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Reçu par la Rédaction le 1. 7. 1959

STUDIA MATHEMATICA, T. XIX. (1960)

On some functional equations in Banach spaces

by

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Throughout this paper $R = \{t, s, u, \ldots\}$ denotes the set of all real numbers, $X = \{x, y, z, \ldots\}$ a complete Banach space, X^* the adjoint of X and $L(X)$ the set of all linear and continuous mappings of X into X endowed with the usual structure of a Banach space.

In § 1 we consider operator-valued mappings F and G of R into $L(X)$ such that

$$F(t+s)+F(t-s) = 2F(t)G(s) \tag{1}$$

holds for all $t, s \in R$. In the case $F = G$ we have a cosine functional equation for which we have proved [6] that a weak measurability of F on one interval implies a weak continuity of F on R in the case of X being a separable Hilbert space. Theorem 1 of this paper extends this result to the case of a reflexive and separable Banach space. If X is a Hilbert space, $F(t) = N(t)$ is a normal operator and the function $N(t)$ is weakly continuous, then $N(t) = \cos tN$ for all $t \in R$, where the normal operator N does not depend on t [6]. In this paper we generalise this result to the functional equation (1).

In § 2 we consider continuous complex-valued functionals f and g defined on X and such that

$$f(x+y)+f(x-y) = 2f(x)g(y) \tag{2}$$

for all $x, y \in X$. We prove that the functionals f and g can be expressed as functions of an additive and continuous functional. Some other functional equations which can be reduced to (2) are also treated.

We note that St. Kaczmarz [3] has considered real-valued functions f, φ defined on R and such that

$$f(x)+f(x+y) = \varphi(y)f(x+y/2).$$

Replacing here x by $x+y/2$ and setting $\varphi(y) = 2g(y/2)$ we find that f and g satisfy (2) and conversely. Kaczmarz has proved that the measurability of f implies the continuity of functions f, φ and he has found all solutions of that equation. It seems to us that his method (lemma 2, p. 144) cannot be generalised to the general case of functional equation (1).

§ 1. THEOREM 1. *Let $t \to F(t)$ be a mapping of R into $L(X)$ such that:*

(3) $$F(t+s)+F(t-s) = 2F(t)F(s), \qquad F(0) = E$$

for all $t, s \in R$, where E is the identity operator.

Suppose that: a) *There is an interval $I \leqslant R$ such that the restriction of F on I is weakly measurable, and* b) *X is a separable and reflexive Banach space.*

Then F is a weakly continuous function on R.

Proof. In the same way as in [6], theorem 1, we find that $F(t)$ is weakly measurable on R and that $\|F(t)\|$ is a bounded function on every finite interval. Hence a function

$$y^*[F(t)x] \qquad (x \in X,\ y^* \in X^*)$$

is summable in every finite interval. This implies that the equation

(4) $$y^*_{ab}[x] = \int_a^b y^*[F(t)x]\,dt$$

defines a linear functional y^*_{ab} for any $a, b \in R$ and $y^* \in X^*$. By X_1^* we denote the set of all functionals $y^*_{ab} \in X^*$ which can be written in the form (4). We assert that X_1^* is dense in X^*, i. e. $\overline{X_1^*} = \bar{X}^*$. Suppose that this is not true. Then a functional $z^* \in X^*$ exists such that $z^* \notin \bar{X}_1^*$ and $z^* \neq 0$. But then a functional $w^{**} \in X^{**}$ can be found ([2], p. 64-65) such that

(5) $$w^{**}[z^*] = 1 \quad \text{and} \quad w^{**}[y^*] = 0$$

for all $y^* \in \overline{X_1^*}$. Since X is a reflexive space ([2], p. 66) one can find an element $w \in X$ such that

$$w^{**}[y^*] = y^*[w]$$

for every $y^* \in X^*$. This and (5) lead to

(6) $$y^*_{ab}[w] = 0$$

for all $y^*_{ab} \in X_1^*$. Now (6) and (4) imply

(7) $$\int_a^b y^*[F(t)w]\,dt = 0$$

for every couple $a, b \in R$ and for any $y^* \in X^*$. From (7) we find

(8) $$y^*[F(t)w] = 0$$

for every $t \notin S(y^*)$ where $mS(y^*) = 0$ (the Lebesgue measure of $S(y^*)$). Since X^* is a separable space there is a set $A = \{y_1^*, y_2^*, \ldots, y_n^*, \ldots\}$ which is countable and dense on X^*. We set $S = \bigcup_{k=1}^{\infty} S(y_k^*)$. Then $mS = 0$ and $y_n^*[F(t)w] = 0$ for every $t \notin S$.

