(C^a(M))$ follows now from theorem A, and each of the sets

$$(5) \qquad M \cap \mathfrak{I}(C^a(M))\,, \quad M \cap \mathfrak{F}(E(C^a(M)))\,, \quad M \cap \mathfrak{F}(B(C^a(M)))$$

are either void or the set M itself. For all three are invariant and are contained in M and the last two are obviously closed. Hence if either of the last two is not void it is the set M. For if not M is not minimal. On the other hand, if the last two sets in (5) are both void, then the first is closed, and the same argument applies.

The first part of corollary 1 follows from the same argument. Then since $\mathfrak{T}(B_0)$ is obviously a component of $B(C^a(M))$, if B_0 contains a fixed point, $\mathfrak{T}(B_0) = B_0$, and so the rest follows.

To prove corollary 2 we first observe that since $I = \mathfrak{F}(I)$, both $\mathfrak{I}(C^a(M)) = \emptyset$ and $B(C^a(M)) = \emptyset$ so that (2) must hold. The remaining part of the corollary follows from the following lemma which we shall use repeatedly:

LEMMA 1. *If $I = \mathfrak{F}(I)$, and C_1, C_2 are any two continua in I, then $C_1 \cap C_2$ is a continuum (possibly degenerate) or void.*

If $C_1 \cap C_2 \neq \emptyset$ and is not connected, there exist two points in the plane separated by $C_1 \cup C_2$. But $C_1 \cup C_2 \subset I$, and so I has interior points which is impossible. Hence the lemma holds.

If $C^a(M)$, $C^\beta(M)$ are two irreducible continua satisfying the hypotheses of the theorem, $C^a(M) \cap C^\beta(M) \supset M \neq \emptyset$. Since by lemma $C^a(M) \cap C^\beta(M)$ is a continuum and invariant, $C^a(M) = C^\beta(M)$. For if not they are not irreducible.

2.2. We need certain general results about sets of N continua which are invariant under $\mathfrak{T}^N$ and permute among themselves under $\mathfrak{T}$. Each continuum must in fact contain a set minimal for $\mathfrak{T}^N$, but we make no assumption about this at present, and the minimal set may consist of a single point with period N.

THEOREM 2. *Suppose that N is an integer greater than 1 and that C_N^a is a continuum in I such that*

$$(1) \qquad \mathfrak{T}^N(C_N^a) = C_N^a\,,$$

$$(2) \qquad C_N^a \cap \mathfrak{T}^\nu(C_N^a) = \emptyset\,, \quad \nu = 1, 2, \ldots, N-1\,.$$

Then (i) $\mathfrak{T}^\nu(C_N^a) \subset E(C_N^a)$, $\nu = 1, 2, \ldots, N-1$, (ii) $I(C_N^a) \cap \mathfrak{T}^\nu(I(C_N^a)) = \emptyset$, $\nu = 1, 2, \ldots, N-1$, (iii) *$I(C_N^a)$ contains a point p_N with least period N,* (iv) *all periodic points in $I(C_N^a)$ have periods which are multiples of N,* (v) *if M is any minimal set in I such that $M_N = C_N^a \cap M \neq \emptyset$ then M_N is minimal for $\mathfrak{T}^N$, but not for $\mathfrak{T}^\nu$, $\nu = 1, 2, \ldots, N-1$.*

Since (2) holds, for each ν such that $1 \leqslant \nu \leqslant N-1$ we have either $\mathfrak{T}^\nu(C_N^a) \subset E(C_N^a)$ or $\mathfrak{T}^\nu(C_N^a) \subset B(C_N^a)$. Suppose that $\mathfrak{T}^\nu(C_N^a) \subset B(C_N^a)$ for some ν such that $1 \leqslant \nu \leqslant N-1$. Then since $\mathfrak{T}$ is $(1,1)$ and continuous

$$\mathfrak{T}^{2\nu}(C_N^a) \subset B(\mathfrak{T}^\nu(C_N^a)) \subset B(C_N^a)\,,$$

and, repeating the argument, we have

$$C_N^a = \mathfrak{T}^{N\nu}(C_N^a) \subset B(C_N^a)\,,$$

which is impossible. Hence (i) holds, and (ii) follows from it.

By theorem A with $\mathfrak{T}^N$ in place of $\mathfrak{T}$ the set $I(C_N^a)$ contains a point p_N with period N and $\mathfrak{T}^\nu(p_N) \subset \mathfrak{T}^\nu(I(C_N^a))$. Hence, by (ii), N is the least period of p_N. If q_N is any periodic point in $I(C_N^\alpha)$, then

$$\mathfrak{T}^{nN+\nu}(q_N) \subset \mathfrak{T}^{nN+\nu}\big(I(C_N^\alpha)\big) = \mathfrak{T}^{\nu}\big(I(C_N^\alpha)\big).$$

It follows from (ii) that $\mathfrak{T}^{nN+\nu}(q_N) \neq q_N$ for $\nu = 1, 2, \ldots, N-1$, and so the period of q_N is a multiple of N.

Finally if M is a minimal set in I such that $M_N = M \cap C_N^\alpha \neq \emptyset$, then M_N is closed and, by (1), $\mathfrak{T}^N(M_N) = M_N$, but, by (2), $\mathfrak{T}^\nu(M_N) \neq M_N$ for $\nu = 1, 2, \ldots, N-1$. Suppose that a proper subset M_N^* of M_N is minimal for $\mathfrak{T}^N$, then

$$M^* = \bigcup_{\nu=0}^{N-1} \mathfrak{T}^\nu(M_N^*)$$

is a proper subset of M minimal for $\mathfrak{T}$ which is impossible and so we have the result required.

2.3. We consider next a minimal set consisting of N disjunct, non-degenerate continua, that is to say an infinite continuous minimal set or a partially discontinuous minimal set. If M_N is one of the N continua of such an M, then

$$M = \bigcup_{\nu=0}^{N-1} \mathfrak{T}^\nu(M_N), \tag{1}$$

$$\mathfrak{T}^N(M_N) = M_N, \tag{2}$$

$$\mathfrak{T}^\nu(M_N) \cap M_N = \emptyset, \quad \nu = 1, 2, \ldots, N-1. \tag{3}$$

THEOREM 3. *Suppose that M satisfies* (1), (2) *and* (3) *where M_N is a continuum. Then* (i) $\mathfrak{T}^\nu(M_N) \subset E(M_N)$, $\nu = 1, 2, \ldots, N-1$, (ii) $B(M_N)$ *contains a point p_N with least period N*, (iii) *all the periodic points in $B(M_N)$ have periods which are multiples of N*, (iv) $M_N = \mathfrak{F}\big(E(M_N)\big) = \mathfrak{F}\big(B(M_N)\big)$, (v) *if $B_0(M_N)$ is a component of $B(M_N)$ such that $\mathfrak{T}^N\big(B_0(M_N)\big) = B_0(M_N)$, then $M_N = \mathfrak{F}\big(B_0(M_N)\big)$, and in particular this holds for the component containing the point p_N with least period N.*

COROLLARY 1. *If M is locally connected, then $\mathfrak{T}^\nu(M_N)$, $\nu = 0, 1, 2, \ldots, N-1$, are closed Jordan curves.*

COROLLARY 2. *If $B(M_N)$ has more than one component, then M_N is an indecomposable continuum or the sum of two indecomposable continua.*

We may obviously apply theorem 2 with $M_N = C_N^\alpha$, and we obtain parts (i), (ii), (iii) at once. For the periodic points in $I(M_N)$ cannot lie on M_N but must lie in $B(M_N)$. Part (iv) follows from theorem 1 with $\mathfrak{T}^N$ in place of $\mathfrak{T}$. For $M_N \cap \mathfrak{F}\big(E(M_N)\big) \neq \emptyset$, $M_N \cap \mathfrak{F}\big(B(M_N)\big) \neq \emptyset$. Part (v) follows from corollary 1 of theorem 1.

