T_t would commute with each R_a. However, T_t would not be bilinear. This feature suggests falling back on an approach which is a priori very natural, and which reduces the problem to two partial problems each similar to that dealt with in §§ 1-5 above. It is merely necessary to decompose u_t into a sum

$$u_t = v_t + v_t',$$

where $v_t = T_t(f, 0)$ is the solution of (6.3) and (6.4) with $f' = 0$, and $v_t' = T_t(0, f')$ is the solution of the same equations with $f = 0$. The mappings

$$S_t: \mathcal{D} \to \mathcal{E}, \qquad S_t f = T_t(f, 0) = v_t,$$
$$S_t': \mathcal{D}' \to \mathcal{E}, \qquad S_t' f' = T_t(0, f') = v_t'$$

are each linear and commute with right-translations. Each may be treated by the methods of §§ 1-5. The resulting representation theorem will be of the form

$$u_t = \mu_t * f + \mu_t' * f',$$

where μ_t and μ_t' are bounded Radon measures. In general μ_t and μ_t' will not be positive measures: this will depend partly on X and partly on the boundary conditions (if any), and also of course on the nature of D.

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Projections in certain Banach spaces

by

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It is a well known fact that every finitely dimensional subspace Y of a B-space X is complemented (1) in X. In general this property of finitely dimensional subspaces does not remain valid for subspaces of an infinite dimension. The first example of a subspace Y which has no complement in $C[0, 1]$ was due to Banach and Mazur [2]. Further examples of non-complemented subspaces in L_p, l_p $(1 \leqslant p \neq 2)$, c_0, m, M and in other spaces were given by Murray [25], Sobczyk [30], Philips [29] and Komatuzaki [21], [22]. In many cases the fact that a subspace Y is not complemented in a B-space X depends only on the isomorphic properties of X and Y. For example: no reflexive infinitely dimensional subspace of $C[0, 1]$ has a complement (Grothendieck [16]); in an arbitrary B-space X each subspace isomorphic to a space $C(S)$, where S is a topological compact Hausdorff space extremally disconnected, has a complement (Nachbin [26], Goodner [15]).

Hence the following two problems arise naturally:

1° Given a B-space X, characterize the isomorphic types of complemented subspaces of X.

2° Given a B-space X, characterize the isomorphic types of such B-spaces Z that every subspace of Z isomorphic to X is complemented in Z.

In section 2 of this paper we prove (Theorem 1) that every infinitely dimensional subspace complemented in l_p $(p \geqslant 1)$ or c_0 is isomorphic to l_p or c_0 respectively. We do not know whether the space m has the same property. Partial results in this direction are given in section 4 (Corollaries 7-9). In section 3 we consider the reciprocal problem. We prove that, if Y is a subspace of c_0, or l_2, or s, or m, or l_p $(1 \leqslant p \neq 2)$ and Y is isomorphic to c_0, l_2, s, m, or is isometrically isomorphic to l_p $(1 \leqslant p \neq 2)$, then Y has a complement.

(1) For the terminology and notation see section 1.

The main result of section 4 states that every linear operator from a space $C(S)$ into an arbitrary B-space X, no subspace of which is isomorphic to c_0, is weakly compact. Applying this fact we obtain some necessary conditions, expressed in terms of isomorphic invariants, for a subspace Y to have a complement in $C(S)$. These results suggest positive solutions of the following problems:

P_1. Is every complemented subspace of $C(S)$ isomorphic to a space $C(S_1)$ (S and S_1 being topological compact Hausdorff spaces)?

P_2. Let X be isomorphic to an abstract L-space and Y be complemented in X. Is Y isomorphic to an abstract L-space?

P_3. Let X be a B-space such that X is complemented in each space in which it is embedded. Is X isomorphic to a space $C(S)$, where S is a topological compact Hausdorff space extremally disconected?

In section 5 we apply the results of section 2 to the investigation of unconditional bases in L_p and l_p.

1. We intend to preserve the notation and terminology of the treatise of Dunford-Schwartz [9]. In particular, the symbols m, c_0, l, L, L_p, l_p and l_p^n for $p>1$ $(n=1,2,\ldots)$, $C[0,1]$ and $C(S)$ have the same meaning as in [9], Chapter IV.

We consider also the F-space s of all real sequences $x=(t_n)$ with the F-norm

$$\|x\| = \sum_{n=1}^{\infty} 2^{-n}|t_n|(1+|t_n|)^{-1}.$$

Let X be an F-space. The term "subspace" of X always means a closed linear manifold in X. Let (x_n) be a sequence in X; by $[x_n]$ we denote the smallest subspace of X spanned on elements $x_1, x_2, \ldots$ By $[x_1, x_2, \ldots, x_n]$ we denote the smallest subspace spanned on elements $x_1, x_2, \ldots, x_n$.

Let E be an F-space consisting of real sequences. We use the symbol e_n $(n=1,2,\ldots)$ to denote the n-th "unit vector" $(0,0,\ldots,1,0,\ldots)$ in E.

