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Remarks on the algebraic derivative in the Operational Calculus

by

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1. Let $\mathcal{C}$ be the ring of continuous complex-valued functions in $0 \leqslant t < \infty$ with ordinary addition and convolution as multiplication. The formal quotients p/q, where $p, q \in \mathcal{C}$, $q \neq 0$, will be called *operators*. The theory of those operators is widely developed in my book *Operational Calculus* (¹).

The algebraic derivative is defined as follows:

$$Df = D\{f(t)\} = \{-tf(t)\} \quad \text{for} \quad f \in \mathcal{C},$$

$$D\left(\frac{p}{q}\right) = \frac{Dp \cdot q - p \cdot Dq}{q^2} \quad \text{for} \quad p, q \in \mathcal{C}, \quad q \neq 0.$$

This derivative has the following properties (see op. cit., p. 261-263):

$$D(a \pm b) = Da \pm Db), \qquad D(a \cdot b) = Da \cdot b + a \cdot Db,$$

$$D\left(\frac{a}{b}\right) = \frac{Da \cdot b - a \cdot Db}{b^2} \quad (b \neq 0), \qquad D(\lambda a) = \lambda Da,$$

where a, b are operators and λ a number. Moreover we have

$$D(a_n s^n + \ldots + a_1 s + a_0) = n a_n s^{n-1} + \ldots + a_1, \tag{1}$$

which suggests that the operation D be considered as derivation with respect to the differential operator s:

$$D = \frac{d}{ds}.$$

The purpose of this paper is to prove some further fundamental properties of D.

2. A particular case of (1) is $Da = 0$, where a is a number. We shall prove the converse.

(¹) See J. Mikusiński, *Operational calculus*, Pergamon Press 1959.

PROPERTY I. *$Dx=0$ implies $x=$ number.*

Proof. Put $x=p/q$, where $p,q\in\mathcal{C}$, $q\neq 0$. From $Dx=0$ we obtain

$$Dp\cdot q-p\cdot Dq=0, \tag{2}$$

and hence

$$D^2\cdot p\cdot q-p\cdot D^2q=0. \tag{3}$$

Since $q\neq 0$, we can eliminate p and q from (2) and (3):

$$D^2p\cdot Dq-Dp\cdot D^2q=0.$$

Differentiating (3), we get by the last equality

$$D^3p\cdot q-p\cdot D^3q=0. \tag{4}$$

Eliminating p and q from (2) and (4) we have

$$D^3p\cdot Dq-Dp\cdot D^3q=0.$$

Thus we obtain on differentiating (4)

$$D^4p\cdot q-p\cdot D^4q=0.$$

Generally we have

$$D^np\cdot q-p\cdot D^nq=0 \qquad (n=1,2,\ldots).$$

which can be written in ordinary symbols as

$$\int_0^t(-\tau)^np(\tau)q(t-\tau)d\tau-\int_0^t p(t-\tau)(-\tau)^nq(\tau)d\tau=0$$

or

$$\int_0^t\tau^n[p(\tau)q(t-\tau)-p(t-\tau)q(\tau)]d\tau=0 \qquad (n=1,2,\ldots).$$

Hence by Lerch's theorem on moments

$$p(\tau)q(t-\tau)-p(t-\tau)q(\tau)=0 \quad \text{for} \quad 0\leqslant\tau\leqslant t<\infty$$

or

$$p(\tau)q(\sigma)-p(\sigma)q(\tau)=0 \quad \text{for} \quad \tau\geqslant 0 \text{ and } \sigma\geqslant 0.$$

Since $q\neq 0$, there is a value of σ such that $q(\sigma)\neq 0$. Thus

$$p(\tau)=aq(\tau) \quad \text{for} \quad 0\leqslant\tau<\infty,$$

where $a=\dfrac{p(\sigma)}{q(\sigma)}$, and the theorem is proved.

The theorem can also be formulated in the following form:

The condition

$$\int_0^t(t-2\tau)f(t-\tau)g(\tau)d\tau=0 \quad \text{for} \quad 0\leqslant t<\infty \tag{5}$$

is sufficient and necessary for the functions f and g to be linearly dependent in $0\leqslant t<\infty$.

In fact, it suffices to remark that (5) is a non-operational form of (2).