Corollary 1 is the theorem of Birkhoff to which reference was made in § 1.2. It follows from (ii). For this is equivalent to saying that M_N is an irreducible cutting (see [9], p. 175, theorem 1, and p. 403, theorem 7) of the plane between p_N and the point at infinity.

To prove corollary 2 we first observe that if M_N is an irreducible cutting (see [9], p. 404, theorem 10) of the plane between each pair of three points a_0, a_1, a_2, then it is an indecomposable continuum or the sum of two indecomposable continua. Let $B_0(M_N)$ be the component of $B(M_N)$ containing the point p_N with least period N, and let $B_1(M_N)$ denote the union of the remaining components of $B(M_N)$. Since $B(M_N)$ and $B_0(M_N)$ are invariant under $\mathfrak{T}^N$, so is $B_1(M_N)$, and also $\mathfrak{F}\big(B_1(M_N)\big)$. Obviously $M_N \cap \mathfrak{F}\big(B_1(M_N)\big) \neq \emptyset$, and so by the usual arguments

$$M_N = M_N \cap \mathfrak{F}\big(B_1(M_N)\big).$$

Hence M_N is an irreducible cutting between each pair of three points, the point at infinity, p_N, and a point of $B_1(M_N)$, and so we have the result.

2.4. Before proceeding to the consideration of minimal sets in general we may observe that theorem 2 can be applied with $\mathfrak{T}^N$ in place of $\mathfrak{T}$ and $I(C_N^\alpha)$ in place of I, and some continuum $C_{N'}^\beta \subset I(C_N^\alpha)$ in place of C_N^α, provided that N' is a multiple of N and C_N^β satisfies hypotheses corresponding to 2.2 (1) and 2.2 (2). Repeated applications of parts (ii)-(v) of theorem 2 give

THEOREM 4. *Suppose that $1 = N_0 < N_1 < N_2 < \ldots$ is a finite or infinite sequence of integers such that N_i/N_{i-1} is an integer for $i = 1, 2, \ldots$ Suppose further that $C_{N_i}^{\alpha_i}$, $i = 0, 1, 2, \ldots$, is a continuum such that*

$$I \supset C_{N_0}^{\alpha_0} \supset C_{N_1}^{\alpha_1} \supset \ldots \supset C_{N_i}^{\alpha_i} \supset \ldots, \tag{1}$$

$$\mathfrak{T}^{N_i}(C_{N_i}^{\alpha_i}) = C_{N_i}^{\alpha_i}, \quad i = 0, 1, 2, \ldots, \tag{2}$$

$$\mathfrak{T}^{\nu N_{i-1}}(C_{N_i}^{\alpha_i}) \cap C_{N_i}^{\alpha_i} = \emptyset, \quad \nu = 1, 2, \ldots, (N_i/N_{i-1})-1, \quad i = 1, 2, \ldots \tag{3}$$

Then (i) *we have*

$$\mathfrak{T}^{\nu N_{i-1}}\big(I(C_{N_i}^{\alpha_i})\big) \cap I(C_{N_i}^{\alpha_i}) = \emptyset, \quad \nu = 1, 2, \ldots, (N_i/N_{i-1})-1, \quad i = 1, 2, \ldots$$

(ii) $I(C_{N_i}^{\alpha_i})$ *contains a point p_{N_i} with least period N_i*, (iii) *all the periodic points in $I(C_{N_i}^{\alpha_i})$ have least periods, which are multiples of N_i*, (iv) *if $M_{N_i} = M \cap C_{N_i}^{\alpha_i} \neq \emptyset$, then M_{N_i} is minimal for $\mathfrak{T}^{N_i}$, but not for $\mathfrak{T}^\nu$, $\nu = 1, 2, \ldots, N_i-1$.*

COROLLARY. *If the sequence N_i, $i = 1, 2, \ldots$, in theorem 4 is infinite, then I contains periodic points with arbitrarily large periods.*

Levinson [10] has given an example of a set I such that $I = \mathfrak{F}(I)$ for which the hypotheses of theorem 2 are satisfied with $N = 3$ and $C_3^a = C_3$, a segment which rotates about its middle point under $\mathfrak{T}^3$, so that the end points of the segments $\mathfrak{T}^\nu(C_3)$, $\nu = 0, 1, 2$, form a set of 6 points $\mathfrak{T}^\nu(p_6)$, $\nu = 0, 1, 2, \ldots, 5$, with period 6. There is a stable fixed point, p_1, and three spirals from p_1, winding round $\mathfrak{T}^\nu(C_3)$, $\nu = 0, 1, 2$, respectively. Each spiral contains a point of period 3 which is a saddle point (or col) for $\mathfrak{T}^3$, and the points with period 6 are stable fixed points for $\mathfrak{T}^6$. This I also satisfies the hypotheses of theorem 4 with $N_0 = 1$, $N_1 = 3$, $N_2 = 6$, $I = C^{a_0}$, $C_3^{a_1} = C_3$, and $C_6^{a_2} = p_6$.

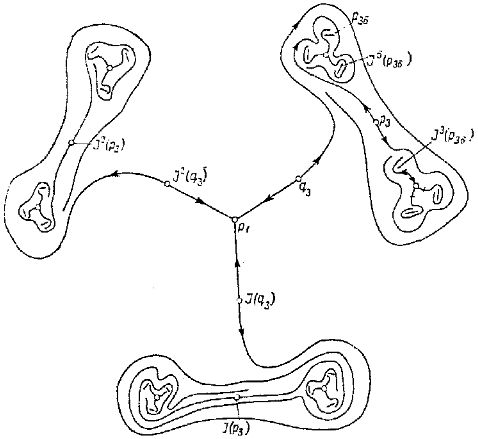

Fig. 1

It is easy to see that this figure can be modified by replacing each of the points $\mathfrak{T}^\nu(p_6)$, $\nu = 0, 1, 2, \ldots, 5$, by a small continuum $\mathfrak{T}^\nu(C_6)$ of the same form as I itself. The end of the segment C_3 near p_6 is then drawn out into a spiral winding round C_6, and which goes into similar spirals for the other continua $\mathfrak{T}^\nu(C_6)$, $\nu = 1, 2, \ldots, 5$. The spiral from p_1 is modified to wind round the new C_3 which includes the two continua C_6, $\mathfrak{T}^3(C_6)$ and spirals from a point p_3 of period 3 winding round each of them. The continua $\mathfrak{T}^\nu(C_6)$, $\nu = 0, 1, 2, \ldots, 5$, contain a set of points $\mathfrak{T}^\nu(p_{36})$, $\nu = 0, 1, 2, \ldots, 35$, with period 36. The point p_{36} can be replaced by a small continuum C_{36} of the same form as I itself, provided that the segment C_{18} in C_6 is drawn out into a spiral round C_{36} just as in the case of the first C_3, and corresponding other modifications are made (see Fig. 1).

This process can be repeated indefinitely, and so we have

THEOREM 5. *There exist $\mathfrak{T}$, I, such that $I = \mathfrak{F}(I)$ and theorem 4 holds for an infinite sequence of integers $1, 3, 6, 18, 36, 6^n, 3.6^n, \ldots$, I contains periodic points with arbitrarily large least periods and an infinite minimal set M which is the limit set of periodic points.*

It follows from theorem 4, (ii), that there exist points in I with arbitrarily large least periods, and the limit set of these points is invariant under $\mathfrak{T}$. If the continuum C_{N_i} replacing a point p_{N_i} is made sufficiently small, the limit set will not contain any periodic points. Hence it contains at least one infinite minimal set M.

2.5. We now seek to obtain a set of N continua containing M and invariant under $\mathfrak{T}^N$ to which theorems 1 and 2 can be applied so as to establish the existence of periodic points associated with a given minimal set. When M contains no continua, we have to use some continuum in I to connect the points of M, and the first method is based on a continuum in I containing two points x and $\mathfrak{T}^N(x)$ of M and irreducible with respect to this property. The set M may contain a continuum, but the result remains valid for all such continua if they are not contained in M.

