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Convex sets invariant under group representations

by

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§ 1. Introduction and summary. An earlier paper [1] began a study of convex sets invariant under translation in certain spaces of functions built over a group, the main emphasis resting on the question of separation of such sets. Here the setting is generalized and an attempt made to discuss the structure of convex sets in a topological vector space E which are invariant under a given representation $s\to A_s$ of the group G by endomorphisms of E. (Most of our results are easily modified to deal with invariant disks rather than convex sets, a disk meaning a closed, convex and circled set in E: such sets were termed mean-invariant in [2]).

The group G is assumed to be locally compact and abelian; it is additively written. Elements of the dual $\hat{G}$ of G are denoted by $\zeta,\chi,\ldots$ Haar measures on G and $\hat{G}$ are adjusted so that the Fourier inversion formula holds without external numerical factors. The Fourier transform of a function f on G is denoted by $\hat{f}(\zeta)=\int_G f(s)\overline{\zeta(s)}\,ds$.

The representation space E is assumed to be separated locally convex. Elements of E are denoted by $x,y,\ldots$, and those of the topological dual E' by $x',y',\ldots$; the bilinear form defining the duality is written $\langle x,x'\rangle$.

Each endomorphism A_s is assumed to be continuous, and the representation is to be bounded and continuous; this entails that $s\to\langle A_s x,x'\rangle\equiv\varphi_{x,x'}(s)$ is a bounded and continuous function on G for each fixed $(x,x')\in E\times E'$. Certain other restrictions on E will be imposed later (see § 2).

The analysis of closed, convex sets invariant under the given representation is attempted in terms of the concept of spectrum applied to elements of E. This concept is defined in § 3, and Theorem 1 characterizes the spectrum of x in terms of the familiar L^∞-spectra (according to Beurling and Godement) of the associated functions $\varphi_{x,x'}$. The main result is Theorem 2: this contains the essence in abstract form of summability properties of Fourier series and integrals, namely the approximation of "arbitrary" functions f by functions belonging to the closed translation-invariant convex set generated by f and having compact spectra contained in the spectrum of f.

The case in which G is compact, discussed in §§ 6 and 7, is the easiest, thanks to the existence of sufficiently many elements with one-point spectra.

§ 2. Conditions on E. The abstract convolution. For the stronger form of Theorem 2 we shall need to assume that E is a t-space (*espace tonnelé*; [3], p. 1). The useful consequences of this assumption are summarized in

PROPOSITION 1. *Suppose that E is a t-space. To each equicontinuous subset H of E' corresponds a neighbourhood U of 0 in E such that*

$$\|\varphi_{x,x'}\|_\infty \equiv \sup_{s\epsilon G} |\varphi_{x,x'}(s)| \leqslant 1$$

for $x \epsilon U$ and $x' \epsilon H$. In particular:

(i) *for fixed x in E, the functions $\varphi_{x,x'}$ $(x' \epsilon H)$ are uniformly bounded on G;*

(ii) *the endomorphisms A_s $(s \epsilon G)$ are equicontinuous on E.*

Proof. For given x in E, the $A_s x$ are bounded (by hypothesis). It follows that $N(x) = \operatorname{Sup}\{|\varphi_{x,x'}(s)| : s \epsilon G,\ x' \epsilon H\}$ is finite. N is a semi-norm which, since the representation is assumed to be weakly continuous, is lower semicontinuous. The set U, defined by $N(x) \leqslant 1$, is therefore a tonneau in E. So U is a neighbourhood of 0, which was to be proved.

We denote by $M(G)$ $(M_+(G))$ the set of bounded (positive) Radon measures on G. An element of $M_+(G)$ is said to be normalised if its total mass is precisely one.

Let $\mu \epsilon M(G)$ and $x \epsilon E$ be given. Under very mild restrictions on E (that it be quasi-complete, for example) the vector-valued integral

$$\mu * x = \int_G (A_{-s} x)\, d\mu(s) \tag{2.1}$$

exists as a member of E. The value of this integral is the abstract convolution of μ and x. Note that if the hypotheses of Proposition 1 are fulfilled, the mapping $(\mu, x) \to \mu * x$ is continuous from $M(G) \times E$ into E, when $M(G)$ is equipped with its customary norm.

It is easily verified that

$$(\mu * \nu) * x = \mu * (\nu * x), \tag{2.2}$$

where on the left $\mu * \nu$ denotes the ordinary convolution of two measures.

The relevance of the abstract convolution ot the study of invariant convex sets is clear from Theorems 1, 2 and 3 of [1].

§ 3. The spectrum. When E is a function-space and A_s means translation by amount s, the structure of invariant vector subspaces of E is customarily discussed in terms of synthesis from elements whose behaviour under translation is especially simple (usually the characters of G). Even though this cannot be done directly in all cases, due to the fact that the characters may not belong to E, the Fourier spectrum of elements of E is almost always crucial. We therefore frame a consistent extension of this concept to the case in hand.

The space E and the representation $s \to A_s$ are assumed fixed. Given $x \epsilon E$, we define the spectrum $\sigma(x)$ of x to be the set of $\zeta \epsilon \hat{G}$ with the property that $f \epsilon L^1(G)$ and $f * x = 0$ together imply $\hat{f}(\zeta) = 0$. In other words, $\sigma(x)$ is the intersection of the zero-sets of Fourier transforms $\hat{f}$ of functions $f \epsilon L^1(G)$ satisfying $f * x = 0$. The latter set of f is an ideal in $L^1(G)$, and $\sigma(x)$ is its co-spectrum. $\sigma(x)$ is therefore always a closed subset of $\hat{G}$. In view of the identity

$$\mu * (A_s x) = A_s(\mu * x), \tag{3.1}$$

$\sigma(x)$ is invariant under each A_s: $\sigma(A_s x) = \sigma(x)$. Besides this, the identity

$$\varphi_{\mu * x, x'} = \mu * \varphi_{x,x'} \tag{3.2}$$

suggests that $\sigma(x)$ is closely related to the L^∞-spectra of the functions $\varphi_{x,x'}$. This expectation is borne out by the following result:

THEOREM 1. *For any x in E, $\sigma(x)$ is the closure in $\hat{G}$ of the union of the L^∞-spectra $\sigma(\varphi_{x,x'})$, as x' ranges over E'.*

Proof. Let S denote closure of the set just mentioned.