3. An operational function (function whose values are operators) $x(\lambda)$ is said to be *continuously derivable in $\lambda_1\leqslant\lambda\leqslant\lambda_2$* if it can be represented in the form

$$x(\lambda)=\frac{\{p(\lambda,t)\}}{\{q(t)\}}, \tag{6}$$

where $q\in\mathcal{C}$ and $p(\lambda,t)$ is a continuous function in $\lambda_1\leqslant\lambda\leqslant\lambda_2$, $0\leqslant t<\infty$, with a continuous derivative $\dfrac{\partial p(\lambda,t)}{\partial\lambda}$. The derivative of $x(\lambda)$ is given by the formula

$$x'(\lambda)=\frac{\left\{\dfrac{\partial p(\lambda,t)}{\partial\lambda}\right\}}{\{q(t)\}}.$$

PROPERTY II. *If $x(\lambda)$ is a continuously derivable operational function, we have $Dx'(\lambda)=\big(Dx(\lambda)\big)'$, i. e.*

$$\frac{d}{ds}\frac{d}{d\lambda}x(\lambda)=\frac{d}{d\lambda}\frac{d}{ds}x(\lambda).$$

Proof. The formula is trivially true if $x(\lambda)$ is a parametric function (i. e. its values are in $\mathcal{C}$), continuously derivable:

$$x(\lambda)=\{x(\lambda,t)\} \qquad \left(\frac{\partial x(\lambda,t)}{\partial\lambda}\ \text{continuous}\right);$$

in fact

$$-t\frac{\partial}{\partial\lambda}x(\lambda,t)=\frac{\partial}{\partial\lambda}\big(-tx(\lambda,t)\big).$$

If $x(\lambda)$ is an arbitrary continuously derivable operational function, write

$$x(\lambda)=\frac{p(\lambda)}{q},$$

which is an abbreviated form of (6). Then

$$Dx'(\lambda) = D\left(\frac{p'(\lambda)}{q}\right) = \frac{1}{q^2}\big(Dp'(\lambda)\cdot q - p'(\lambda)\cdot Dq\big)$$
$$= \frac{1}{q^2}\big([Dp(\lambda)]'\cdot q - p'(\lambda)\cdot Dq\big) = \frac{1}{q^2}\big(Dp(\lambda)\cdot q - p(\lambda)\cdot Dq\big)'$$
$$= \left(\frac{Dp(\lambda)}{q}\right)' = \big(Dx(\lambda)\big)'.$$

4. The operator e^w is defined as the value at $\lambda = 1$ of the solution $x(\lambda) = e^{w\lambda}$ of the differential equation

$$(7) \qquad x'(\lambda) = wx(\lambda)$$

such that $x(0) = 1$. Thus the operator e^w exists for a given operator w if and only if there is a non-vanishing solution of (7).

PROPERTY III. *If the operator e^w exists, we have*

$$De^w = e^w\cdot Dw.$$

Proof. By (7) we have

$$Dx'(\lambda) = Dw\cdot x(\lambda) + w\cdot Dx(\lambda).$$

If we put

$$y(\lambda) = Dx(\lambda)$$

it follows that $y'(\lambda) = Dx'(\lambda)$ and therefore

$$(8) \qquad y'(\lambda) - wy(\lambda) = Dw\cdot x(\lambda).$$

In order to find $y(\lambda)$ we consider (8) as a differential equation with the unknown function $y(\lambda)$, the function $x(\lambda) = e^{w\lambda}$ being known. Put

$$y(\lambda) = c(\lambda)\cdot x(\lambda);$$

substituting that expression into (8), we obtain

$$c'(\lambda)x(\lambda) = Dw\cdot x(\lambda).$$

Since $x(\lambda) = e^{w\lambda}$ is different from 0, this implies $c'(\lambda) = Dw$ and consequently $c(\lambda) = c + \lambda Dw$, where c is a constant operator. The function

$$y(\lambda) = (c + \lambda Dw)x(\lambda)$$

is the general solution of (8). To determine c, it suffices to remark that

$$y(0) = Dx(0) = D1 = 0.$$

Thus $c = 0$ and, eventually,

$$y(\lambda) = \lambda Dw\cdot x(\lambda).$$

In particular, we have $y(1) = Dw\cdot x(1)$, which proves the theorem.