THEOREM 6. *Let $x \in M$; let N be a positive integer, and let $C^a(x, \mathfrak{T}^N(x))$ be a continuum in I containing x and $\mathfrak{T}^N(x)$ and irreducible with respect to these properties. Then*

$$C_N^a(x) = \overline{\bigcup_{-\infty<n<\infty} \mathfrak{T}^{nN}\big(C^a(x, \mathfrak{T}^N(x))\big)} \tag{1}$$

is a continuum in I such that (i) $\mathfrak{T}^N\big(C_N^a(x)\big) = C_N^a(x)$, (ii) $M_N(x) = M \cap C_N^a(x)$ *is minimal for $\mathfrak{T}^N$, and*

$$M = \bigcup_{\nu=0}^{N-1} \mathfrak{T}^\nu\big(M_N(x)\big), \tag{2}$$

(iii) *$I\big(C_N^a(x)\big)$ contains a point $p_N = p_N(x, a)$ with period N.*

COROLLARY 1. *Let $A_N^a(x)$, $\Omega_N^a(x)$ denote the limit sets of $\mathfrak{T}^{nN}\big(C^a(x, \mathfrak{T}^N(x))\big)$ as $n \to -\infty$ and as $n \to \infty$ respectively. Then $A_N^a(x)$ and $\Omega_N^a(x)$ are continua, and* (i), (ii) *and* (iii) *of theorem 6 hold with $A_N^a(x)$, and $\Omega_N^a(x)$ in place of $C_N^a(x)$.*

COROLLARY 2. *If $I = \mathfrak{F}(I)$, then $I\big(C_N^a(x)\big) = C_N^a(x)$ and p_N lies on $C_N^a(x)$.*

COROLLARY 3. *If $I = \mathfrak{F}(I)$, then there is one and only one continuum $C^a(x, \mathfrak{T}^N(x))$ in I containing x and $\mathfrak{T}^N(x)$ and irreducible with respect to these properties, and $C_N^a(x) = C_N^a(y)$ where y is any point of $M_N(x)$. In particular $C_1^a(x)$ is independent of the particular point x, and $A_N^a(x) = \Omega_N^a(x) = C_N^a(x)$.*

It should be observed that $C^a(x, \mathfrak{T}^N(x))$ certainly exists, and so theorem 6 establishes the existence of a continuum $C^a_N(x)$ satisfying 2.2 (1) and results corresponding to (iii) and (v) of theorem 2, but, since $C^a_N(x)$ does not necessarily satisfy 2.2 (2) the period of p_N may be less than N and M_N may be minimal for $\mathfrak{T}^\nu$ with $1 \leqslant \nu < N$. In fact the possibility that

$$C^a(x, \mathfrak{T}^N(x)) = C^a_N(x) = I$$

is not ruled out. It should also be observed that if M is a set of points of period N, $C^a(x, \mathfrak{T}^N(x)) = C^a_N(x) = x$.

To prove the theorem we observe that $C^a(x, \mathfrak{T}^N(x))$ and therefore $C^a_N(x)$ lies in I, and $C^a_N(x)$ is closed by definition. Since

$$\mathfrak{T}^{nN}\big(C^a(x, \mathfrak{T}^N(x))\big) \frown \mathfrak{T}^{(n+1)N}\big(C^a(x, \mathfrak{T}^N(x))\big) \supset \mathfrak{T}^{(n+1)N}(x)$$

$C^a_N(x)$ is connected, and therefore a continuum. Obviously $\mathfrak{T}^N(C^a_N(x)) = C^a_N(x)$, and (ii) follow from this. Part (iii) is obtained by applying theorem A with $\mathfrak{T}^N$ in place of $\mathfrak{T}$.

In corollary 1 we suppose that $C^a(x, \mathfrak{T}^N(x))$ is a non degenerate continuum, and then it is easy to show that $A^a_N(x)$, $\Omega^a_N(x)$ being limit sets of continua are themselves closed and connected. They are obviously invariant under $\mathfrak{T}^N$, and since $M_N(x)$ is minimal for $\mathfrak{T}^N$ every point y of $M_N(x)$ is a limit point of as $\mathfrak{T}^{nN}(x)$ and as $n \to \pm\infty$. Hence $M_N(x) \subset M \frown A^a_N(x)$ and $M_N(x) \subset M \frown \Omega^a_N(x)$, and so (ii) holds. Part (iii) follows as usual from part (i) and theorem A.

Corollary 2 is obvious, and corollary 3 follows from lemma 1. For if $C^a(x, \mathfrak{T}^N(x))$ and $C^\beta(x, \mathfrak{T}^N(x))$ are irreducible continua in I containing x and $\mathfrak{T}^N(x)$ so is

$$C^a(x, \mathfrak{T}^N(x)) \frown C^\beta(x, \mathfrak{T}^N(x)),$$

and since both continua are irreducible they must coincide. The rest of corollary 3 follows from this, and the fact that y is a limit point of $\mathfrak{T}^{nN}(x)$ as $n \to \pm\infty$.

2.6. Since there is necessarily a point p_N with period N associated with the subset $M_N(x)$ containing x minimal for $\mathfrak{T}^N$, we may use a periodic point to determine the continuum which we require.

THEOREM 7. *Let $x \in M$; let N be a positive integer; let p_N be any point in I with period N, and let $C^a(x, p_N)$ be a continuum in I containing x and p_N and irreducible with respect to these properties. Then*

$$C^a_N(x : p_N) = \overline{\bigcup_{-\infty<n<\infty} \mathfrak{T}^{nN}(C^a_N(x, p_N))} \tag{1}$$

is a continuum in I, and (i), (ii) *and* (iii) *of theorem* 6 *hold with $C^a_N(x : p_N)$ in place of $C^a_N(x)$.*

COROLLARY 1. *If $A^a_N(x : p_N)$, $\Omega^a_N(x : p_N)$ are the limit sets of $\mathfrak{T}^{nN}(C^a(x, p_N))$ as $n \to -\infty$ and as $n \to \infty$ respectively, then* (i), (ii), (iii), *of theorem* 6 *hold with $A^a_N(x : p_N)$ and $\Omega^a_N(x : p_N)$ in place of $C^a_N(x)$.*

COROLLARY 2. *If $I = \mathfrak{F}(I)$, then $I(C^a_N(x : p_N)) = C^a_N(x : p_N)$, and there is only one continuum $C^a(x, p_N)$ in I containing x and p_N and irreducible with respect to these properties, and $C^a_N(x : p_N) = C^a_N(y : p_N)$ where y is any point of $M_N(x)$.*

COROLLARY 3. *If M consists of $N'N$ points, and*

$$C^a(x, p_N) = \mathfrak{T}^{N'N}(C^a(x, p_N)), \tag{2}$$

then

$$C^a_N(x : p_N) = \bigcup_{\nu=0}^{N'-1} \mathfrak{T}^{nN}(C^a(x, p_N)); \tag{3}$$

in particular if M consists of $N'N$ points and $I = \mathfrak{F}(I)$, then (3) *holds.*

COROLLARY 4. *If $C^a(x, p_N)$ contains a point y belonging to some other set $M^*_N(y)$ minimal for $\mathfrak{T}^N$, then $M^*_N(y) \subset C^a_N(x : p_N)$.*

The proof of the theorem and corollaries 1 and 2 follows similar lines to that of theorem 6 and its corollaries, except that the connectedness of $C^a_N(x : p_N)$ follows from the fact that all the continua $\mathfrak{T}^{nN}(C^a(x, p_N))$ contain the point p_N.