Since A is dense on X^* we find

(9) $$F(t)w = 0$$

for every $t \notin S$. Obviously $mS = 0$ implies the existence of a number $u \in R$ such that $u \notin S$ and $2u \notin S$. From (9) we find $F(2u)w = F(u)w = 0$, which together with the functional equation (3) leads to

$$w = F(0)w = [F(2u) - 2F^2(u)]w = 0,$$

which contradicts (5). Thus X_1^* is dense on X^*. Replacing x by $2F(s)x$ in (4) and using (3) we get

$$2y^*_{ab}[F(s)x] = \int_a^b y^*[2F(s)F(t)x]\,dt$$

$$= \int_a^b y^*[(F(t+s)+F(t-s))x]\,dt = \int_{a+s}^{b+s} y^*[F(t)x]\,dt + \int_{a-s}^{b-s} y^*[F(t)x]\,dt,$$

from which we see that $z^*[F(t)x]$ is a continuous function for any $z^* \in X_1^*$. Since X_1^* is dense on X^* and $\|F(t)\|$ is locally bounded, we find that $y^*[F(t)x]$ is continuous for any pair $x \in X$ and $y^* \in X^*$, i. e. $F(t)$ is a weakly continuous function on R, q. e. d.

THEOREM 2. *Let X be a Hilbert space (separable or not), $t \to N(t)$ a mapping of R into the set of all bounded normal operators defined on X and $t \to F(t)$ a mapping of R into $L(X)$. Suppose that:*

a) *1 is not an eigenvalue of $N(s)$ for all s,*
b) *there is at least one $t_0 \in R$ such that $F^{-1}(t_0)$ is a bounded and everywhere defined operator,*
c) *$F(t)$ is a weakly continuous function on R,*
d) *$F(t)$ and $N(s)$ satisfy the functional equation*

(10) $$F(t+s)+F(t-s) = 2F(t)N(s).$$

Then,

$$F(t) = A\cos tN + B\sin tN \quad \textit{and} \quad N(t) = \cos tN$$

for every $t \in R$, where N is a normal operator which does not depend on t and A, B are bounded linear operators.

Proof. Setting $s = 0$ and $t = t_0$ in (10) we get $F(t_0) = F(t_0)N(0)$, which implies $N(0) = E$. Further (11) leads to

(11) $$N(s) = \tfrac{1}{2}F^{-1}(t_0)[F(t_0+s)+F(t_0-s)],$$

from which we immediatelly deduce

$$N(t+s)+N(t-s) = 2N(t)N(s) \tag{12}$$

for all $t, s \in R$. The weak continuity of $F(t)$ and (11) imply that $N(t)$ is a weakly continuous function. This, $N(0) = E$ and theorem 2 of [6] imply the existence of a normal operator N such that

$$N(t) = \cos tN \tag{13}$$

for every $t \in R$. In such a way (10) is reduced to

$$F(t+s)+F(t-s) = 2F(t)\cos sN. \tag{14}$$

If $F_1(t)$ and $F_2(t)$ denote the even and the odd part of $F(t)$ respectively, then (14) leads to

$$F_1(t+s)+F_1(t-s) = 2F_1(t)\cos sN, \tag{15}$$

$$F_2(t+s)+F_2(t-s) = 2F_2(t)\cos sN. \tag{16}$$

For $t = 0$, (15) gives $F_1(s) = A\cos sN$ with $A = F_1(0) = F(0)$. If we interchange t and s in (16) and add the result obtained to (16), we get

$$F_2(t+s) = F_2(t)\cos sN+F_2(s)\cos tN. \tag{17}$$

Using the identity

$$F_2[t_1+(t_2+t_3)] = F_2[(t_1+t_2)+t_3] \tag{18}$$

and (17) we find

$$F_2(t_1)\sin t_3N\sin t_2N = F_2(t_3)\sin t_1N\sin t_2N.$$

For $t_1 = t$, $t_2 = t_3 = s/2$ we get

$$F_2(t)\cdot\big(E-N(s)\big) = 2F_2(s/2)\sin s/2N\sin tN,$$

from which follows $F_2(t) = B\sin tN$. Thus $F(t) = F_2(t)+F_2(t) = A\cos tN + B\sin tN$ and $N(t) = \cos tN$. It is easily seen that these functions satisfy functional equation (10).