Let X and Y be F-spaces. We shall write $X\sim Y$ if the spaces X and Y are *isomorphic*. If X and Y are B-spaces and there exist an isomorphism T from X onto Y, and a constant $k\geqslant 1$ such that

$$\|x\| \leqslant \|Tx\| \leqslant k\|x\|, \quad x \in X, \tag{1}$$

then we shall write $X \overset{k}{\sim} Y$. In particular, to express the fact that X and Y are *isometrically isomorphic* we shall write $X \overset{1}{\sim} Y$.

The symbol $X\oplus Y$ denotes the *product* of F-spaces X and Y, i. e. the space Z of all pairs (x,y), where $x\in X$ and $y\in Y$ with the F-norm $\|(x,y)\| = (\|x\|^2+\|y\|^2)^{1/2}$.

Let E be an F-space consisting of real sequences with the F-norm $\|\cdot\|_E$ satisfying the condition

(*) if $(t_i)\in E$ and $|s_i|\leqslant |t_i|$ $(i=1,2,\ldots)$, then $(s_i)\in E$ and $\|(t_i)\|_E \geqslant \|(s_i)\|_E$.

Let $X_1, X_2, \ldots$ be F-spaces with F-norms $\|\cdot\|_{X_1}, \|\cdot\|_{X_2}, \ldots$ respectively. By $(X_1\oplus X_2\oplus\ldots)_E$ we denote the space of all sequences (x_i) where $x_i\in X_i$ $(i=1,2,\ldots)$ such that $(\|x_i\|_{X_i})\in E$ with the norm $\|(x_i)\| = \|(\|x_i\|_{X_i})\|_E$. The definition of finite product $(X_1\oplus X_2\oplus\ldots\oplus X_n)_E$ is analogous. It is easily seen that $(X_1\oplus X_2\oplus\ldots)_E$ is an F-space. Moreover if $E, X_1, X_2, \ldots$ are B-spaces, then $(X_1\oplus X_2\oplus\ldots)_E$ is also a B-space.

Let X be an F-space. A subspace Y of X is said to be *complemented* in X (to have a *complement in* X) if there is a subspace Y_1 (a *complement to* Y) such that for each x in X there exist a y in Y and a y_1 in Y_1 such that $x=y+y_1$ and if $0=y+y_1$, then $y=0$ and $y_1=0$.

We recall (²) that:

A. *The subspace Y is complemented in X if and only if there is a projection (i. e. a linear idempotent operator) from X onto Y.*

B. *If Y has a complement Y_1 in X, then $X\sim Y\oplus Y_1$.*

Let X be a B-space. The symbol $X\in\mathfrak{P}_1$ denotes that X is complemented in each B-space Y which contains X as a subspace. We write $X\in\mathfrak{P}_1$ if for each space Y which contains X as a subspace there is a projection P from Y onto X with the norm $\|P\|=1$.

In the sequel we shall need a few propositions.

PROPOSITION 1. *Let X and Y_1 be B-spaces and let X contain a subspace Y isomorphic to Y_1. Then there are a B-space X_1 and an isomorphic mapping T from X onto X_1 such that the subspace $T(Y)$ is isometrically isomorphic to Y_1.*

Proof. We define the space X_1 as the set of elements of X with the same operations of addition and of multiplication by scalars, under the norm defined as the Minkowski functional of a suitable convex set W. Namely, we set

$$W = \mathrm{co}\left(\{y\in Y: \|Uy\|\leqslant 1\} \cup \left\{x\in X: \|x\|\leqslant \frac{1}{K}\right\}\right) (^3),$$

where U is an isomorphism from Y onto Y_1 and K is a positive constant such that $\|y\|\leqslant \|U(y)\|\leqslant K\|y\|$, $y\in Y$ and

$$\|x\|_1 = \inf\left\{\lambda>0; \frac{x}{\lambda}\in W\right\}.$$

(²) For the proofs see [8], p. 480-482.

(³) By co(A) we denote the intersection of all convex sets containing a given set A.

We leave it to the reader to show that the functional $\|\cdot\|_1$ is a norm, X_1 under this norm is a B-space and the identity operator T $(Tx = x,$ $x \in X)$ is the required isomorphism, q. e. d.

From Proposition 1 and the fact that the notion of complement is an isomorphic invariant we obtain.