The first part of corollary 3 is obvious, and the rest follows from the fact that, since $\mathfrak{T}^{N'N}(x) = x$, the continuum $\mathfrak{T}^{N'N}(C^a(x, p_N))$ is a continuum containing x and p_N, and so by lemma 1 it contains $C^a(x, p_N)$. On the other hand, if $C^a(x, p_N)$ is a proper subcontinuum of $\mathfrak{T}^{N'N}(C^a(x, p_N))$, $\mathfrak{T}^{-NN'}(C^a(x, p_N))$ is a proper subcontinuum of $C^a(x, p_N)$, and then $C^a(x, p_N)$ is not irreducible. Hence (2) holds when $I = \mathfrak{F}(I)$.

Corollary 4 follows from the fact that since $M^*_N(y) \frown C^a_N(x : p_N)$ is a closed set invariant for $\mathfrak{T}^N$ contained in $M^*_N(y)$, it is $M^*_N(y)$. For if not $M^*_N(y)$ is not minimal for $\mathfrak{T}^N$.

2.7. The relationship between theorems 6 and 7 is given by the following theorems which are easily verified:

THEOREM 8. *If the hypotheses of theorem* 6 *hold for some $C^a(x, \mathfrak{T}^N(x))$ and $p_N = p_N(x)$ is a point with period N in $I(C^a_N(x))$, then there is a continuum $C^\beta(x, p_N)$, where $\beta = \beta(a)$ in $I(C^a_N(x))$ containing x and p_N and irreducible with respect to these properties, and*

$$C^\beta_N(x : p_N) \subset I(C^a_N(x)),$$

where $C^\beta_N(x : p_N)$ is defined by 2.6 (1) *with β instead of a.*

THEOREM 9. *If the hypotheses of theorem 7 hold for some* $C^{\alpha}(x, p_N)$, *then there is a continuum* $C^{\beta}(x, \mathfrak{T}^N(x))$, *where* $\beta = \beta(a)$ *in* $C_N^{\alpha}(x: p_N)$ *containing* x *and* $\overline{\mathfrak{T}^N(x)}$ *and irreducible with respect to these properties, and*

$$C_N^{\beta}(x) \subset C_N^{\alpha}(x: p_N)\,,$$

where $C_N^{\beta}(x)$ *is defined by* 2.5 (1) *with* β *instead of* a.

COROLLARY. *If* $I = \mathfrak{F}(I)$, $C_N^{\alpha}(x) = C_N^{\beta}(x: p_N)$, *where* p_N *is any point in* $C_N(x)$ *with period* N, *for all* a *and* β.

2.8. The question remains whether these results can be improved when I contains interior points. For instance if $C^{\alpha}(M)$ in theorem 1 is the continuum $A_1^{\alpha}(x)$ of $\Omega_1^{\alpha}(x)$ of theorem 7 Corollary 1 can 2.1 (1) hold, and can 2.1 (3) hold without 2.1 (2)? The answers to these questions are contained in the following theorems:

THEOREM 10. *There exist* $\mathfrak{T}$, I *and* M *such that for every* $C^{\alpha}(M)$ *satisfying the hypotheses of theorem* 1 2.1 (1) *holds.*

THEOREM 11. *There exist* $\mathfrak{T}$, I *and* M *such that for every* $C^{\alpha}(M)$ *satisfying the hypotheses of theorem* 1 2.1 (3) *holds and not* 2.1 (2).

Consider first a $(1, 1)$ continuous transformation of the unit disc which is a rotation about the origin through an irrational multiple of π. For every continuum $C^{\alpha}(x, 0)$ joining the origin to a point x on the circumference the limit set of $\mathfrak{T}^n(C^{\alpha}(x, 0))$ as $n \to -\infty$ and as $n \to \infty$ will be the whole unit disc. For the limit set of any point at a distance r from the origin is the circle $\xi^2+\eta^2 = r^2$ and the continuum $C^{\alpha}(x, 0)$ meets every circle $\xi^2+\eta^2 = r^2$ for which $0 < r \leqslant 1$ in at least one point.

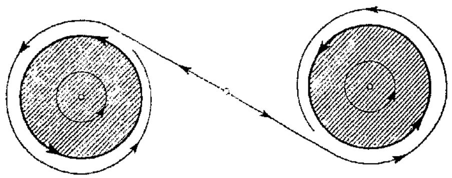

Fig. 2

Now consider two discs of unit radius with centres $(\pm 2, 0)$, and a spiral from the origin winding round each counterclockwise (see Fig. 2). We may suppose that $\mathfrak{T}$ leaves the origin fixed, and takes each circle with its spiral into the other so that the invariant set I consists of the two discs and their spirals. We may suppose that $\mathfrak{T}$ moves all points

of the plane towards I. The essential feature is that $\mathfrak{T}^2$ rotates each disc counterclockwise about its centre through an irrational multiple of π; the points on the spirals in virtue of the continuity of $\mathfrak{T}^2$ must move along the spirals, either towards or away from the disc. We suppose that they move towards the discs.

If M is the pair of points $(\pm 2, 0)$, any continuum in I joining them must contain a continuum joining each point to the circumference of its disc, and as we have seen the limit set of each of these is the whole of the corresponding disc. Hence if $\mathfrak{T}(C^{\alpha}(M)) = C^{\alpha}(M)$, $C^{\alpha}(M)$ contains both discs and $M \subset \mathfrak{J}(C^{\alpha}(M))$ which proves theorem 10.

For theorem 9 we use the same $\mathfrak{T}$ and I and take M to be the two circles with centres $(\pm 2, 0)$ and radii equal to $\frac{1}{2}$. Any continuum $C^{\alpha}(M)$ such that $\mathfrak{T}(C^{\alpha}(M)) = C^{\alpha}(M)$ must, as before, contain the annuli

$$\tfrac{1}{4} \leqslant (\xi+2)^2+\eta^2 \leqslant 1\,,$$

and these with the spirals satisfy the conditions of theorem 1. In this case $B(C^{\alpha}(M))$ consists of the discs $(\xi\pm 2)^2+\eta^2 < \frac{1}{4}$, and 2.1 (3) holds, but not 2.1 (2) (nor of course 2.1 (1) which is incompatible with 2.1 (3)).

2.9. It may be asked whether the continuum $C_N^{\alpha}(x)$ of theorem 6 or the continuum $C_N^{\alpha}(x: p_N)$ of theorem 7 is likely to be the simplest, or whether the limit sets of either will give still simpler continua. If $M_N(x)$ is a continuum, we may take $C^{\alpha}(x, \mathfrak{T}^N(x))$ to be a continuum in $M_N(x)$ and then $C_N^{\alpha}(x) = M_N(x)$. On the other hand, if we are mainly interested in the simplest invariant continuum containing $M_N(x)$ and not separating the plane, the construction of theorem 7 may lead to a simpler result for some types of minimal set, but in virtue of theorems 10 and 11 unless $I = \mathfrak{F}(I)$ it depends on the particular continua $C^{\alpha}(x, \mathfrak{T}^N(x))$, $C^{\alpha}(x, p_N)$ chosen. Some possible effects of various choices are shown in the following examples.

Let $\mathfrak{T}$ be a $(1, 1)$ continuous transformation which leaves the unit circle and the axes invariant, taking the positive ξ and η axes into the negative and vice versa. Then the points $x = (1, 0)$, $\mathfrak{T}(x) = (-1, 0)$, $y = (0, 1)$, $\mathfrak{T}(y) = (0, -1)$ have period 2, and the origin is fixed. We may suppose that all other points move towards the unit circle, and in particular move towards x or $\mathfrak{T}(x)$ under $\mathfrak{T}$ (see Fig. 3). Consider the minimal set x, $\mathfrak{T}(x)$, and theorem 6 with $N = 1$. If $C^{\alpha}(x, \mathfrak{T}(x))$ is the semi-circle $\xi^2+\eta^2 = 1$, $\eta \geqslant 0$, $C_1^{\alpha}(x)$ is $\xi^2+\eta^2 = 1$, and $I(C_1^{\alpha}(x))$ is the unit disc. Further $A_1^{\alpha}(x) = \Omega_1^{\alpha}(x) = C_1^{\alpha}(x)$. If $C^{\alpha}(x, \mathfrak{T}(x))$ is any continuum joining x to $\mathfrak{T}(x)$ lying in $0 < \xi^2+\eta^2 < 1$, $\eta > 0$ except for its end points, then $C_1^{\alpha}(x)$ includes $A_1^{\alpha}(x)$ which is the segment $-1 \leqslant \xi \leqslant 1$, $\eta = 0$ and $\Omega_1^{\alpha}(x)$ which is the unit circle, and $I(C_1^{\alpha}(x))$ is the unit disc, but if $C^{\alpha}(x, \mathfrak{T}(x))$

is the segment $-1 \leqslant \xi \leqslant 1$, $\eta = 0$ then $C^\alpha(x, \mathfrak{T}(x)) = C_1^\alpha(x) = A_1^\alpha(x) = \Omega_1^\alpha(x) = I(C_1^\alpha(x))$.