§ 2. THEOREM 3. *Let X be a Banach space and $x \to f(x)$ a complex-valued functional defined on X and such that*

$$f(x+y)+f(x-y) = 2f(x)f(y), \quad f(0) = 1 \tag{19}$$

for all $x, y \in X$.

If the functional f is continuous, then an additive and continuous functional $a(x)$ exists such that

$$f(x) = \cos a(x) \tag{20}$$

for all $x \in X$.

Proof. We divide the proof in four steps.

I. For a given $x \in X$, $x \neq 0$ and a real number t set

$$f_x(t) = f(tx).$$

Then

$$f_x(t+s)+f_x(t-s) = 2f_x(t)f_x(s), \quad f_x(0) = 1 \tag{21}$$

for all $t, s \in R$. Since a complex-valued function $f_x(t)$ of a real wariable t is continuous, we have ([7], p. 172, lemma 4) $f_x(t) = \cos tb(x)$ for every $t \in R$, where the complex number $b(x)$ is determined up to the sign by $f_x(t)$. Thus

$$f(tx) = \cos tb(x). \tag{22}$$

The continuity of the functional f implies the continuity of the functional $b^2(x)$. By $a(x)$ we denote one of two possible continuous functionals such that $a^2(x) = b^2(x)$. Then (22) implies

$$f(tx) = \cos ta(x) \tag{23}$$

for all $x \in X$, $t \in R$, where $a(x)$ is a continuous and obviously $a(tx) = ta(x)$. It remains to prove that

$$a(x+y) = a(x)+a(y). \tag{24}$$

II. Suppose that X is a real Hilbert space. Setting (23) in (19) we get

$$\cos ta(x+y)+\cos ta(x-y) = 2\cos a(x)\cos a(y). \tag{25}$$

If we take the second derivative of (25) with respect to t we find

$$\begin{aligned} &a^2(x+y)\cos ta(x+y)+a^2(x-y)\cos ta(x-y) \\ &\quad = 2[a^2(x)+a^2(y)]\cdot\cos ta(x)\cdot\cos ta(y)-4a(x)a(y)\sin ta(x)\sin ta(y). \end{aligned} \tag{26}$$

For $t = 0$ (26) implies

$$a^2(x+y)+a^2(x-y) = 2a^2(x)+2a^2(y). \tag{27}$$

Since $a^2(x)$ is a continuous functional we have ([5], theorem 3)

$$a^2(x) = (Ax, x)+i(Bx, x) \tag{28}$$

where A and B are symmetrical and bounded linear operators. Formula (21) implies

$$a^2(x\pm y) = a^2(x)+a^2(y)\pm 2[A(x, y)+i(Bx, y)]. \tag{29}$$

Now (29), (28), (26) and (25) lead to

$$\begin{aligned} &[(Ax, y)+i(Bx, y)]\cdot[\cos ta(x+y)-\cos ta(x-y)] \\ &\qquad = -2a(x)a(y)\sin ta(x)\sin ta(y). \end{aligned} \tag{30}$$

If we take the second derivative of (30) and if we set $t=0$ we get

$$(Ax,y)+i(Bx,y) = \pm a(x)a(y), \tag{31}$$

from which it follows that $a(x)$ is a linear functional.