PROPOSITION 2. *If $Y_1 \in \mathfrak{P}$ and $Y \sim Y_1$, then $Y \in \mathfrak{P}$.*

PROPOSITION 3. *Let E be one of the spaces s, l_p where $1 \leqslant p < +\infty$, c_0, m. Then*

a) $(E \oplus E \oplus \ldots)_E \sim E$,

b) *for each of the F-spaces X and Y*

$$\big((X\oplus Y)\oplus(X\oplus Y)\oplus\ldots\big)_E \sim (X\oplus X\oplus\ldots)_E\oplus(Y\oplus Y\oplus\ldots)_E,$$

c) *If (X_n) and (Y_n) are sequences of B-spaces such that there is a constant $K \geqslant 1$ that $X_n \underset{K}{\sim} Y_n$ $(n = 1, 2, \ldots)$ then $(X_1 \oplus X_2 \oplus \ldots)_E \sim (Y_1 \oplus Y_2 \oplus \oplus \ldots)_E$,*

c') *If X and Y are F-spaces and $X \sim Y$ then $(X\oplus X\oplus\ldots)_E \sim (Y\oplus Y\oplus\ldots)_E$,*

d) *If (X_n) is a sequence of B-spaces and Y_n is a subspace of X_n such that there is a projection P_n from X_n onto Y_n $(n = 1, 2, \ldots)$ and $\sup\|P_n\| < \infty$, then there is a projection P from $(X_1\oplus X_2\oplus\ldots)_E$ onto its subspace $(Y_1\oplus Y_2\oplus\ldots)_E$.*

Proof. a) The required isomorphism may be given by the linear extension of an arbitrary one-to-one mapping from the set of vectors $e_{in} = (0, 0, \ldots, \overbrace{0, e_i, 0}^{n\text{-th place}}, \ldots) \in (E\oplus E\oplus\ldots)_E$ $(i, n = 1, 2, \ldots)$ onto the set of unit vectors in E.

b) The required isomorphism may be given by the following formula: $T\big((x_i), (y_i)\big) = \big((x_i, y_i)\big)$ where $x_i \in X$, $y_i \in Y$ $(i = 1, 2, \ldots)$.

c) Let T_n be an isomorphism from X_n onto Y_n satisfying (1). Then the required isomorphism may be given by the formula

$$T(x_n) = (T_n x_n), \quad (x_n) \in (X_1\oplus X_2\oplus\ldots)_E.$$

c') If T' is an isomorphic mapping from X onto Y, then the required isomorphism is $T(x_i) = (T' x_i)$, $(x_i) \in (X\oplus X\oplus\ldots)_E$.

d) The required projection is

$$P(x_n) = (P_n x_n), \quad (x_n) \in (X_1\oplus X_2\oplus\ldots)_E.$$

PROPOSITION 4. *If E has the same meaning as in Proposition 3, X is a subspace complemented in E and X contains a subspace Y complemented in X and isomorphic to E, then X is isomorphic to E.*

Proof. From the assumption of Proposition 4 it follows that there exist F-spaces X, X_1, Y and Y_1 such that

$$\text{(i) } E \sim X\oplus X_1, \quad \text{(ii) } E \sim Y, \quad \text{(iii) } X \sim Y\oplus Y_1.$$

Thus using the fact that the operation of the product of F-spaces is associative, according to Proposition 3 a), b), c') we have

$$E \sim X\oplus X_1 \sim (Y\oplus Y_1)\oplus X_1 \sim (E\oplus Y_1)\oplus X_1 \sim E\oplus(Y_1\oplus X_1)$$

$$\sim (E\oplus E\oplus\ldots)_E\oplus(Y_1\oplus X_1) \sim \big((X\oplus X_1)\oplus(X\oplus X_1)\oplus\ldots\big)_E\oplus(Y_1\oplus X_1)$$

$$\sim \big((X\oplus X\oplus\ldots)_E\oplus(X_1\oplus X_1\oplus\ldots)_E\oplus(X_1\oplus Y_1)\sim$$

$$\sim \big((X\oplus X\oplus\ldots)_E\oplus(X_1\oplus X_1\oplus\ldots)_E\big)\oplus Y_1 \sim \big((X\oplus X_1)\oplus(X\oplus X_1)\oplus\ldots\big)_E\oplus Y_1$$

$$\sim (E\oplus E\oplus\ldots)_E\oplus Y_1 \sim E\oplus Y_1 \sim Y\oplus Y_1 \sim X,$$

q. e. d.

2. THEOREM 1. *Let E be one of the spaces s, l_p, where $1 \leqslant p < \infty$, or c_0. Then each subspace complemented in E is isomorphic to E or is of finite dimension* (4).

For the space s our statement immediately follows from the result of Mazur and Orlicz, which states that each infinitely dimensional subspace of s is isomorphic to s (see e. g. [5]). In the sequel we shall only consider the case of the space l_p $(1 \leqslant p < \infty)$. The proof for c_0 is analogous.

LEMMA 1. *Let (z_n) be a sequence in l_p such that there is an increasing sequence of indices $0 = p_0 < p_1 < \ldots$ such that the expansion of z_m in the unit vector basis in l_p is of the form*

$$z_m = \sum_{i=p_{m-1}+1}^{i=p_m} t_i^m e_i \neq 0 \quad (m = 1, 2, \ldots).$$

Then

a) *the subspace $[z_m]$ is isometrically isomorphic to l_p,*

b) *$[z_m]$ is complemented in l_p.*

Proof. First we observe that for arbitrary scalars $\lambda_1, \lambda_2, \ldots, \lambda_k$ the equality

$$(2) \quad \Big\|\sum_{m=1}^{k} \lambda_m z_m\Big\| = \Big\|\sum_{m=1}^{k}\sum_{i=p_{m-1}+1}^{p_m} \lambda_m t_i^m e_i\Big\| = \Big(\sum_{m=1}^{k}\sum_{i=p_{m-1}+1}^{p_m} |\lambda_m t_i^m|^p\Big)^{1/p}$$

$$= \Big(\sum_{m=1}^{k} |\lambda_m|^p \|z_m\|^p\Big)^{1/p} \quad (k = 1, 2, \ldots)$$

holds.