Suppose first that ζ does not belong to $\sigma(x)$. The identity

$$\varphi_{f * A_s x, x'}(0) = f * \varphi_{x,x'}(s), \tag{3.3}$$

combined with (3.1), shows that if f is chosen such that $f * x = 0$ and $\hat{f}(\zeta) \neq 0$, then f is orthogonal (in the duality between $L^1(G)$ and $L^\infty(G)$) to all translates of $\varphi_{x,x'}$, and this for each x' in E'. Consequently, ζ lies outside $\sigma(\varphi_{x,x'})$ for each x' in E'. It follows that $S \subset \sigma(x)$, since $\sigma(x)$ is known to be closed.

Conversely suppose that ζ is not in S. Since S is closed, there is a closed neighbourhood N of ζ in $\hat{G}$ such that N is disjoint from $\sigma(\varphi_{x,x'})$ for all x'. Choose $f \epsilon L^1(G)$ such that $\hat{f}(\zeta) = 1$ and $\hat{f} = 0$ outside N. Reference to Lemma 1 below will show that then $f * \varphi_{x,x'} = 0$ for all x', hence $f * x = 0$. Since $\hat{f}(\zeta) \neq 0$, it follows that $\zeta \notin \sigma(x)$. Thus $\sigma(x) \subset S$.

The two opposing inclusion relations now available complete the proof. It remains to deal with the lemma.

LEMMA 1. *Let $\varphi \in L^\infty(G)$ and $f \in L^1(G)$ be such that $\hat{f} = 1$ (resp. $\hat{f} = 0$) on a neighbourhood W of $\sigma(\varphi)$. Then $f*\varphi = \varphi$ (resp. $f*\varphi = 0$). The analogous assertion, in which φ is replaced by an $x \in E$, is also valid.*

Proof. The analogous assertion about elements of E follows from that about $\varphi \in L^\infty(G)$ on taking scalar components. To prove the latter we use a theorem due independently to Segal and Godement ([4], Theorem 2.2; [5], Théorème 7), which says that φ is the weak limit in $L^\infty(G)$ of "trigonometric polynomials"

$$\theta_i(s) = \sum_{\zeta \in F_i} c_\zeta \cdot \zeta(s),$$

where F_i is a *finite* subset of W. For each i we shall have $f*\theta_i = \theta_i$ (resp. $f*\theta_i = 0$), since in fact

$$f*\theta_i(s) = \sum_{\zeta \in F_i} c_\zeta \hat{f}(\zeta) \cdot \zeta(s).$$

Convolution with f being weakly continuous on $L^\infty(G)$, the conclusion follows.

The use of the concept of spectrum is illustrated by the next proposition.

PROPOSITION 2. (i) *If $x \in E$ and $\mu \in M(G)$, and if S is the support of $\hat{\mu}$, then*

$$\sigma(\mu * x) = \sigma(x) \cap S. \tag{3.4}$$

(ii) *If $x \in E$ has spectrum $\sigma(x)$ contained in the union of two disjoint compact sets A_1 and A_2, there is a unique decomposition*

$$x = x_1 + x_2, \quad \sigma(x_i) \subset A_i \quad (i = 1, 2). \tag{3.5}$$

Proof. (i) Let $\zeta \in \sigma(\mu * x)$. It is clear that then $\zeta \in \sigma(x)$. If ζ were not also in S, $\hat{\mu}$ would be 0 on a neighbourhood N of ζ, and one could take an f in $L^1(G)$ for which $\hat{f}$ has the value 1 at ζ and is 0 outside N. This would arrange that $f*\mu = 0$ and hence $f*(\mu*x) = 0$. However, the relations $\zeta \in \sigma(\mu*x)$ and $\hat{f}(\zeta) \neq 0$ are then in conflict. Thus $\sigma(\mu*x) \subset \sigma(x) \cap S$.

Reciprocally, suppose that $\zeta \in \sigma(x) \cap S$. Let $f \in L^1(G)$ satisfy $f*(\mu*x) = 0$: we need to show that $\hat{f}(\zeta) = 0$. Now, since $\zeta \in \sigma(x)$, (2.2) shows that $(f*\mu)\hat{}\,(\zeta) = 0$, i. e. $\hat{f}(\zeta)\hat{\mu}(\zeta) = 0$. Since $\zeta \in S$ and $\hat{f}$ is continuous, $\hat{f}(\zeta) = 0$ follows. Thus $\sigma(x) \cap S \subset \sigma(\mu*x)$ and (i) is established.

(ii) Let us first establish the existence of a decomposition (3.5). Take disjoint compact neighbourhoods U_i of A_i and then $f_i \in L^1(G)$ such that $\hat{f}_i = 1$ on a neighbourhood of A_i and has support contained in U_i. Put $x_i = f_i * x$. Reference to Lemma 1 and Proposition 2 shows that (3.5) holds; notice that $(f_1+f_2)\hat{}$ is 1 on a neighbourhood of $\sigma(x)$.

To prove uniqueness, it suffices to show that if y_i has spectrum contained in A_i $(i = 1, 2)$, and if $y_1 = y_2$, then $y_1 = y_2 = 0$. But, taking f_i as above, we shall have $f_1 * f_2 = 0$ and Lemma 1 will give (j denoting that index, 1 or 2, different from i):

$$0 = (f_1*f_2)*y_i = f_j*(f_i*y_i) = f_j*y_i = f_j*y_j = y_j,$$

remembering that $y_i = y_j$. This establishes the uniqueness.