Using the same example with the origin as p_1 and the segment $0 \leqslant \xi \leqslant 1$, $\eta = 0$ as $C^\alpha(x, 0)$, we obtain the segment $-1 \leqslant \xi \leqslant 1$, $\eta = 0$ as $C_1^\alpha(x:0) = A_1^\alpha(x:0) = \Omega_1^\alpha(x:0)$. If $C^\alpha(x, 0)$ lies in the quadrant $0 < \xi^2 + \eta^2 < 1$, $\xi > 0$, $\eta > 0$, except for the points x and 0 then $A^\alpha(x:0)$ is the segment $-1 \leqslant \xi \leqslant 1$, $\eta = 0$, but $\Omega_1^\alpha(x:0)$ is the segment $\xi = 0$, $-1 \leqslant \eta \leqslant 1$, and the pair of arcs of $\xi^2+\eta^2 = 1$ in the first and third quadrants. Of course by suitable choice of $C^\alpha(x, \mathfrak{T}(x))$, we can obtain $C_1^\alpha(x)$ which is also of this form.

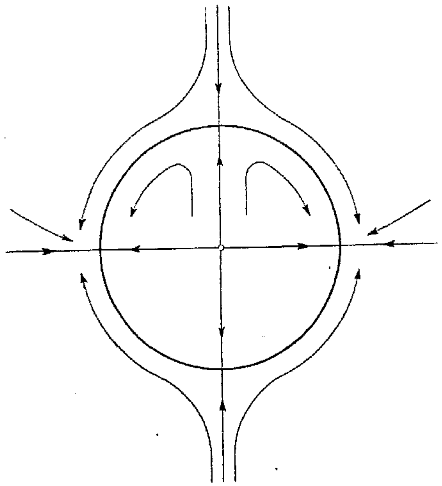

Fig. 3

Other choices of $C^\alpha(x, 0)$ lead to a $C_1^\alpha(x:0)$ consisting of the unit circle together with the segment $-1 \leqslant \xi \leqslant 1$, $\eta = 0$. We may sum up the examples discussed above as follows:

THEOREM 12. *There is a $\mathfrak{T}$ such that I is the unit disc, the pair of points $x = (1, 0)$, $\mathfrak{T}(x) = (-1, 0)$ is a minimal set M, and the origin is a fixed point. Further this $\mathfrak{T}$ may be chosen so that for a suitable $C^\alpha(x, \mathfrak{T}(x))$ the continuum $C_1^\alpha(x)$ defined by* 2.5 (1) *with $N = 1$ is the unit circle or so that $C_1^\alpha(x)$ is the segment $-1 \leqslant \xi \leqslant 1$, $\eta = 0$, and $C^\alpha(x, 0)$ may be chosen so that $C_1^\alpha(x:0)$ defined by* 2.6 (1) *is the segment $-1 \leqslant \xi \leqslant 1$, $\eta = 0$, or so that $C_1^\alpha(x:0)$ is the segment $\xi = 0$, $-1 \leqslant \eta \leqslant 1$, together with the arcs of $\xi^2+\eta^2 = 1$ from x to $y = (0, 1)$ and from $\mathfrak{T}(x)$ to $\mathfrak{T}(y) = (0, -1)$.*

3.1. In connection with theorem 6 it was pointed out that for some $\mathfrak{T}$, I, M, N and a non-degenerate irreducible continuum $C^\alpha(x, \mathfrak{T}^N(x))$ we may (so far as we know) have

$$C^\alpha(x, \mathfrak{T}^N(x)) = C_N^\alpha(x), \tag{1}$$

and similarly in theorem 7 it is possible that for an irreducible $C^\alpha(x, p_N)$

$$C^\alpha(x, p_N) = C_N^\alpha(x:p_N). \tag{2}$$

The consequences of these and similar relations will be discussed in the next group of theorems.

THEOREM 13. *Suppose that the hypotheses of theorem 6 hold, and that $C^\alpha(x, \mathfrak{T}^N(x))$ is non-degenerate and irreducible and satisfies* (1). *Then if $C_N(x)$ is irreducible between x and $\mathfrak{T}^{2N}(x)$, it is an indecomposable continuum.*

Since

$$\mathfrak{T}^N\big(C^\alpha(x, \mathfrak{T}^N(x))\big) = \mathfrak{T}^N\big(C_N^\alpha(x)\big) = C_N^\alpha(x) = C_N^\alpha(x, \mathfrak{T}^N(x)),$$

it is irreducible between $\mathfrak{T}^N(x)$ and $\mathfrak{T}^{2N}(x)$. For if not, it contains a proper sub-continuum $C^\beta(\mathfrak{T}^N(x), \mathfrak{T}^{2N}(x))$ irreducible between $\mathfrak{T}^N(x)$ and $\mathfrak{T}^{2N}(x)$, and $\mathfrak{T}^{-N}\big(C^\beta(\mathfrak{T}^N(x), \mathfrak{T}^{2N}(x))\big)$ is a proper sub-continuum of $C^\alpha(x, \mathfrak{T}^N(x))$ containing x and $\mathfrak{T}^N(x)$ which is impossible because $C^\alpha(x, \mathfrak{T}^N(x))$ is irreducible. Hence if $C_N^\alpha(x)$ is irreducible between x and $\mathfrak{T}^{2N}(x)$, it is irreducible between each pair (see [9], p. 150, theorem 7) of the three points x, $\mathfrak{T}^N(x)$, $\mathfrak{T}^{2N}(x)$, and is therefore an indecomposable continuum.

THEOREM 14. *Suppose that the hypotheses of theorem 7 hold and that $C^\alpha(x, p_N)$ is non-degenerate and irreducible and satisfies* (2). *Then if $C_N^\alpha(x:p_N)$ is irreducible between x and $\mathfrak{T}^N(x)$, it is an indecomposable continuum.*

Since

$$\mathfrak{T}^N\big(C^\alpha(x, p_N)\big) = \mathfrak{T}^N\big(C_N^\alpha(x:p_N)\big) = C_N^\alpha(x:p_N) = C^\alpha(x, p_N)$$

by the usual argument it is irreducible between $\mathfrak{T}^N(x)$ and p_N. If $C^\alpha(x, p_N)$ is irreducible between x and $\mathfrak{T}^N(x)$, it is irreducible between each pair of the three points x, $\mathfrak{T}^N(x)$, p_N, and the result follows as before.

3.2. It seems difficult to construct an example for which the hypotheses of theorem 13 or theorem 14 hold, but the so-called "curves" of Birkhoff (see [2] and [6]) provide an example of a continuum of measure zero which separates the plane into two domains and remains invariant under a certain analytic transformation. These curves have different external and internal rotation numbers, and are indecomposable continua.

So far as I know the minimal sets lying on them have not been studied; it seems possible that the hypotheses of theorem 13 or 14 might hold for some minimal set on such a curve.

The invariant continuum F studied by Cartwright and Littlewood ([5], § 7.1) also separates the plane and has no interior points. It contains two sets of periodic points, one with period $2N-1$, the other with period $2N+1$ as well as various other minimal sets (see [10]), but it is not known whether it is a Birkhoff "curve", or whether it is indecomposable.