(4) The assumption that X is complemented in E is essential in the case of $E = c_0$ and $E = l_p$ for $1 \leqslant p < 2$, is not essential in the case of $E = s$ and $E = l_2$, and seems to be essential in the remaining cases. We can construct in the space c_0 and in the space l_p $(1 \leqslant p < 2)$ infinitely dimensional manifolds which are not isomorphic to the whole space c_0, or l_p respectively.

From (2) we infer that

1° (z_m) is a basis in $[z_m]$ (by a result of Nikolskiĭ in [27]),

2° $\sum_{m=1}^{\infty} t_m \frac{z_m}{\|z_m\|}$ converges if and only if $\sum_{m=1}^{\infty} |t_m|^p < +\infty$.

By (2), 1° and 2° it is easily seen that the mapping

$$T(t_m) = \sum_{m=1}^{\infty} t_m \frac{z_m}{\|z_m\|} \qquad (t_m) \epsilon l_p$$

is an isometric isomorphism from l_p onto $[z_m]$.

For $m = 1, 2, \ldots$ denote by E_m the subspace spanned on the vectors $e_{p_{m-1}+1}, e_{p_{m-1}+2}, \ldots, e_{p_m}$. Since $z_m \epsilon E_m$ $(m = 1, 2, \ldots)$ there is a linear functional w_m^* in E_m^* such that $w_m^*(z_m) = 1$ and $\|w_m^*\| = 1/\|z_m\|$. Let us put for $x = \sum_{i=1}^{\infty} t_i e_i \epsilon l_p$,

$$Px = \sum_{m=1}^{\infty} w_m^* \Big(\sum_{i=p_{m-1}+1}^{p_m} t_i e_i \Big) \cdot z_m .$$

It immediately follows from (2), 2° and the inequality

$$\Big| w_m^* \Big(\sum_{i=p_{m-1}+1}^{p_m} t_i e_i \Big) \Big| \leqslant \frac{1}{\|z_m\|} \Big(\sum_{i=p_{m-1}+1}^{p_m} |t_i|^p \Big)^{1/p} \qquad (m = 1, 2, \ldots), \tag{3}$$

that P maps l_p into z_m. Since the sequence (z_m) forms a basis in $[z_m]$ and $Pz_m = z_m$, P is a projection from l_p onto $[z_m]$. The continuity of P and moreover the fact that $\|P\| = 1$ follow from the inequality

$$\|Px\| = \Big(\sum_{m=1}^{\infty} \Big| w_m^* \Big(\sum_{i=p_{m-1}+1}^{p_m} t_i e_i \Big) \Big|^p \|z_m\|^p \Big)^{1/p} \tag{4}$$

$$\leqslant \Big(\sum_{m=1}^{\infty} \frac{1}{\|z_m\|^p} \sum_{i=p_{m-1}+1}^{p_m} |t_i|^p \|z_m\|^p \Big)^{1/p} = \Big(\sum_{m=1}^{\infty} \sum_{i=p_{m-1}+1}^{p_m} |t_i|^p \Big)^{1/p} = \|x\|,$$

where $x = \sum_{i=1}^{\infty} t_i e_i \epsilon l_p$ (to establish inequality (4) we use formulae (2) and (3)).

LEMMA 2. *Let X be an infinitely dimensional subspace of l_p. Then X contains a subspace Y which has a complement in l_p and is isomorphic to l_p.*

Proof. According to the fact that X is of an infinite dimension we may choose a sequence (y_m) in X and a sequence of indices $0 = p_0 < p_1 < \ldots$ in such a way that

$$y_m = \sum_{i=p_{m-1}+1}^{p_m} t_i^m e_i, \tag{5}$$

$$\|y_m\| = 1 \qquad (m = 1, 2, \ldots), \tag{6}$$

$$\Big\| \sum_{i=p_{m-1}+1}^{\infty} t_i^m e_i \Big\| \leqslant \frac{1}{2^{m+1}}. \tag{7}$$

Let us set

$$z_m = \sum_{i=p_{m-1}+1}^{p_m} t^m e_i \qquad (m = 1, 2, \ldots).$$

Clearly $\|z_m - y_m\| \leqslant 1/2^{m+1}$ and so $z_m \neq 0$ $(m = 1, 2, \ldots)$. Let P be the projection from l_p onto $[z_m]$ and (z_m^*) be the sequence in $[z_m]^*$ orthonormal to (z_m). It is easily seen by 2° that

$$\|z_m^*\| = \frac{1}{\|z_m\|} \leqslant \frac{1}{\|y_m\| - \|y_m - z_m\|} = \frac{1}{1 - \frac{1}{2^{m+1}}} \qquad (m = 1, 2, \ldots).$$

Since

$$\|P\| \sum_{m=1}^{\infty} \|z_m^*\| \|y_m - z_m\| \leqslant \sum_{m=1}^{\infty} \frac{1}{1 - 2^{-m-1}} < 1,$$

the sequence (y_m) fulfils the assumptions of Theorems 2 and 3 of [4]. Applying these theorems we infer that $[y_m]$ is the required subspace.