§ 4. Elements with compact spectra. Examination of simple examples shows that it is too much to expect that a general closed invariant convex set (or disk) shall be generated by the elements with point spectra belonging to it, and this even in the most favourable case in which G is compact and such elements exist in abundance (see §§ 5, 6). However, we shall show in this section that each $x \in E$ is the limit of elements with compact spectra belonging to the closed invariant convex set generated by x. As a consequence, every closed invariant convex set is generated by elements with compact spectra belonging to it.

The proof of the main result, Theorem 2, depends upon some simple lemmas which show, not only that the approximation is possible, but that it may be effected by means of a simple operation closely akin to each of several standard summability methods in harmonic analysis.

LEMMA 2. *Let G be a locally compact space, A a closed subset of G. Let (μ_i) be a directed family in $M_+(G)$ such that*

$$\lim \mu_i = \mu \tag{4.1}$$

vaguely, and

$$\limsup \int_G d\mu_i \leqslant 1, \tag{4.2}$$

μ being a normalized measure supported by A. Then, for each open neighbourhood U of A, one has

$$\lim_i \int_{U'} d\mu_i = 0, \quad \lim_i \int_G d\mu_i = 1, \tag{4.3}$$

U' being the complement of U relative to G.

Remark. The vague topology of measures is the weak topology defined by duality with $C_c(G)$, the set of continuous functions on G with compact supports.

Proof. We make use of the following observation: If φ is a positive lower semicontinuous function on G (positivity may be replaced by majorization of some member of $C_c(G)$), then $\mu \to \int \varphi d\mu$ is lower semicontinuous for the vague topology. This is so because $\mu \to \int \varphi d\mu$ is by definition the upper envelope of the functions $\mu \to \int k d\mu$ as k ranges over all functions in $C_c(G)$ minorizing φ. It follows that

$$\int_G \varphi d\mu \leqslant \liminf \int_G \varphi d\mu_i \tag{4.4}$$

for each such φ, whenever (4.1) holds vaguely. If further $\varphi \leqslant 1$ and $\varphi = 1$ on A, (4.4) and (4.2) and the condition that μ is concentrated on A combine to yield

$$\lim \int_G \varphi d\mu_i = 1. \tag{4.5}$$

Taking in turn $\varphi = 1$ and $\varphi = \varphi_U$ (the characteristic function of U), and then subtracting, we are led to (4.3).

Lemma 3. *Let (μ_i) be a directed family in $M_+(G)$ which converges vaguely to the Dirac measure placed at a point $s_0 \in G$, and suppose that* (4.2) *holds. Then*

$$\lim_i \int_G f(s) d\mu_i(s) = f(s_0), \tag{4.6}$$

uniformly for f ranging over any set C of continuous [1] *functions on G which is uniformly bounded and equicontinuous at s_0.*

Proof. Lemma 2 is to be applied, taking $A = \{s_0\}$. Suppose $|f(s)| \leqslant M$ $(s \in G, f \in C)$. By equicontinuity, an open neighbourhood U of s_0 may be chosen so that

$$|f(s) - f(s_0)| \leqslant \varepsilon \qquad (s \in U, f \in G).$$

Then

$$\left|\int f d\mu_i - f(s_0)\int d\mu_i\right| \leqslant \int |f(s) - f(s_0)|\, d\mu_i(s);$$

splitting the integral into parts taken over U and U' separately, we see that there results the majorization

[1] Continuity may be replaced by measurability for each μ_i.

$$\left|\int f d\mu_i - f(s_0)\int d\mu_i\right| \leqslant \varepsilon \cdot \int d\mu_i(s) + 2M \int_{U'} d\mu_i(s).$$

The result therefore follows from (4.2) and (4.3).

Lemma 4. *Let f be positive, continuous, such that $\int_G f(s) ds = 1$, and let $\varepsilon > 0$. There exists a function p satisfying the same conditions, together with the requirement that $\hat{p}$ shall have a compact support, and such that $\|f - p\|_1 \leqslant \varepsilon$.*

Proof. We may write $f = g^2$, where g is a positive, continuous function in $L^2(G)$. By the Plancherel theory ([9], p. 145), $\hat{g} \in L^2(\hat{G})$. Choose $Q \in L^2(\hat{G})$ with a compact support satisfying $\|\hat{g} - Q\|_2 \leqslant \alpha$, where $\alpha > 0$ will be chosen later depending upon ε. Since $\hat{g}$ is "symmetric" (i. e. $\hat{g}(\zeta) = \overline{\hat{g}(-\zeta)}$ a. e.), we may assume that the same is true of Q: if not, replace Q by its symmetric part $\frac{1}{2}(Q + \tilde{Q})$, where $\tilde{Q}(\zeta) = \overline{Q(-\zeta)}$. Let q be the inverse Fourier transform of Q; q is then real and $\|g - q\|_2 = \|\hat{g} - Q\|_2 \leqslant \alpha$. $p_0 = q^2$ is positive, continuous, belongs to $L^1(G)$, and $\hat{p}_0 = Q * Q$ has a compact support. Also

$$\begin{aligned}\|f - p_0\|_1 &= \|g^2 - q^2\|_1 = \|(g - q)(g + q)\|_1 \\ &\leqslant \|g - q\|_2 \cdot \|g + q\|_2 \leqslant \alpha(2\|g\|_2 + \alpha) \\ &= \alpha(2\|f\|_1 + \alpha) = \alpha(2 + \alpha).\end{aligned}$$

From this it follows that $c = \int_G p_0(s) ds = \|p_0\|_1$ satisfies $|c - 1| \leqslant \alpha(2 + \alpha)$. It therefore suffices to take $p = c^{-1}p_0$ and arrange that $2\alpha(2 + \alpha) \leqslant \varepsilon$. The preceding proof is a slight refinement of that given by Loomis ([9], Section 37A).