I do not know of an indecomposable continuum I such that $I = \mathfrak{F}(I)$ which is invariant under a continuous (1, 1) transformation other than the identical transformation.

3.3. The remaining theorems concern the special case in which $I = \mathfrak{F}(I)$. By theorem 6 corollary 3 and theorem 7 corollary 2 the continua $C^a(x, \mathfrak{T}^N(x))$, $C^a(x, p_N)$ are uniquely defined when $I = \mathfrak{F}(I)$, and so we shall omit the a. In future we shall denote generally by $C(a, b)$ a continuum in I containing the points a and b and irreducible with respect to these properties. The continua $C_N(x)$, $C_N(x: p_N)$ are defined by 2.5 (1) and 2.6 (1) with a omitted.

THEOREM 15. *Suppose that $I = \mathfrak{F}(I)$ and let $x \in M$. Then either $C(x, \mathfrak{T}^N(x)) = C_N(x)$ for some N, or $C(x, \mathfrak{T}^N(x))$ does not contain $\mathfrak{T}^{2N}(x)$ for any $N > 0$.*

Suppose that for $N > 0$

$$\mathfrak{T}^{2N}(x) \in C(x, \mathfrak{T}^N(x)).$$

Then $\mathfrak{T}^{3N}(x) \in \mathfrak{T}^N\big(C(x, \mathfrak{T}^N(x))\big) = C(\mathfrak{T}^N(x), \mathfrak{T}^{2N}(x))$. For if $\mathfrak{T}^N\big(C(x, \mathfrak{T}^N(x))\big)$ is not irreducible between $\mathfrak{T}^N(x)$ and $\mathfrak{T}^{2N}(x)$, then $\mathfrak{T}^{-N}\big(C(\mathfrak{T}^N(x), \mathfrak{T}^{2N}(x))\big)$ is a proper sub-continuum of $C(x, \mathfrak{T}^N(x))$ which is impossible. Hence

$$C(x, \mathfrak{T}^N(x)) \supset \mathfrak{T}^N\big(C(x, \mathfrak{T}^N(x))\big) \supset \mathfrak{T}^{2N}\big(C(x, \mathfrak{T}^N(x))\big) \supset \ldots,$$

and so

$$C(x, \mathfrak{T}^N(x)) \supset \overline{\bigcup_{1 \leqslant n < \infty} \mathfrak{T}^{nN}\big(C(x, \mathfrak{T}^N(x))\big)}.$$

The right-hand side is a continuum, and, since the limit points of $\mathfrak{T}^{nN}(x)$, $n = 1, 2, \ldots$, include all the points of the set $M_N(x) = C_N(x) \cap M$ which is minimal for $\mathfrak{T}^N$,

$$C(x, \mathfrak{T}^N(x)) \supset \overline{\bigcup_{-\infty < n < \infty} \mathfrak{T}^{nN}\big(C(x, \mathfrak{T}^N(x))\big)} = C_N(x).$$

For each continuum $\mathfrak{T}^{nN}\big(C(x, \mathfrak{T}^N(x))\big)$ is irreducible and therefore contained in any continuum containing $\mathfrak{T}^{nN}(x)$ and $\mathfrak{T}^{(n+1)N}(x)$, and since $C(x, \mathfrak{T}^N(x))$ is closed, it contains the closure of the continua. But $C(x, \mathfrak{T}^N(x)) \subset C_N(x)$, and so $C(x, \mathfrak{T}^N(x)) = C_N(x)$. This proves the theorem.

THEOREM 16. *Suppose that $I = \mathfrak{F}(I)$ and let $x \in M$. Then for each integer $N > 0$, either there is a point p_N with period N such that $C(x, p_N)$ does not contain $\mathfrak{T}^N(x)$, or $C(x, p_N) = C_N(x: p_N)$ for every point p_N with period N.*

Suppose that, for a given N, $C(x, p_N)$ contains $\mathfrak{T}^N(x)$ for every point p_N in I with period N. Then $C(x, p_N) \supset C(\mathfrak{T}^N(x), p_N)$, and since $I = \mathfrak{F}(I)$ the usual argument shows that $C(\mathfrak{T}^N(x), p_N) = \mathfrak{T}^N(C(x, p_N))$. Hence

$$C(x, p_N) \supset \mathfrak{T}^N\big(C(x, p_N)\big) \supset \mathfrak{T}^{2N}\big(C(x, p_N)\big) \supset \ldots,$$

and so, as in the proof of theorem 15, we have

$$C(x, p_N) \supset \overline{\bigcup_{1 \leqslant n < \infty} \mathfrak{T}^{nN}\big(C(x, p_N)\big)},$$

and

$$C(x, p_N) = C_N(x: p_N).$$

3.4. Theorem 15 may be regarded as a special case of the following theorem with $\mathfrak{T}^N$ in place of $\mathfrak{T}$ and $N = 2$:

THEOREM 17. *Suppose that $I = \mathfrak{F}(I)$, and let $x \in M$. Suppose further that*

$$(1) \qquad \mathfrak{T}^N(x) \in C(x, \mathfrak{T}(x)), \qquad N > 1.$$

Then

$$C_1(x) = \bigcup_{0 \leqslant n \leqslant N-1} \mathfrak{T}^n\big(C(x, \mathfrak{T}(x))\big).$$

It follows from (1) that $\mathfrak{T}^{N+1}(x) \in \mathfrak{T}\big(C(x, \mathfrak{T}(x))\big)$, and so by lemma 1

$$(2) \qquad C(x, \mathfrak{T}(x)) \cup \mathfrak{T}\big(C(x, \mathfrak{T}(x))\big) \supset C(\mathfrak{T}^N(x), \mathfrak{T}^{N+1}(x)) = \mathfrak{T}^N\big(C(x, \mathfrak{T}(x))\big).$$

For since $C(x, \mathfrak{T}(x))$ and $\mathfrak{T}\big(C(x, \mathfrak{T}(x))\big)$ both contain $\mathfrak{T}(x)$ the left hand side is a continuum containing $\mathfrak{T}^N(x)$ and $\mathfrak{T}^{N+1}(x)$. But, by (1), $\mathfrak{T}^{2N}(x) \in \mathfrak{T}^N\big(C(x, \mathfrak{T}(x))\big)$, and so by (2) either $\mathfrak{T}^{2N}(x) \in C(x, \mathfrak{T}(x))$, or $\mathfrak{T}^{2N}(x) \in \mathfrak{T}\big(C(x, \mathfrak{T}(x))\big)$. In the latter case $\mathfrak{T}^{2N-1}(x) \in C(x, \mathfrak{T}(x))$.

Now applying the method with $\mathfrak{T}^{2N-1}(x)$, or $\mathfrak{T}^{2N}(x)$, as the case may be, in place of $\mathfrak{T}^N(x)$, we have either

$$C(x, \mathfrak{T}(x)) \cup \mathfrak{T}\big(C(x, \mathfrak{T}(x))\big) \supset \mathfrak{T}^{2N-1}\big(C(x, \mathfrak{T}(x))\big),$$

or

$$C(x, \mathfrak{T}(x)) \cup \mathfrak{T}\big(C(x, \mathfrak{T}(x))\big) \supset \mathfrak{T}^{2N}\big(C(x, \mathfrak{T}(x))\big),$$

and so by a similar argument $C(x, \mathfrak{T}(x))$ contains either $\mathfrak{T}^{3N-2}(x)$ or $\mathfrak{T}^{3N-1}(x)$ or $\mathfrak{T}^{3N}(x)$. Repeating the process we find that $C(x, \mathfrak{T}(x))$ contains a sequence of points $\mathfrak{T}^{n_i}(x)$ such that $n_{i+1}-n_i \leqslant N$, $N = n_1 < n_2 < \ldots$, $n_i \to \infty$ as $i \to \infty$. Hence