Proof of Theorem 1. Let X be an infinitely dimensional subspace complemented in l_p. By lemma 2 there is a subspace Y of X such that Y is isomorphic to l_p and is complemented in l_p. Therefore Y has a complement in X. Now we apply Proposition 4, q. e. d.

Remark. In the space L_p $(1 < p \neq 2)$ there is a subspace isomorphically different from L_p and l_p which has a complement. This follows from

PROPOSITION 5. *Let $\psi_n(t) = \operatorname{sign} \sin(2^n \pi t)$ $(n = 0, 1, \ldots)$ be Rademacher functions and $p > 1$. Then the subspace $[\psi_n]$ is complemented in L_p and $[\psi_n]$ is isomorphic to l_2.*

Proof. By Khintchin's inequality (see [17], p. 131-132) it follows that there are constants A_p and B_p such that for each of the scalars $t_0, t_1, \ldots, t_k$ $(k = 0, 1, \ldots)$

$$A_p \Big(\sum_{i=0}^{k} t_i^2 \Big)^{1/2} \leqslant \Big(\int_0^1 \Big| \sum_{i=0}^{k} t_i \psi_i(t) \Big|^p dt \Big)^{1/p} = \Big\| \sum_{i=0}^{k} t_i \psi_i \Big\| \leqslant B_p \Big(\sum_{i=0}^{k} t_i^2 \Big)^{1/2}. \tag{8}$$

Hence by the results of [26] it follows that the sequence (ψ_n) is a basis (in the sense of the metric of L_p) of the subspace $[\psi_n]$ and we can easily verify that $[\psi_n]\sim l_2$.

From well-known properties of Rademacher functions (see [17], Kapitel VII) it follows that the mapping

$$Px=\sum_{i=0}^{\infty}\int_0^1 x(t)\psi_i(t)\,dt\,\psi_i(t), \qquad x\in(L_p),$$

is a projection from L_p onto $[\psi_n]$, q. e. d.

Proposition 5 shows that Lemma 2 cannot be generalized to the case of the space L_p for $1<p\neq 2$. But, in view of the fact that the space L is an abstract L-space, from Corollary 6 (see p. 222) we obtain:

Each infinitely dimensional subspace complemented in L contains a subspace isomorphic to l and complemented in L.

3. In this section we shall give a few results concerning the converse of Theorem 1.

THEOREM 2. *Let X be a subspace of l_p $(1\leqslant p<+\infty)$. If X is isometrically isomorphic to l_p, then X is complemented in l_p.*

LEMMA 3. *Let $1\leqslant p\neq 2$ and $x=(t_i)$, $y=(s_i)$ be two elements in l_p such that*

$$\|x-y\|^p+\|x+y\|^p=2(\|x\|^p+\|y\|^p); \tag{9}$$

then $s_it_i=0$ for $i=1,2,\ldots$

Proof. For $p>1$ this lemma is proved in [16], p. 239. For $p=1$ it follows immediately from an elementary inequality, which states that for each real a,b,

$$|a+b|+|a-b|\leqslant 2(|a|+|b|); \tag{10}$$

moreover the equality sign holds if and only if $a\cdot b=0$. (We omit the proof of inequality (10)).

Indeed, if $s_{i_0}\cdot t_{i_0}\neq 0$, then by (10) $|s_{i_0}+t_{i_0}|+|s_{i_0}-t_{i_0}|<2(|s_{i_0}|+|t_{i_0}|)$. Hence by (10)

$$\|x-y\|+\|x+y\|=\sum_{i=0}^{\infty}|s_i+t_i|+|s_i-t_i|<2\Big(\sum_{i=1}^{\infty}|s_i|+|t_i|\Big)=2(\|x\|+\|y\|).$$

Proof of Theorem 2. The case of $p=2$ is well known. Let $p\neq 2$ and X be a subspace of l_p isometrically isomorphic to l_p. Let (z_n) be the sequence in X which corresponds under an isometric isomorphism to the unit vector basis (e_n) in l_p.

Let

$$z_n=\sum_{i=1}^{\infty}t_i^n e_i \qquad (n=1,2,\ldots),$$

$$N_n=\{i\in N: t_i^n\neq 0\}$$

(N denotes the set of all integers).