Proposition 3. *There exists a directed family (p_i) of functions on G with the following properties:*

(1) *p_i is positive, continuous and in $L^1(G)$;*

(2) *$\hat{p}_i$ has a compact support in $\hat{G}$;*

(3) $\int_G p_i(s) ds = 1$;

(4) *for each open neighbourhood U of 0 in G one has $\lim_i \int_{U'} p_i(s) ds = 0$;*

(5) $\lim_i \int_G p_i(s) f(s) ds = f(0)$, *uniformly as f ranges over any set C of continuous (or even merely measurable) functions on G which is uniformly bounded and equicontinuous at 0.*

Proof. In view of Lemma 3, it is enough to show that δ, the Dirac measure placed at 0, is vaguely adherent to the set of measures μ defined symbolically by $d\mu(s) = p(s) ds$, where p is positive, continuous, belongs to $L^1(G)$, such that $\int_G p(s)\,ds = 1$, and such that $\hat{p}$ has a compact support. And, by virtue of Lemma 4, it is enough to prove the same assertion with the final restriction deleted. But this is very simple.

Remark. (p_i) plays the role of a sort of "approximate identity" for convolution. Unlike the usual approximate identities, however, the functions of the family have compact spectra rather than compact supports.

We can now establish without further difficulty the main theorem.

THEOREM 2. *Let E be quasi-complete and let $x \in E$, and let the p_i be as in Proposition 3. The elements $p_i * x$ each belong to the closed invariant convex set generated by x, each has a compact spectrum, and*

$$\lim_i (p_i * x) = x \tag{4.7}$$

weakly in E. If further E is a t-space, (4.7) *holds in the sense of the initial topology of E.*

Proof. We have for each x' in E'

$$\langle p_i * x, x' \rangle = \int_G \varphi_{x,x'}(-s) p_i(s)\,ds. \tag{4.7}$$

This shows first that by virtue of (1) and (3) of Proposition 3,

$$\langle p_i * x, x' \rangle \leqslant \sup_{s \in G} \langle A_{-s} x, x' \rangle$$

and hence, via the Hahn-Banach Theorem, that $p_i * x$ belongs to the closed convex set generated by the $A_s x$ $(s \in G)$, i. e. to the closed invariant convex set generated by x. Moreover, since $\varphi_{x,x'}$ is bounded and continuous, (5) of Proposition 3 shows that $p_i * x \to x$ weakly in E. The Hahn-Banach Theorem (in the form asserting that a closed convex subset of E is weakly closed), together with Proposition 2 (i), now goes to show that x is adherent in E to the set of elements with compact spectra belonging to the closed invariant convex set generated by x.

Moreover, if E is a t-space, we know from Proposition 1 that the functions $\varphi_{x,x'}$, with x' ranging over any equicontinuous subset H of E', are equicontinuous at 0 and uniformly bounded (Proposition 1). Hence, by (5) of Proposition 3, $\lim \langle p_i * x, x' \rangle = \langle x, x' \rangle$ holds uniformly for $x' \in H$. This means exactly that $p_i * x \to x$ in the sense of the initial topology of E. The proof of Theorem 2 is thus complete.

Remarks. (i) Even without the restriction of quasicompleteness on E, and assuming merely that the representation $s \to A_s$ is bounded and weakly continuous, it is still true that (4.7) holds weakly, provided the integrals defining $p_i * x$ exist in a suitable sense. This will certainly be the case if E is a separable Fréchet space, or if E is a Fréchet space and the function $s \to A_s x$ is almost separably-valued for each $x \in E$. In such cases, the main conclusion stands: each x is the limit of elements with compact spectra belonging to the closed invariant convex set generated by x.

(ii) Theorem 2 does not cover all cases of interest, amongst which there are instances where the representation involved, $s \to A_s$, is unbounded. In such a case, $\mu * x$ will not be defined for *all* measures μ in $M(G)$ or $M_+(G)$, and the preceding considerations call for modification. As an example, take for G the additive group of real numbers, for E the space $\mathcal{S}'$ of temperate distributions [7], and for A_s the translation operators. In this case E' is the space $\mathcal{S}$ of indefinitely differentiable functions on the real axis, each function and each of its derivatives being rapidly decreasing at infinity. The functions $\varphi_{x,x'}$ are now generally unbounded like some positive power of s. The preceding arguments can be rehabilitated as soon as it is arranged that the functions p_i be each rapidly decreasing. This may be attended to without difficulty, and the theorem remains valid.

§ 5. Elements with point spectra. For a given representation we shall say that an element x of E is *elementary* and is *associated* with a point ζ of $\hat{G}$ if $\sigma(x) \subset \{\zeta\}$. The element $x = 0$ is elementary and is associated indifferently with all points of $\hat{G}$. It seems that for bounded representations of non-compact groups, elementary elements of E other than 0 are rare.

THEOREM 3. *An element x of E is elementary and associated with $\zeta \in \hat{G}$ if and only if $A_s x = \zeta(s) \cdot x$ for all $s \in G$.*

Proof. If $A_s x = \zeta(s) \cdot x$, then $\varphi_{x,x'} = \langle x, x' \rangle \cdot \zeta$, hence $\sigma(\varphi_{x,x'}) \subset \{\zeta\}$ and so (Theorem 1) $\sigma(x) \subset \{\zeta\}$.

Conversely, if x is elementary and associated with ζ, then $\sigma(x) \subset \{\zeta\}$. Hence (Theorem 1 again) $\sigma(\varphi_{x,x'}) \subset \{\zeta\}$. According to a theorem of Kaplansky about primary ideals in $L^1(G)$ (see [8]), this implies that $\varphi_{x,x'}$ is a scalar multiple of ζ. Necessarily, therefore, $\varphi_{x,x'}(s) = \langle x, x' \rangle \cdot \zeta(s)$ for all s, and hence $A_s x = \zeta(s) \cdot x$ for all s. Thus x is elementary and associated with ζ. This completes the proof.