III. X is a complex n-dimensional space. Obviously we can "represent" X by use of a $2n$-dimensional unitary (Hilbert) space $\hat{X}=\{\hat{x},\ldots\}$ in such a way that this representation is one-one, additive and continuous. If x and $\hat{x}$ are corresponding elements we set $\hat{f}(\hat{x})=f(x)$. Then $\hat{f}$ is a continuous functional on $\hat{X}$ and it satisfies (19). Since $\hat{X}$ is a unitary space (by use of II) we have $\hat{f}(t\hat{x})=\cos t\hat{a}(\hat{x})$, where $\hat{a}(\hat{x})$ is a continuous and linear functional on $\hat{X}$. Thus

$$\cos t\hat{a}(\hat{x}) = \cos ta(x) \tag{32}$$

for all $t\in R$, $x\in X$ and $\hat{x}\in\hat{X}$. Obviously (32) implies

$$\hat{a}(\hat{x}) = \varepsilon a(x) \tag{33}$$

for all x and $\hat{x}$, where ε does not depend on x, $\hat{x}$ and $\varepsilon=1$ or $\varepsilon=-1$. This and (33) imply that $a(x)$ is an additive functional on X.

IV. The general case: X is a complex Banach space. In order to prove (24) for given $x,y\in X$ we consider a two-dimensional Banach subspace X_2 whose elements are x and y. The restriction of f on X_2 we denote by f_2. Then f_2 is a continuous functional on X_2 and it satisfies (19). Using III we have $f_2(tz)=\cos ta_2(z)$ for every $t\in R$ and for each $z\in X_2$, where $a_2(z)$ is a continuous and additive functional. On the other hand, f_2 is the restriction of f. Thus $\cos ta_2(z)=\cos ta(z)$, which implies $a(z)=\varepsilon a_2(z)$ for all $z\in X_2$, where ε does not depend on z and $\varepsilon=1$, or $\varepsilon=-1$. Therefore we have $a(x+y)=\varepsilon a_2(x+y)=\varepsilon a_2(x)+\varepsilon a_2(y)=a(x)+a(y)$. Since x and y are arbitrary vectors of X, (24) is proved. In such a way theorem 3 is completely proved.

THEOREM 4. *Let $M=\{x\}$ be the set of all real square matrices of the order n, and $x\to f(x)$ a complex-valued functional defined on M and such that:*

a) *for all x, $y\in M$*

$$f(x+y)+f(x-y) = 2f(x)f(y), \quad f(0)=1; \tag{34}$$

b) *$f(x)$ is a continuous functional;*

c) *$f(s^{-1}xs)=f(x)$ for each $x\in M$ and for every non-singular matrix $s\in M$.*

Then $f(x)=\cos(a\operatorname{Tr}x)$ for all $x\in M$, where the complex number a does not depend on x and

$$\operatorname{Tr}x=\sum_{k=1}^{n}x_{kk}$$

is the trace of the matrix x.

Proof. As in theorem 3, we have $f(tx)=\cos ta(x)$ for all $t\in R$, where $a(x)$ is a continuous functional such that

$$a^2(x+y)+a^2(x-y) = 2a^2(x)+2a^2(y).$$

Since f is invariant we find $a^2(s^{-1}xs)=a^2(s)$, which leads to

$$a^2(x) = a_1\cdot(\operatorname{Tr}x)^2+a_2\cdot\sum_{1\leqslant i<j\leqslant n}\begin{vmatrix}x_{ii} & x_{ij}\\ x_{ji} & x_{jj}\end{vmatrix} \tag{35}$$

for every $x\in M$, where the complex numbers a_1 and a_2 do not depend on x ([4], theorem 9). Now (35) implies

$$a^2(x\pm y) = a^2(x)+a^2(y)\pm 2a_1\operatorname{Tr}x\cdot\operatorname{Tr}y\pm b_2\sum_{1\leqslant i<j\leqslant n}\left\{\begin{vmatrix}x_{ii} & y_{ij}\\ x_{ji} & y_{jj}\end{vmatrix}+\begin{vmatrix}y_{ii} & x_{ij}\\ y_{ji} & x_{jj}\end{vmatrix}\right\},$$

from which in the same way as in theorem 3 we get

$$2a_1\cdot\operatorname{Tr}x\cdot\operatorname{Tr}y+a_2\cdot\sum_{1\leqslant i<j\leqslant n}\left\{\begin{vmatrix}x_{ii} & y_{ij}\\ x_{ji} & y_{jj}\end{vmatrix}+\begin{vmatrix}y_{ii} & x_{ij}\\ y_{ji} & x_{jj}\end{vmatrix}\right\} = \pm 2a(x)a(y). \tag{36}$$