As each pair e_n,e_m $(n\neq m;\ n,m=1,2,\ldots)$ satisfies (9), each pair z_n,z_m also satisfies (9). Hence $N_n\cap N_m=\Phi$ $(n\neq m;\ n,m=1,2,\ldots)$. Let E_n be the smallest subspace spanned on the sequence (e_i) where $i\in N_n$. Since $z_n\in E_n$, there is a linear functional w_n^* in E_n^* such that $w_n^*(z_n)=1$, $\|w_n^*\|=1/\|z_n\|=1$ $(n=1,2,\ldots)$. For each

$$x=\sum_{i=1}^{\infty}t_ie_i\in l_p \quad\text{let}\quad P(x)=\sum_{n=1}^{\infty}w_n^*\Big(\sum_{i\in N_n}t_ie_i\Big)\cdot z_n.$$

By the same arguments as in the proof of Lemma 1 we infer that P is a projection from l_p onto X with the norm $\|P\|=1$.

Remark. By the same method as in Theorem 2 one may establish the following statement:

Let $1<p<\infty$ and X be a subspace of L_p. If X is isometrically isomorphic to l_p or L_p, then X is complemented in L_p.

THEOREM 3. *Let E be one of the spaces s, l_2, c_0 or m and Y a subspace of E isomorphic to E. Then Y has a complement in E.*

Proof. The case of l_2 is well known. In the case of s it follows from the results of [5] and in the case of m if follows from the fact that $m\in\mathfrak{P}_1$ and by Corollary 1. The validity of Theorem 3 in the case of c_0 is an immediate consequence of the next theorem.

THEOREM 4 (Sobczyk [31]). *Let X be a separable B-space and Y be a subspace of X isomorphic to c_0. Then Y is complemented in X.*

Moreover if $Y\overset{1}{\sim}c_0$, then there is a projection P from X onto Y with the norm $\|P\|\leqslant 2$ (5).

Proof. Let us note that the first part of this theorem is an immediate consequence of the second one and of Proposition 1.

To prove the second part of this theorem we observe that it is sufficient to restrict our attention to the case of the space $C[0,1]$. Indeed according to the Banach-Mazur theorem on universality ([1], p. 185) we may realize the space X as a subspace of $C[0,1]$. If P is a projection

(5) Added in print. For the generalization and other proofs of Sobczyk's theorem see [33] and references in [33].

from $C[0,1]$ onto Y with the norm $\|P\| \leqslant 2$ then, according to the inclusion $Y \subset X \subset C$, the operator P restricted to X is the required projection.

Let Y be a subspace of $C[0,1]$ and let us suppose that there is an isometric isomorphism between Y and c_0. Let the functions f_n ($n = 1, 2, \ldots$) corespond under this isometric isomorphism to the unit vectors in c_0. Since $\|f_n\| = 1$, there is a point t_n such that $|f_n(t_n)| = 1$ ($n = 1, 2, \ldots$). Let Z denote the set of all limit-points of the sequence (t_n). Clearly Z is a closed subset of $[0,1]$. According to the obvious properties of the unit vectors in c_0, $\|f_n \pm f_m\| = 1$ ($n \neq m$; $n, m = 1, 2, \ldots$). Thus

$$|f_n(t_m)| = \begin{cases} 0 & \text{if} \quad n \neq m, \\ 1 & \text{if} \quad n = m. \end{cases}$$

Hence if $t \in Z$, then $f_n(t) = \lim\limits_k f_n(t_{m_k}) = 0$ ($n = 1, 2, \ldots$). Finally according to the fact that the sequence (f_n) is a basis in Y we infer that for each $y \in Y$, if $t \in Z$, then $y(t) = 0$. Let C_Z denote the subspace of $C[0,1]$ consisting of all functions vanishing at each point $t \in Z$. Let us put

$$Tx = \sum_{n=1}^{\infty} x(t_n) \operatorname{sign} f_n(t_n) f_n, \qquad x \in C_Z,$$

since $x \in C_Z$, $\lim\limits_n x(t_n) = 0$. Hence in view of the fact that the functions f_n corespond to the unit vectors in c_0, T is a well-defined linear operator from C_Z onto Y. Since $Y \subset C_Z$ and the sequence (f_n) forms a basis in Y and $T(f_n) = f_n$, we infer that T is a projection from C_Z onto Y. Moreover

$$\|T\| = \sup_{\|x\| \leqslant 1} \|Tx\| = \sup_n |x(t_n)| = 1.$$

To complete the proof it is sufficient to show that there exists a projection Q from $C[0,1]$ onto C_Z with the norm $\|Q\| \leqslant 2$. Indeed if such a projection exists, then the required mapping may be given by the formula $P = QT$. The existence of the projection Q is a consequence of the next proposition.

PROPOSITION 6. *Let Z be a closed set in a compact metric space S. Then there exists a projection Q with the norm $\|Q\| \leqslant 2$ from $C(S)$ onto its subspace $C(S/Z)$ of all continuous functions on S vanishing at each point of Z.*

Proof. According to Borsuk's extension theorem (see [6]]) there is a linear operation U preserving the norm which corresponds each function in $C(Z)$ its continuous extension on S. Let us put $Qx = x - URx$, $x \in C(S)$, where R is the restriction operator which corresponds the function x its restriction to Z. It is obvious that Q is a linear mapping from $C(S)$ into $C(S/Z)$. If $x \in C(S/Z)$ then $Rx = 0$, and thus $URx = 0$ and finally $Qx = x$. Hence Q is a projection operator with the norm $\|Q\| = \sup\limits_{\|x\| \leqslant 1} \|x\| + \|U\| \|R\| = 2$, q. e. d.