Let us denote by M_ζ the set of $x \in E$ which are elementary and associated with ζ. Plainly, M_ζ is a closed vector subspace of E which is invariant under the given representation. Further, $M_\zeta \cap M_{\zeta'} = \{0\}$ if $\zeta \neq \zeta'$. It is thus natural to investigate when E is the direct sum of the M_ζ. For reasons already given, this happens but rarely (perhaps never) when G is non-compact. The contrary case is discussed in some detail in the next section.

§ 6. The case G compact. The functions $s \to \zeta(s)$ are now integrable over G, so that $\zeta * x$ is defined.

THEOREM 4. *If G is compact, $\zeta \in \sigma(x)$ if and only if $\zeta * x \neq 0$.*

Proof. If $\zeta \in \sigma(x)$ and yet $\zeta * x = 0$ one would have by definition $\hat{\zeta}(\zeta) = 0$: this would conflict with the orthogonality relations for characters. Hence $\zeta * x$ must be distinct from 0. Conversely, suppose that $\zeta * x \neq 0$; we wish to show that $\zeta \in \sigma(x)$. Now for any $f \in L^1(G)$, $f * \zeta = \hat{f}(\zeta) \cdot \zeta$ and so $f*(\zeta * x) = (f*\zeta)*x = \hat{f}(\zeta)\cdot\zeta * x$. Thus if $\zeta * x \neq 0$ and $f*(\zeta*x) = 0$, then $\hat{f}(\zeta) = 0$, showing that $\zeta \in \sigma(x)$. The proof of Theorem 4 is complete.

To discuss the possibility of decomposing E into some sort of direct sum of the subspaces M_ζ, it is convenient to introduce the operator-valued Fourier transform of the operator-valued function $s \to A_s$. This transform is the function $\zeta \to \hat{A}_\zeta$ defined by the equation

$$\langle \hat{A}_\zeta x, x' \rangle = \int_G \langle A_s x, x' \rangle \overline{\zeta(s)}\, ds,$$

which is required to hold identically in x and x'. Symbolically:

$$\hat{A}_\zeta = \int_G A_s \overline{\zeta(s)}\, ds.$$

An alternative way of framing this definition amounts to writing $\hat{A}_\zeta x = \zeta * x$. The ζ-th Fourier coefficient of the function $\varphi_{x,x'}$ is just $\langle \hat{A}_\zeta x, x' \rangle$.

This last remark leads, not only to the representation of E as a type of direct sum of the M_ζ, but also to the corresponding expansion theorem. We shall have in fact

$$\langle A_s x, x' \rangle = \varphi_{x,x'}(s) \sim \sum_{\zeta \in \hat{G}} \zeta(s) \cdot \langle \hat{A}_\zeta x, x' \rangle,$$

where the series can be rendered uniformly convergent in s by the insertion of suitable summation factors. More precisely, one can find numerous fixed directed families (S_p) of functions on $\hat{G}$ such that

$$0 \leqslant S_p \leqslant 1, \quad \sum_{\zeta \in \hat{G}} S_p(\zeta) < +\infty, \quad \lim_p S_p(\zeta) = 1,$$

and such that for each x in E one has

$$A_s x = \lim_p \sum_{\zeta \in \hat{G}} S_p(\zeta) \cdot \zeta(s) \cdot \hat{A}_\zeta x$$
$$= \lim_p \sum_{\zeta \in \hat{G}} S_p(\zeta) \cdot \zeta(s) \cdot (\zeta * x),$$

the series converging weakly in E and uniformly with respect to $s \in G$. In particular, for $s = 0$, one obtains

$$x = \lim_p \sum_{\zeta \in \hat{G}} S_p(\zeta) \cdot \hat{A}_\zeta x = \lim_p \sum_{\zeta \in \hat{G}} S_p(\zeta) \cdot (\zeta * x)$$

weakly in E. The operator $\hat{A}_\zeta$ appears as a projection of E onto M_ζ, and we have therefore a direct sum expansion theorem.

Note that the expansion formula, although it shows that each $x \in E$ is the limit of finite linear combinations of elementary elements associated with points $\zeta \in \sigma(x)$, does *not* show that x belongs to the closed invariant convex disk generated by such elements. As we shall see by example in the next section, this latter assertion is generally false even when G is compact. However, we can derive a weaker conclusion.

COROLLARY. *Let G be compact, and let D be any closed invariant disk, $D \neq \{0\}$. Then D contains at least one elementary element other than* 0.

Proof. By Theorem 2, D contains some $x \neq 0$ having a finite spectrum. Theorem 4 then shows that

$$x = \sum_{n=1}^{N} x_n,$$

where x_n is elementary and associated with ζ_n, the ζ_n being the distinct points of $\sigma(x)$. Let d be the distance in $L^\infty(G)$ of ζ_1 from the vector subspace generated by $\zeta_2, \ldots, \zeta_N$; $d > 0$. By the Hahn-Banach Theorem, we can find $\mu \in M(G)$ such that

$$\int_G \zeta_1 d\mu = d, \quad \int_G \zeta_n d\mu = 0 \quad (2 \leqslant n \leqslant N),$$
$$\int_G d|\mu| = 1.$$

Then $z = \mu * x = d \cdot x$ is elementary and associated with ζ_1, and it belongs to D. This completes the proof.

§ 7. An example: G compact, $E = L^2(G)$. The representation operators A_s are here to be translation operators. Since E is now a Hilbert space, the criteria provided by Theorems 2 and 4 of [1] and their analogues become very simple to handle in conjunction with the Parseval formula.

Theorem 3 of the present paper shows at once that the elementary elements of E associated with ζ are precisely the scalar multiples of ζ (considered as a function on G).