$$C^* = \bigcup_{0 \leqslant n \leqslant N-1} \mathfrak{T}^n\big(C(x, \mathfrak{T}(x))\big)$$

contains $\mathfrak{T}^n(x)$ for $n = 0, 1, 2, \ldots$ For $\mathfrak{T}^{n_i}(x) \in C^*$, and so $\mathfrak{T}^{n_i - n}(x) \in C^*$ for $n = 0, 1, 2, \ldots, N-1$, and this includes $\mathfrak{T}^n(x)$ for $n_i \leqslant n \leqslant n_{i+1}$. Since all other points of M are limit points of $\mathfrak{T}^n(x)$ as $n \to \infty$, and since C^* is closed, $M \subset C^*$. Since C^* is obviously connected, and $I = \mathfrak{F}(I)$,

$$\mathfrak{T}^n\big(C(x, \mathfrak{T}(x))\big) = C\big(\mathfrak{T}^n(x), \mathfrak{T}^{n+1}(x)\big) \subset C^*$$

for $n = -1, -2, \ldots$, and so, since C^* is closed, $C_1(x) \subset C^*$. But $C^* \subset C_1(x)$ and so $C_1(x) = C^*$ which is the required result.

3.5. We now return to the special case in which

$$C\big(x, \mathfrak{T}^N(x)\big) = C_N(x)\,. \tag{1}$$

THEOREM 18. *Suppose that $I = \mathfrak{F}(I)$ and that $x \in M$. Suppose further that* (1) *holds, and that p_N has period N and $p_N \in C_N(x)$. Then*

$$C(x, p_N) \supset C_{2N}(x)\,. \tag{2}$$

We observe first that for all integers n

$$C\big(\mathfrak{T}^{nN}(x), \mathfrak{T}^{(n+1)N}(x)\big) = \mathfrak{T}^{nN}\Big(C\big(x, \mathfrak{T}^N(x)\big)\Big) = \mathfrak{T}^{nN}\big(C_N(x)\big) = C(x)\,. \tag{3}$$

Since $C(x, p_N) \cup \mathfrak{T}^N\big(C(x, p_N)\big)$ is obviously a continuum containing x and $\mathfrak{T}^N(x)$,

$$C(x, p_N) \cup \mathfrak{T}^N\big(C(x, p_N)\big) \supset C\big(x, \mathfrak{T}^N(x)\big) = C_N(x)\,.$$

Hence for all integers n either $\mathfrak{T}^{nN}(x) \in C(x, p_N)$ or $\mathfrak{T}^{nN}(x) \in \mathfrak{T}^N C(x, p_N)$. If, for some n, $\mathfrak{T}^{nN}(x) \in C(x, p_N)$ and $\mathfrak{T}^{(n+1)N}(x) \in C(x, p_N)$, it follows from (3) that

$$C_N(x) = C\big(\mathfrak{T}^{nN}(x), \mathfrak{T}^{(n+1)N}(x)\big) \subset C(x, p_N)$$

and a fortiori (2) holds.

Suppose that there is no pair of consecutive integers n, $n+1$ such that

$$C\big(\mathfrak{T}^{nN}(x), \mathfrak{T}^{(n+1)N}(x)\big) \subset C(x, p_N)\,. \tag{4}$$

Then, if for some n we have $\mathfrak{T}^{nN}(x) \in C(x, p_N)$, it follows that $\mathfrak{T}^{(n+1)N}(x) \in \mathfrak{T}^N\big(C(x, p_N)\big)$ and $\mathfrak{T}^{(n+2)}(x) \in C(x, p_N)$. For if $\mathfrak{T}^{(n+2)}(x) \in \mathfrak{T}^N\big(C(x, p_N)\big)$, then $\mathfrak{T}^{(n+1)N}(x) \in C(x, p_N)$, and (4) holds. But $x \in C(x, p_N)$, and so $\mathfrak{T}^{2N}(x) \in C(x, p_N)$, and $\mathfrak{T}^{2nN}(x) \in C(x, p_N)$ for $n = 1, 2, \ldots$, and also for $n = -1, -2, \ldots$ by a similar argument. Hence

$$C(x, p_N) \supset \overline{\bigcup_{-\infty < n < \infty} \mathfrak{T}^{nN}\Big(C\big(x, \mathfrak{T}^{2N}(x)\big)\Big)} = C_{2N}(x)\,.$$

3.6. It might be supposed from consideration of simple examples such as those given by Levinson [10] that if $I = \mathfrak{F}(I)$ the irreducible continuum $C(x, \mathfrak{T}(x))$ must contain a fixed point, and similarly the continuum $C(x, \mathfrak{T}^N(x))$ a point with period N, but it seems difficult to prove anything more then the following result:

THEOREM 19. *Suppose that $I = \mathfrak{F}(I)$ and that $x \in M$. Then either*

$$K = \bigcap_{-\infty < n < \infty} \mathfrak{T}^n\big(C(x, \mathfrak{T}(x))\big) = \emptyset \tag{1}$$

or K is an invariant continuum which does not separate the plane, and therefore contains a fixed point.

If $K \neq \emptyset$ it follows from Lemma 1 that it is a continuum, and since it is contained in I it cannot separate the plane. Obviously $\mathfrak{T}(K) = K$, and so by theorem A it contains a fixed point. K may be a single fixed point.

From this we obtain a special form of theorem 13:

THEOREM 20. *Suppose that $I = \mathfrak{F}(I)$ and $x \in M$. If for some N such that $x \neq \mathfrak{T}^N(x)$*

$$K_N = \bigcap_{-\infty < n < \infty} \mathfrak{T}^{nN}\Big(C\big(x, \mathfrak{T}^N(x)\big)\Big) \neq \emptyset$$

and $C(x, p_N) = C_N(x)$, where p_N is a point with period N in K_N, then $C_N(x)$ is an indecomposable continuum.

By theorem 19 with $\mathfrak{T}^N$ in place of $\mathfrak{T}$ the point p_N exists and lies on $C(x, \mathfrak{T}^N(x))$. By definition $C(x, p_N)$ is irreducible between x and p_N, and it follows as usual that $\mathfrak{T}^N\big(C(x, p_N)\big) = C_N(x) = C(x, p_N)$ is irreducible between $\mathfrak{T}^N(x)$ and p_N. Hence $C(x, p_N)$ is irreducible between each pair of the three points x, $\mathfrak{T}^N(x)$, p_N and so it is an indecomposable continuum.

3.7. We sum up some of the results about the case in which $C(x, \mathfrak{T}^N(x)) = C_N(x)$ and $I = \mathfrak{F}(I)$ in the following theorem:

THEOREM 21. *Suppose that* $I = \mathfrak{F}(I)$, $x \in M$ *and* $x \neq \mathfrak{T}^N(x)$ *and that* 3.5 (1) *holds. Then either* (i) $C(x, \mathfrak{T}^N(x))$ *is irreducible between* x *and* $\mathfrak{T}^{2N}(x)$, *and therefore an indecomposable continuum, or* (ii)

$$C(x, \mathfrak{T}^{2N}(x)) \cup \mathfrak{T}^N\big(C(x, \mathfrak{T}^{2N}(x))\big) = C_N(x)\,, \tag{1}$$

and $C(x, \mathfrak{T}^{2N}(x))$ *is an indecomposable continuum, or* (iii)

$$C(x, \mathfrak{T}^{2N}(x)) \cap \mathfrak{T}^N\big(C(x, \mathfrak{T}^{2N}(x))\big) = \emptyset\,, \tag{2}$$

and $C(x, \mathfrak{T}^{(2n+1)N}(x)) = C_N(x)$ *for every positive integer* n.