4. The main result of this section is Theorem 5, which will be applied to the investigation of the properties of complemented subspaces of $C(S)$.

THEOREM 5. *Let S be a topological compact Hausdorff space and X a B-space such that no subspace of X is isomorphic to c_0. Then every linear operator T from $C(S)$ into X is weakly compact* ([6]).

Proof. This proof is a modification of the proof of Theorem 6 in [9], p. 494.

According to Theorem 2 in [9], p. 492, there is a unique set function $\mu(\cdot)$, defined on the Borel sets in S and having values in X, such that

(a) $\mu(\cdot)x^*$ is a real countably additive set function defined on the Borel sets in S for each x^* in X^*,

(b) $x^*Tf = \int\limits_S f(s)\mu(ds)x^*, \quad f \in C(S), \quad x^* \in X^*$ ([7]).

By Theorem 3 in [9], p. 493, it is sufficient to prove that $\mu(E)$ is in X for every Borel set E.

By the same arguments as in [9], p. 496, we show that it is sufficient to restrict our attention for metric compact spaces.

Let S be a metric compact space with the metric function ϱ and let $\mathcal{B}$ denote the family of all Borel sets in S. Let $\mathfrak{B}_0$ be the intersection of all sets $\mathfrak{B}$ of functions of S, such that

(i) $C(S) \subseteq \mathfrak{B}$,

(ii) if f_n be a sequence of functions in $\mathfrak{B}$ such that $\sup\limits_{s \in S} \sum\limits_{n=1}^{\infty} |f_n(s)| < \infty$, then $\sum\limits_{n=1}^{\infty} f_n \in \mathfrak{B}$ (where $\left(\sum\limits_{n=1}^{\infty} f_n\right)(s) = \sum\limits_{n=1}^{\infty} f_n(s)$, $s \in S$).

By the same consideration as in [9], p. 495, we prove that $\mathfrak{B}_0$ is an algebra under the natural product $fg(s) = f(s) \cdot g(s)$, $s \in S$.

Denoting the characteristic function of a set E by χ_E we let $\mathcal{B}_0 = \{E \in \mathcal{B} : \chi_E \in \mathfrak{B}_0\}$. Let E_n be a sequence of disjoint sets in $\mathcal{B}_0$. Since

$$1 \geqslant \chi_{\cup E_n}(s) = \sum_{n=1}^{\infty} \chi_{E_n}(s) = \sum_{n=1}^{\infty} |\chi_{E_n}(s)|, \qquad s \in S,$$

according to (ii), $\bigcup E_n \in \mathcal{B}_0$. Hence, by the fact that $\mathcal{B}_0$ is an algebra, it is easily seen that $\mathcal{B}_0$ is a σ-field contained in $\mathcal{B}$. We now show that $\mathcal{B}_0 = \mathcal{B}$ by proving that $\mathcal{B}_0$ contains all the closed sets.

([6]) The operator T from X onto Y is said to be *weakly compact* if for every bounded sequence (x_n) the sequence (Tx_n) contains a subsequence (Tx_{n_k}) weakly convergent to an element y in Y.

([7]) The symbol $\int\limits_S f(s)\mu(ds)x^*$ denotes the integral of the function f with respect to the σ-additive set function $\mu(\cdot)x^*$.

Let F be an arbitrary closed set in S. Let us put $g_0(s)=1$; $g_n(s) = e^{-n\varrho(s,F)}-e^{-(n-1)\varrho(s,F)}$, $s\in S$ $(n=1,2,\ldots)$ where $\varrho(s,F)=\inf\limits_{t\in F}\varrho(s,t)$. Clearly $g_n\in C(S)$ and, for each s in S,

$$\chi_F(s)=\sum_{n=0}^{\infty}g_n(s)\leqslant\sum_{n=0}^{\infty}|g_n(s)|\leqslant 2.$$

Hence, by (ii), $\lambda_F\in\mathfrak{B}_0$.