THEOREM 5. *Let D be a closed translation invariant disk in $L^2(G)$. Then*

(1) *for given $\zeta \in \hat{G}$, the elementary elements of $L^2(G)$ belonging to D and associated with ζ are precisely those of the form $c \cdot \zeta$, where c is a scalar not greater in modulus than*

$$d(\zeta) \equiv \mathrm{Sup}_{x \in D} |\hat{x}(\zeta)| \quad (\leqslant +\infty); \tag{7.1}$$

(2) *the closed translation invariant disk in $L^2(G)$ generated by the elementary elements belonging to D comprises those and only those $y \in L^2(G)$ such that*

$$\sum_{\zeta \in \hat{G}} |\hat{y}(\zeta)|/d(\zeta) \leqslant 1. \tag{7.2}$$

Remark. In the left hand member of (7.2) we adopt the conventions: $a/(+\infty) = 0$ if $a \geqslant 0$, $a/0 = +\infty$ if $a > 0$, and $0/0 = 0$.

Proof. This follows directly from Theorem 2 of [1], modified so as to apply to disks rather than to convex sets, together with the Parseval formula.

If D_0 denotes the disk defined in (2), we see that $D = D_0$ only in the extreme case in which D is generated by a single character ζ. Further, if D is bounded, D_0 contains only functions with absolutely convergent Fourier series.

Theorem 5 also renders it a simple matter to construct examples of distinct pairs D, D' of closed translation invariant disks which contains precisely the same multiples of characters. For example, the pair D = unit ball in $L^2(G)$, D' = the closed translation invariant disk generated by all characters. This lays to rest any hopes of characterizing completely invariant disks by multiples of characters they contain — a characterization which is well known to be effective for closed invariant vector subspaces.

§ 9. Remarks on the non-commutative case. The preceding sections have made frequent essential use of the assumption that G is commutative. It is natural to ask to what extent this assumption may be dispensed with. The answer would appear to be that the difficulties become very great when G is both non-commutative and non-compact, and we shall here confine our remarks to the compact case. Even here there are defects in the available theory of harmonic analysis for such groups which are serious enough to upset some of the results proved above in the commutative case. Some of these defects will be indicated below.

We assume henceforth that G is compact and choose the Haar measure so that G has total mass 1.

If the representation $t \to A_t$ is commutative, nothing essentially novel appears. The set of t for which $A_t = I$ is a closed normal subgroup G_0 of G, G/G_0 is commutative, and there is an obvious induced representation of G/G_0. This case may, therefore, be dismissed.

It is now better to make a slight change in notation, writing the abstract convolution

$$\int A_t x \cdot d\mu(t)$$

as $x * \mu$ (rather than $\mu * x$); we also adopt the multiplicative notation for G to stress its non-commutativity. The analogue of (2.2) stands in the new notation.

As we shall see in due course, a happy definition of spectrum for functions in $L^\infty(G)$ and for elements of E is by no means obvious. Nevertheless, we can define the meaning of "compact spectrum" in an entirely satisfactory way, which we shall now explain.

Introduce the set Γ of normalised, elementary, continuous, positive-definite functions on G: these turn out to be the building-bricks for harmonic synthesis, in much the same way as are the characters in the commutative case. (In the non-commutative case the characters are adequate only for the synthesis of *central* functions). Γ may be regarded as a subset of $L^\infty(G)$ and equipped with the topology induced by the weak topology of $L^\infty(G)$. Γ is then locally compact; it is discrete if and only if G is commutative. The spectrum of a function in $L^\infty(G)$, whatever definition is adopted, will be a subset of Γ: the meaning of "compact spectrum" is thus quite definite. The relationship between elements of Γ and equivalence classes of continuous, irreducible unitary representations of G will be noted below, and it will appear that a subset of Γ is relatively compact if it is associated with (or derived from) only a *finite* number of inequivalent such representations of G.

It is convenient at this stage to record that *the main result* (Theorem 2) *remains valid, even with the widest of several possible definitions of $\sigma(x)$ described below*. The proof of this is basically the same as before, but the details depend on the use of the ζ's in the Fourier expansion of functions, the basic facts about which will now be set forth.

Select arbitrarily one representative from each equivalence class of continuous, irreducible unitary representations of G, and denote by $\mathcal{R}$ the set of elected candidates. A member of $\mathcal{R}$ is thus a representation U: $t \to U(t)$, $U(t)$ being a unitary operator on some Hilbert space $\mathcal{H}(U)$ of finite dimension $d(U)$ $(=1, 2, \ldots)$. Each function $f \in L^1(G)$ admits an operator-valued Fourier transform $\hat{f}$ defined by

$$\hat{f}(U) = \int f(t)\, U(t)\, dt \quad (U \in \mathcal{R}); \tag{9.1}$$

$\hat{f}(U)$ is thaus an endomorphism of $\mathcal{H}(U)$. An entirely analogous definition applies if f is replaced by a bounded measure μ. Note that $\hat{f}(U) = \hat{f}(U)^*$ ($* =$ adjoint), and that $(f*g)\hat{}(U) = \hat{f}(U)\hat{g}(U)$ for each $U \in \mathcal{R}$. The character χ_U of the representation U may be defined by

$$\chi_U(t) = \operatorname{Tr} U(t)^*, \tag{9.2}$$

where Tr signifies the trace defined for endomorphisms of $\mathcal{H}(U)$.

The Peter-Weyl theory of the Fourier expansion in $L^2(G)$ (see Weil [10], Ch. V) may be summarised as follows. The formal Fourier series of f, namely

$$f(t) \sim \sum_{U \in \mathcal{R}} d(U) \cdot \operatorname{Tr}[\hat{f}(U)\, U(t)^*], \tag{9.3}$$

is convergent in $L^2(G)$ (according to the increasing directed family of finite subsets of $\mathcal{R}$) whenever $f \in L^2(G)$, and one has the Parseval formulae

$$\int |f(t)|^2 dt = \sum_{U \in \mathcal{R}} d(U) \operatorname{Tr}[\hat{f}(U)\hat{f}(U)^*], \tag{9.4}$$

$$\int f(t)\overline{g(t)}\, dt = \sum_{U \in \mathcal{R}} d(U) \operatorname{Tr}[\hat{f}(U)\hat{g}(U)^*], \tag{9.4'}$$

whenever $f, g \in L^2(G)$. Note that $\operatorname{Tr}(TT^*) \geqslant 0$ for any endomorphism T, so that the series on the right of (9.4) consists of positive terms. Note also that (9.3) can be written in the equivalent form

$$f(t) \sim \sum_{U \in \mathcal{R}} d(U) \cdot f * \chi_U(t), \tag{9.3'}$$

involving explicitly the group characters.