In virtue of theorem 13 it is sufficient to suppose that $C(x, \mathfrak{T}^{2N}(x))$ is a proper subcontinuum of $C_N(x)$ and prove that either (ii) or (iii) hold. Suppose first that

$$C(x, \mathfrak{T}^{2N}(x)) \cap \mathfrak{T}^N\big(C(x, \mathfrak{T}^{2N}(x))\big) \neq \emptyset\,.$$

Then by lemma 1 the left hand side of (1) is a continuum containing x and $\mathfrak{T}^N(x)$ and therefore $C(x, \mathfrak{T}^N(x)) = C_N(x)$. Since it is contained in $C_N(x)$, (1) holds. Hence $C(x, \mathfrak{T}^{2N}(x))$ contains the point p_N of period N which lies in $C_N(x)$, and so

$$C(x, p_N) \cup \mathfrak{T}^{2N}\big(C(x, p_N)\big) \subset C(x, \mathfrak{T}^{2N}(x))\,.$$

By theorem 18, $C(x, p_N) \supset C_{2N}(x) \supset C(x, \mathfrak{T}^{2N}(x)) \supset C(\mathfrak{T}^{2N}(x), p_N)$ $= \mathfrak{T}^{2N}\big(C(x, p_N)\big)$, and so

$$\mathfrak{T}^{2N}\big(C(x, p_N)\big) \supset \mathfrak{T}^{2N}\big(C_{2N}(x)\big) = C_{2N}(x) \supset C(x, \mathfrak{T}^{2N}(x))\,.$$

Hence $C(x, p_N) = \mathfrak{T}^{2N}\big(C(x, p_N)\big) = C(x, \mathfrak{T}^{2N}(x)) = C_{2N}(x)$, and, being a continuum irreducible between each pair of the three points $x, p_N, \mathfrak{T}^{2N}(x)$, it is an indecomposable continuum.

It remains to consider the case in which (2) holds. Let n be any positive integer, and consider

$$C(n) = \bigcup_{0 \leqslant \nu \leqslant n-1} \mathfrak{T}^{2\nu N}\big(C(x, \mathfrak{T}^{2N}(x))\big) \cup C(x, \mathfrak{T}^{(2n+1)N}(x))\,.$$

It is easy to verify that $C(n)$ is a continuum containing $\mathfrak{T}^{2nN}(x)$ and $\mathfrak{T}^{(2n+1)N}(x)$, and so by Lemma 1

$$C_N(x) = \mathfrak{T}^{2nN}\big(C(x, \mathfrak{T}^N(x))\big) = C(\mathfrak{T}^{2nN}(x), \mathfrak{T}^{(2n+1)N}(x)) \subset C(n)\,.$$

Hence $p_N \in C(n)$. But, by (2), p_N cannot belong to $C(x, \mathfrak{T}^{2N}(x))$ and therefore cannot belong to $\mathfrak{T}^{2\nu N}\big(C(x, \mathfrak{T}^{2N}(x))\big)$. Hence $p_N \in C(x, \mathfrak{T}^{(2n+1)N}(x))$, and it follows from theorem 18 that

$$C_{2N}(x) \subset C(x, p_N) \subset C(x, \mathfrak{T}^{(2n+1)N}(x))\,,$$

and remembering that $C_{2N}(x) = C_{2N}(\mathfrak{T}^{2nN}(x))$, we have

$$\mathfrak{T}^N(x) \in \mathfrak{T}^N\big(C_{2N}(x)\big) \subset \mathfrak{T}^N\big(C(\mathfrak{T}^{2nN}(x), p_N)\big) = C(\mathfrak{T}^{(2n+1)N}(x), p_N) \subset C(x, \mathfrak{T}^{(2n+1)N}(x))$$

for $p_N \in C(x, \mathfrak{T}^{(2n+1)N}(x))$. Hence

$$C_N(x) = C(x, \mathfrak{T}^N(x)) \subset C(x, \mathfrak{T}^{(2n+1)N}(x))$$

which is the required result.

3.8. It should be observed that 3.7 (2) does not imply

$$C_{2N}(x) \cap \mathfrak{T}^N\big(C_{2N}(x)\big) = \emptyset\,. \tag{1}$$

Even the hypothesis that

$$C(x, \mathfrak{T}^{2N}(x)) \cap \mathfrak{T}^{(2n+1)N}\big(C(x, \mathfrak{T}^{2N}(x))\big) = \emptyset$$

for all integers n, does not necessarily imply that (1) holds. However a slightly different type of result can be obtained by using (1) as a criterion for distinguishing the different cases.

THEOREM 22. *Suppose that the hypotheses of theorem 21 hold. Then either* (i) $C(x, \mathfrak{T}^N(x))$ *is irreducible between* x *and* $\mathfrak{T}^{2N}(x)$ *and therefore an indecomposable continuum, or* (ii) $C(x, p_N) = C_{2N}(x)$ *and*

$$C_{2N}(x) \cup \mathfrak{T}^N\big(C_{2N}(x)\big) = C_N(x) \tag{2}$$

or (iii) (1) *holds and* $C_{2N}(x)$ *contains a point* p_{2N} *of least period* $2N$, *and all the periodic points in* $C_{2N}(x)$ *have periods which are multiples of* $2N$.

If (1) is false, the left hand side of (2) is obviously a continuum containing $C(x, \mathfrak{T}^N(x)) = C_N(x)$, and since it is contained in $C_N(x)$, (2) holds. Hence the point p_N of period N in $C_N(x)$ is contained in $C_{2N}(x)$, and so by theorem 18 (ii) holds.

Part (iii) follows from parts (iii) and (iv) of theorem 2 in virtue of (1).

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Addendum. After further consideration I have come to the conclusion that Birkhoff may have intended the name *continuous minimal set* to cover all sets consisting of a finite number of continua, whether locally connected or not. As is evident from theorem 3 minimal sets which consists of a finite number of continua have many of the properties of the infinite continuous minimal sets defined in § 1.2.

I ought also to have pointed out that partially discontinuous minimal sets include all minimal sets containing an infinity of non-degenerate continua. Birkhoff wrote that the existence of partially discontinuous minimal sets seemed doubtful, but Floyd's example [1] of a non-homogeneous minimal set (although for a transformation which does not satisfy the conditions of § 1.1) strongly suggests that there exists minimal sets with an infinity of non-degenerate continua and also an infinity of points as components.

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Extension of mappings on metric spaces

by

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Introduction. If $\mathfrak{M}$ is a subset of Hilbert space and φ is a topological map of $\mathfrak{M}$ onto $\mathfrak{M}$, it is, in general, impossible to extend φ topologically (or even continuously) over the closure $\overline{\mathfrak{M}}$ of $\mathfrak{M}$. However, is it possible to find a suitable topological re-embedding M of $\mathfrak{M}$ in Hilbert space such that φ may be extended over $\overline{M}$? The answer to this question is in the affirmative. Actually, we shall prove far more: If Φ is a countable set of continuous mappings of the separable metrizable space M into itself, one can find a compact metric space $\overline{M}$ in which M is densely embedded, such that every continuous map of the given set may be extended continuously over $\overline{M}$.

In order to avoid unnecessary repetitions it is useful to introduce the notion of Φ-compactification. If M is a separable metrizable space and Φ a set of continuous maps of M into M, the space $\overline{M}$ is called a *Φ-compactification of M*, if $\overline{M}$ is a compactum (compact metric space) containing M densely, such that every element of Φ may be extended continuously over $\overline{M}$.

We shall investigate Φ-compactifications in § 2 and we shall e.g. find, for every set Φ closed under multiplication, necessary and sufficient conditions under which such a Φ-compactification exists (Theorem 2.12). The way in which Φ operates on M enters into these conditions. It is shown by examples and counterexamples that this is essential, and the authors believe that it is practically impossible to find necessary and sufficient conditions in terms of M alone. Applications to the case where Φ is a set of autohomeomorphisms of M are obtained as corollaries.

It is of interest to ask for conditions under which *every* autohomeomorphism of M can be extended to a suitable metric compactification $\overline{M}$ of M. We shall give in § 3 sufficient conditions — improving a little on already known results — which are believed to be rather general. Also a number of examples is given to clarify the situation.

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