Properties I and III yield a simple proof [2] that $2k\pi$ (k integer) are the only operators such that $e^w = 1$ (see op. cit., p. 192). In fact, $e^w = 1$ implies $e^w\cdot Dw = 0$ and consequently $Dw = 0$. Thus w is a number and the assertion follows.

PROPERTY IV. *If the equation*

$$(9) \qquad Dx = wx$$

is solvable, its solution is determined up to a numerical factor.

Proof. Let $Dx = wx$ and $Dy = wy$; then $Dx\cdot y - x\cdot Dy = 0$. If $x \neq 0$, it follows that $D\left(\dfrac{y}{x}\right) = 0$ and $\dfrac{y}{x} = a$. If there is no solution $\neq 0$, the assertion is trivially true.

As a Corollary to Property IV, we have

PROPERTY V. *If $Du = w$ and the equation $Dx = wx$ is solvable, every solution is of the form ae^u, a being a number.*

5. We have

$$\frac{d}{ds}\{e^{\omega t^2}\} = \{-te^{\omega t^2}\} \qquad (\omega \text{ number})$$

and

$$s\{e^{\omega t^2}\} = 2\omega\{te^{\omega t^2}\} + 1.$$

Thus the operator $x = \{e^{\omega t^2}\}$ satisfies the equation

$$(10) \qquad \frac{d}{ds}x + \frac{s}{2\omega}x = \frac{1}{2\omega}.$$

It is easy to show that $x = \{e^{\omega t^2}\}$ is the only solution of (10). In fact, suppose that y is another solution. Then the function $z = x - y$ satisfies the homogeneous equation

$$\frac{d}{ds}z + \frac{s}{2\omega}z = 0.$$

If $z \neq 0$, it should be of the form $z = ae^{-s^2/4\omega}$ ($a \neq 0$); but the operator $e^{-s^2/4\omega}$ does not exist (see op. cit., p. 410). Consequently, z must be 0, which proves the assertion. Equation (10) determines the operator $\{e^{\omega t^2}\}$ completely.

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[2] This remark is due to C. Ryll-Nardzewski.

6. Now, let us consider s as a complex variable and x as a function of s. Then equation (10) can easily be solved; we obtain

$$x = ae^{-s^2/4\omega} + \frac{1}{2\omega}\int_0^1 e^{-s^2(1-\sigma^2)/4\omega}\,d\sigma, \tag{11}$$

where a is a number.

In this manner, the class of analytic functions (11), depending on the parameter a, is assigned to $e^{\omega t^2}$. In the case of a non-positive real part of ω, we can determine a in such a way that x is the Laplace transform of $e^{\omega t^2}$. If the real part is positive, the Laplace transform does not exist. In that case another condition may be used to determine a, e. g. that x should be an odd function of s.

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The form of the solution of the Cauchy Problem over a group

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§ 0. Introduction and summary. Apart from certain extensions to be discussed briefly in § 6, we shall for definiteness concentrate on what may be termed the "parabolic case" of the Cauchy Problem suggested by the heat (or diffusion) equation. The Cauchy Problem to be considered is therefore of the type

$$Du_t = \dot{u}_t \qquad (t > 0),$$

the dot indicating differentiation with respect to t, with the initial condition

$$\lim_{t\to+0} u_t = f.$$

Herein u_t is, for each $t > 0$, an element of some pre-assigned locally convex space $\mathcal{E}$ of real-valued functions on a group X, D is a given endomorphism of $\mathcal{E}$, and the initial "data" f is an element of a second pre-assigned locally convex space $\mathcal{D}$ of real-valued functions on X. We assume: (i) the existence and uniqueness of the solution; (ii) certain simple properties of the mapping $f \to u_t$; (iii) a few conditions of a very general nature concerning $\mathcal{D}$ and $\mathcal{E}$; and (iv) the crucial condition that D commutes with right translations. From this we deduce that the solution is necessarily of the form

$$u_t = \mu_t * f,$$

each μ_t being a Radon measure on X. There are close connections between the results established below and those set forth by Hille ([4], p. 400-410), whose main aim is to exhibit the relations between solutions of the Cauchy Problem and the theory of semigroups. By comparison the present method is in some senses more general, uses fewer special assumptions, and accords to the convolution a more fundamental role.

The proof of the main theorem, which is given immediately after the hypotheses have been set forth at length in § 1, occupies § 2. The method is suggested by arguments used elsewhere (Edwards [2], [3]).