If $a(y)=0$ for every $y\in M$, then theorem 4 is satisfied with $a=0$. If one can find $y_0\in M$ for which $a(y_0)\neq 0$, then from (36) after dividing by $2a(y_0)$ one easily concludes that $a(x)$ is an additive functional, i. e. $a(x+y)=a(x)+a(y)$ for all $x,y\in M$. Since $a(x)$ is continuous and $a(s^{-1}xs)=a(x)$, we find $a(x)=a\operatorname{Tr}x$ ([4], theorem 3). Thus $f(x)=\cos a(x)=\cos(a\operatorname{Tr}x)$ for all $x\in M$, q. e. d.

THEOREM 5. *Let X be a Banach space, and let $x\to f(x)$ and $x\to g(x)$ be complex-valued functionals defined on X and such that*

$$f(x+y)+f(x-y) = 2f(x)g(y) \tag{37}$$

holds for all $x,y\in X$.

If the functional f is not identically zero and if it is continuous, then

$$f(x)=a(x)+B, \quad g(x)\equiv 1$$

or

$$f(x)=A\cos a(x)+B\sin a(x), \quad g(x)=\cos a(x)$$

for all $x\in X$, where A, B are constants and $a(x)$ is a bounded and additive functional on X.

Proof. Since $f\not\equiv 0$, there is at least one vector $x_0\in X$ such that $f(x_0)\neq 0$. From (37) we find

$$g(y) = [f(x_0+y)+f(x_0-y)]/2f(x_0),$$

which implies that the functional g satisfies all conditions of theorem 3. Thus $g(x) = \cos a(x)$ for all $x \in X$, where $a(x)$ is a bounded and additive functional. If $a(x) = 0$ for all x then (37) reads

$$f(x+y)+f(x-y) = 2f(x),$$

from which one easily finds that $h(x) = f(x)-f(0)$ is an additive functional. Since it is continuous, we have $f(x) = f(0)+h(x)$ and $g(x) \equiv 1$, i. e. we have the first case of theorem 5. If $a(x) \not\equiv 0$, then writing f as a sum of a symmetrical and an antisymmetrical functionals, in the same way as in theorem 2, we prove the rest of theorem 5.

THEOREM 6. *Let X be a Banach space, and let $x \to f(x)$, $x \to g(x)$ be complex-valued functionals defined on X, not identicaly zero and such that*

$$f(x-y) = f(x)g(y)-f(y)g(x) \tag{38}$$

for all $x, y \in X$.

If the functional f is continuous, then

a) $f(x) = a(x)$, $g(x) = 1+A\cdot a(x)$

or

b) $f(x) = A\sin a(x)$, $g(x) = \cos a(x)+B\sin a(x)$,

where A, B are constants and $a(x)$ is a continuous and additive functional on X.

Proof. If we interchange x and y in (38) we find $f(x-y) = -f(y-x)$, which implies $f(-x) = -f(x)$ for all $x \in X$. Replacing $-y$ by y in (38) we get

$$f(x+y) = f(x)g(-y)+f(y)g(x). \tag{39}$$

If we add (38) and (39) we get

$$f(x+y)+f(x-y) = 2f(x)h(y), \quad h(y) = [g(y)+g(-y)]/2. \tag{40}$$

Since f is a continuous and antisymmetrical functional, theorem 5 and (40) lead to

a′) $h(x) = 1$, $\quad f(x) = a(x)$

or

b′) $h(x) = \cos a(x)$, $\quad f(x) = A\sin a(x)$,

where $a(x)$ is a bounded and additive functional on X and A is a constant. In order to find g we set $-x$, $-y$ in (38) instead of x, y respectively and add the result obtained to (38). We find

$$f(x)[g(y)-g(-y)] = f(y)[g(x)-g(-x)].$$

Thus

$$g(x)+g(-x) = 2h(x) \quad \text{and} \quad g(x)-g(-x) = 2cf(x) \tag{41}$$

with some constant c. Now a′), b′) and (41) imply theorem 6.

THEOREM 7. *Let X be a Banach space, and let $x \to f(x)$ and $x \to g(x)$ be complex-valued functionals defined on X and such that*

$$f(x-y) = f(x)f(y)+g(x)g(y) \tag{42}$$

for all $x, y \in X$.