Turning to positive-definite functions, it is quite easily shown (using the Parseval formulae) that a function $p \in L^1(G)$ is positive-definite (i. e. $f*p*\tilde{f}(e) \geqslant 0$ for arbitrary f, $e =$ neutral element of G) if and only if $\hat{p}(U)$ is a positive self-adjoint endomorphism of $\mathcal{H}(U)$ for each U; moreover, if p is continuous, then

$$\sum_{U \in \mathcal{R}} d(U) \operatorname{Tr}[\hat{p}(U)] < +\infty \tag{9.4}$$

and

$$p(t) = \sum_{U \in \mathcal{R}} d(U) \operatorname{Tr}[\hat{p}(U)\, U(t)^*], \tag{9.5}$$

the Fourier series of p, is absolutely and uniformly convergent. These statements are analogues of the famous Bochner Theorem for commutative groups, and the correspondence is made even closer when the right hand side of (9.5) is shown to be expressible as a sum, with positive coefficients, of elementary, continuous positive-definite functions. We recall that a continuous positive-definite function p is said to be *elementary* if the only functions p' of the same category such that $p-p'$ is also positive-definite, are scalar multiples of p. p is said to be *normalised* if $p(e) = 1$. This being so, (9.5) shows almost at once that the normalised, elementary, continuous positive-definite functions ζ (i. e. the elements of Γ) are precisely those of the form

$$\zeta(t) = \operatorname{Tr}[P\, U(t)^*], \tag{9.6}$$

where $U \in \mathcal{R}$ and P is a one-dimensional projector on $\mathcal{H}(U)$. This being so, if we return to (9.5), recall that $\hat{p}(U)$ is positive self-adjoint and apply the spectral decomposition to each $\hat{p}(U)$, we see that (9.5) can be written as

$$p(t) = \sum_i c_i \cdot \zeta_i(t), \tag{9.5'}$$

where $\zeta_i \in \Gamma$, $c_i \geqslant 0$, and $\sum_i c_i = p(e) < +\infty$ (the series being therefore absolutely and uniformly convergent): this is an almost complete analogue of Bochner's theorem, "almost" because there is in the non-commutative case an inevitable lack of uniqueness in the expansion, due to the fact that the ζ's lack the independence of the characters in the commutative case (due there to the orthogonality relations); this uniqueness can be restored only for central functions and expansions thereof in terms of the χ_U.

If we denote by Γ_U the set of Γ obtained via (9.6), when U is fixed and P varies, then Γ is partioned into subsets Γ_U. Also, as is easily seen, Γ_U is both open and compact in Γ. Moreover, Γ_U is discrete if and only if $d(U) = 1$. Confirmation of a statement made earlier is now forthcoming: a subset of Γ is relatively compact if and only if it is contained in finitely many Γ_U.

The remainder of this section is devoted to remarks concerning the concept of spectrum. When speaking of functions, we shall for definiteness deal with the space $L^\infty(G)$ with its weak topology. But the results apply to $L^p(G)$ ($p \neq \infty$) with its normed topology. It turns out that there is little difficulty in defining satisfactorily the spectrum of a *normal* function φ, i. e. one for which $\varphi * \tilde{\varphi} = \tilde{\varphi} * \varphi$. The choice of the name results from the fact that φ is normal if and only if $\hat{\varphi}(U)$ is a normal endomorphism of $\mathcal{H}(U)$ for each U, and the relatively satisfactory spectral theory for such functions owes its existence almost entirely to the spectral decomposition of normal endomorphisms.

We shall define the left (right) spectrum $\sigma_L(\varphi)(\sigma_R(\varphi))$ of φ, normal or not, to be the set of $\zeta \epsilon \Gamma$ which are limits of finite linear combinations of left (right) translates of φ. Both these spectra are closed subsets of Γ. If ζ is given by (9.6), then $\zeta \epsilon \sigma_L(\varphi)(\sigma_R(\varphi))$ if and only if P is a left (right) multiple of $\hat{\varphi}(U)$ (for the particular $U \epsilon \mathcal{R}$ appearing in (9.6)). Let us define further $\sigma_0(\varphi)$ to be the set of $\zeta \epsilon \sigma_L(\varphi) \cap \sigma_R(\varphi)$ which satisfy

(9.7) $$\zeta * \varphi = \varphi * \zeta = c \cdot \zeta \quad (c = \mathrm{const} \neq 0).$$

Then the spectral resolution theorem for normal endomorphisms shows that: *if φ is normal, it is the limit of "trigonometric polynomials"*

(9.8) $$\theta = \sum_i c_i \zeta_i \quad (\textit{finite sum}),$$

where $\zeta \epsilon \sigma_0(\varphi)$ and $\zeta_i * \zeta_j = 0$ whenever $i \neq j$.

It is also worth noting that: *if φ is normal and $\zeta \epsilon \Gamma$ satisfies $\zeta * \varphi = \varphi * \zeta \neq 0$, then $\zeta \epsilon \sigma_0(\varphi)$; and that if φ is positive-definite, this latter condition is equivalent to $\zeta * \varphi = \varphi * \zeta$ and*

$$\int \varphi(t) \overline{\zeta(t)}\, dt \neq 0.$$

These results break down completely for non-normal functions.

We now turn to the proof of the cited analogue of Theorem 2, which has been delayed until now in order to make clearer the difficulties associated with the concept of spectrum in the non-commutative case. It is natural to proceed in three steps:

(a) Taking the cue from Theorem 4 for the commutative case, we define $\sigma(x)$ to be the closure in Γ of the set of ζ satisfying $x * \zeta \neq 0$. (Other definitions might be chosen).

(b) If μ is a bounded measure on G, and if S is the closure in Γ of the set of ζ satisfying $\mu * \zeta \neq 0$, then $\sigma(x * \mu) \subset S$. The proof is immediate. Equally clear is the inclusion $\sigma(x * \mu) \subset \sigma(x)$, provided μ is central. These assertions are weak analogues of Proposition 2 (i).

(c) The construction of functions p_i on G is analogous to those in Proposition 3. It is enough to show that the p_i may be chosen so that p_i is positive, continuous, such that $\int_G p_i(t)\, dt = 1$, $\hat{p}_i(U) = 0$ for all but a finite set of $U \epsilon \mathcal{R}$, whilst $\lim_i p_i = \delta$ in a suitable sense. Apart from the penultimate condition, such functions are shown to exist by Weil ([10], p. 85-86); and to satisfy the remaining restriction, it suffices merely to approximate each p_i sufficiently closely in $L^2(G)$ by a finite partial sum of its Fourier series. It may be noted that we can further arrange that each p_i is positive-definite, by replacing p_i by $p_i * \tilde{p}_i$ if necessary. When this is done the p_i act well as summation kernels for Fourier series, this being in fact the use contemplated by Weil.

There is one further respect in which non-commutativity involves inevitable complications. In the commutative case we have made essential use of the fact that one can find functions $f \epsilon L^1(G)$ whose Fourier transforms $\hat{f}$ take the value 1 at a given point ζ and vanish outside any pre-assigned neighbourhood N of ζ. If G is non-commutative, such functions will not exist for general ζ and N, $\hat{f}$ being now defined as a function on Γ by the equation

$$\hat{f}(\zeta) = \int f(t) \overline{\zeta(t)}\, dt,$$

which is formally identical with the definition in the commutative case.

The source of this difficulty becomes apparent if we restrict attention (as we may) to the behaviour of f on a typical component Γ_U of Γ. If U is fixed, Γ_U is homeomorphic with the unit sphere in $\mathcal{H}(U)$ after identifying points which are scalar multiples of each other. In other words, Γ_U is essentially a complex projective space of dimension $d-1$ ($d = d(U)$). Coordinates z_i in $\mathcal{H}(U)$, relative to some chosen orthonormal base, yield "homogeneous coordinates" in Γ_U, and $\hat{f}$ is then expressible as a sesquilinear form

$$\sum_{i,j=1}^{d} a_{ij} z_i \bar{z}_j$$

with coefficients a_{ij} depending upon f. It is clear that if $d > 1$ such a sesquilinear form cannot vanish outside small neighbourhoods without vanishing identically.

These considerations make it clear that, in order to salvage results like Proposition 2 (ii), it would be necessary to define $\sigma(\varphi)$ in terms, not of single points of Γ_U, but of subspaces of Γ_U considered as a projective space.

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Ergodische Funktionale und individueller ergodischer Satz

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$(S, \mathfrak{B}, m)$, $m(S) = 1$, sei ein festgesetzter Maßraum und T eine meßbare nichtsinguläre Transformation von S in S (es ist also $T^{-1}B \epsilon \mathfrak{B}$, wenn $B \epsilon \mathfrak{B}$, und $m(T^{-1}B) = 0$, wenn $m(B) = 0$). Der Körper von meßbaren und T-invarianten Mengen soll mit $\mathfrak{B}_T$ oder genauer mit $\mathfrak{B}_T(m)$ bezeichnet sein ($B \epsilon \mathfrak{B}_T$ wenn $B \epsilon \mathfrak{B}$ und $m(T^{-1}B \dot{-} B) = 0$), und der Raum von linearen Kombinationen der charakteristischen Funktionen der Mengen aus $\mathfrak{B}$ mit $X(\mathfrak{B})$.

Es ist bekannt, daß man die Voraussetzung der Invarianz des Maßes in dem individuellen ergodischen Satze weit schwächen kann [2], [3]. Für die individuelle Konvergenz m-fast überall (weiter auch m-f. ü. oder $[m]$ bezeichnet) genügt es z. B., wenn es ein solches K gibt, daß

$$\lim_n \frac{1}{n} \sum_{k=0}^{n-1} m(T^{-k}B) \leqslant Km(B), \qquad B \epsilon \mathfrak{B}.$$

Ebenso, kann man solche Voraussetzungen durch andere ersetzen, z. B. durch starke Konvergenz der ergodischen Mitteln

$$f_n(s) = \frac{1}{n} \sum_{k=0}^{n-1} f(T^k s)$$

in $L^r(m)$, $r \geqslant 1$ [2].

In dieser Arbeit ist die Bedeutung aufgeklärt, welche bei solchen Sätzen die hier ergodisch genannten Funktionale, besitzen. Die Existenz eines solchen Funktionals auf $L^1(m)$ ist mit der f. ü.-Konvergenz von $f_n \to f^* \epsilon L^1(m)$ gleichbedeutend. Daraus folgt sofort, daß die individuelle f. ü.-Konvergenz nicht nur eine Konsequenz der starken, sondern auch der schwachen und dabei nach einem einzigen Funktionale $\int \cdot\, dm$ ist. Wie bekannt [2], umgekehrt verursacht die f. ü. -Konvergenz noch nicht die starke. Es entsteht die Frage ob dann wenigstens die Konvergenz der Integrale $\int f_n dm$ folgt. Dann wäre $\lim \int f_n dm$ ein natürliches ergodisches Funktional. Beispiel 3 (in 3) zeigt, daß im allgemeinen dies auch nicht stattfindet.