If f and g are not constants and if f is a continuous functional, then

$$f(x) = \cos a(x) \quad \textit{and} \quad g(x) = \sin a(x)$$

for all $x \in X$, where $a(x)$ is an additive and bounded functional on X.

Proof. Obviously

$$f(-x) = f(x). \tag{43}$$

Setting $-x$ and $-y$ in (42) instead of x and y respectively and using (43) we find

$$g(x)g(y) = g(-x)g(-y),$$

from which follows $g(-x) = g(x)$ for all x or $g(-x) = -g(x)$. The first possibility leads to the conclusion that f and g are constants. Since this trivial case is excluded by the conditions of theorem 7, we have $g(-x) = -g(x)$. Replacing $-y$ by y in (42) and adding the result obtained to (42) we find that f satisfies all the conditions of theorem 3. Thus

$$f(x) = \cos a(x), \tag{44}$$

where $a(x)$ is a bounded and additive functional on X. Setting (44) in (42) we get $g(x) = \sin a(x)$ for all $x \in X$ or $g(x) = \sin(-a(x))$ for all $x \in X$, q. e. d.

Remark 1. Functional equation (42) has been considered by H. V. Vaughan [8] in a completely different way, under the assumption that f and g are real-valued functions of a real variable x and that $\lim(g(x)/x) = 1$, as $x \to 0^+$.

Remark 2. If X is the set of all real numbers, then in theorem 3 continuity may be replaced by measurability in the sense of Lebesgue (theorem 1). This implies that in theorems 5, 6 and 7 continuity can be replaced by measurability. Obviously in this case $a(x) = ax$, where a is a complex number.

Remark 3. Theorems 5, 6 and 7 can be suitably generalized to the corresponding invariant functionals, which are defined on the set M of matrices introduced in theorem 4. In this case $a(x) = a\cdot \mathrm{Tr}x$ with some complex constant a.

Remark 4. In some real Banach spaces the general form of an additive and continuous functional is well known ([9], p. 137-142). Thus for those spaces theorems 3, 5, 6 and 7 give the general form of the functionals treated there.

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Reçu par la Rédaction le 4. 7. 1959

STUDIA MATHEMATICA, T. XIX. (1960)

Convex sets invariant under group representations

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§ 1. Introduction and summary. An earlier paper [1] began a study of convex sets invariant under translation in certain spaces of functions built over a group, the main emphasis resting on the question of separation of such sets. Here the setting is generalized and an attempt made to discuss the structure of convex sets in a topological vector space E which are invariant under a given representation $s \to A_s$ of the group G by endomorphisms of E. (Most of our results are easily modified to deal with invariant disks rather than convex sets, a disk meaning a closed, convex and circled set in E: such sets were termed mean-invariant in [2]).

The group G is assumed to be locally compact and abelian; it is additively written. Elements of the dual $\hat{G}$ of G are denoted by $\zeta, \chi, \ldots$ Haar measures on G and $\hat{G}$ are adjusted so that the Fourier inversion formula holds without external numerical factors. The Fourier transform of a function f on G is denoted by $\hat{f}(\zeta) = \int_G f(s)\overline{\zeta(s)}\,ds$.

The representation space E is assumed to be separated locally convex. Elements of E are denoted by $x, y, \ldots$, and those of the topological dual E' by $x', y', \ldots$; the bilinear form defining the duality is written $\langle x, x'\rangle$.

Each endomorphism A_s is assumed to be continuous, and the representation is to be bounded and continuous; this entails that $s \to \langle A_s x, x'\rangle \equiv \varphi_{x,x'}(s)$ is a bounded and continuous function on G for each fixed $(x, x') \in E \times E'$. Certain other restrictions on E will be imposed later (see § 2).

The analysis of closed, convex sets invariant under the given representation is attempted in terms of the concept of spectrum applied to elements of E. This concept is defined in § 3, and Theorem 1 characterizes the spectrum of x in terms of the familiar L^∞-spectra (according to Beurling and Godement) of the associated functions $\varphi_{x,x'}$. The main result is Theorem 2: this contains the essence in abstract form of summability