We now show that $\mu(E)\in X$ for $E\in\mathcal{B}$. Consider the collection $\mathfrak{B}_1$ of bounded B-measurable function f such that there is an x_f in X such that

$$x^*(x_f)=\int_S f(s)\mu(ds)x^* \quad \text{for each} \quad x^*\in X^*.$$

The collection $\mathfrak{B}_1$ forms a linear manifold which, by (b), contains $C(S)$. Now we show that $\mathfrak{B}_1$ satisfies (ii). Let (f_n) be a sequence in $\mathfrak{B}_1$ such that

$$\sup_{s\in S}\sum_n |f_n(s)|<\infty. \tag{11}$$

Let x^* be fixed and let, for $n=1,2,\ldots$, $\varepsilon_n=\operatorname{sign}x^*(x_{f_n})$. By (11), according to Lebesgue's theorem on integration term by term we have

$$\sum_n|x^*(x_{f_n})|=\sum_{n=1}^{\infty}\varepsilon_n\int_S f_n(s)\mu(ds)x^*=\int_S\Big(\sum_n\varepsilon_n f_n(s)\Big)\mu(ds)x^*$$
$$\leqslant\sup_{s\in S}\sum_n|f_n(s)|\cdot\operatorname{Var}\mu(\cdot)x^*<\infty.$$

Hence the series $\sum\limits_n x_{f_n}$ is weakly unconditionally convergent ([8]). Since no subspace of X is isomorphic to c_0, according to Theorem 5 in [4] the series $\sum\limits_n x_{f_n}$ is unconditionally convergent to an element x in X. It is easily seen that for each x^* in X^*,

$$x^*(x)=\int_S\sum_n f_n(s)\mu(ds)x^*.$$

Hence $\sum\limits_n f_n\in\mathfrak{B}_1$. Thus $\mathfrak{B}_1$ satisfies (i) and (ii). Hence $\mathfrak{B}_1\supset\mathfrak{B}_0$ and thus for arbitrary $E\in\mathcal{B}=\mathcal{B}_0$

$$\mu(E)=x_{\chi_E}\in X, \qquad \text{q. e. d.}$$

([8]) The series $\sum\limits_n x_n$ in B-space X is said to be *weakly unconditionally convergent* if for each $x^*\in X^*$, $\sum\limits_n|x^*(x_n)|<\infty$.

COROLLARY 1. *Let X be a B-space. Then each linear operator from $C(S)$, where S is an infinite compact metric space, into X is weakly compact if and only if X contains no subspace isomorphic to c_0.*

Proof. It immediately follows from Theorem 5 and the fact that for each infinite compact metric space there is a non-weakly complete linear operator from $C(S)$ onto c_0. An operator having this property may be given by the formula

$$Tf=\big(f(s_n)-f(s_0)\big),\qquad f\in C(S),$$

where (s_n) is a convergent sequence of different points in S and $s_0=\lim s_n$.

Remark. The assumption of metrisability of S is essential. It follows from the fact that each linear operator from m into c_0 is weakly compact ([15], p. 168).

By Theorem 5 and by the Dunford-Pettis theorem (see [9], p. 494), which states that the square of an arbitrary weakly compact linear operator from $C(S)$ into itself is a compact operator, we obtain

COROLLARY 2. *Let S be a topological compact Hausdorff space and X a subspace of $C(S)$ complemented in $C(S)$. Then X contains a subspace isomorphic to c_0, or X is of a finite dimension.*

If a B-space X contains a subspace isomorphic to c_0 and X is isomorphic to a conjugate space Y^* of a B-space Y, then according to Theorem 4 in [4] X contains a subspace isomorphic to m. Thus by Corollary 2 we obtain

COROLLARY 3. *Let X be a subspace of $C(S)$ which is complemented in $C(S)$. Then, if X is isomorphic to a conjugate space Y^* of a B-space Y, X contains a subspace isomorphic to m or X is of a finite dimension.*

COROLLARY 4. *If E is a B-space isomorphic to an abstract L-space and X is a subspace of E complemented in E, then either X contains a complemented subspace Y which is isomorphic to l or X is of finite dimension.*

Proof. According to Kakutani's representation theorem [19], E^* is isomorphic to a space $C(S)$. On the other hand, if X is complemented in E then X^* is complemented in E^* (see e. g. [9], p. 481). Thus, by Corollary 2, if X is of an infinite dimension then X^* contains a subspace isomorphic to c_0 and, according to Theorem 4 in [4], X contains a complemented subspace which is isomorphic to l.

THEOREM 6 ([9]). *Let X be a B-space, $X\in\mathfrak{P}$ and satisfy one of the conditions*

([9]) Grothendieck [16], p. 169, has proved in a different way that if $X\in\mathfrak{P}$ and X satisfies (c) or X is reflexive, then X is of a finite dimension. Clearly our condition (a) is essentially more general than the assumption of reflexivity or weak completness of X.

(a) *no subspace of* X *is isomorphic to* c_0,

(b) X *is isomorphic to a conjugate space* Y^* *of a* B*-space* Y *and no subspace of* X *is isomorphic to* m,

(c) X *is separable.*

Then X *is of a finite dimension.*

Proof. Since each B-space may be embedded in a space $C(S)$ and $Y\in\mathfrak{P}$, then X is isomorphic to a complemented subspace of $C(s)$. Thus in case (a) by Corollaries 2 and in case (b) by Corollaries 3 it follows that X is of a finite dimension.

Proof in case (c). We shall now show that every separable B-space X belonging to $\mathfrak{P}$ contains no subspace isomorphic to c_0. If it were not so, there would exist a subspace Y of X isomorphic to c_0. In view of the fact that X belongs to $\mathfrak{P}$, Theorem 4 and a result of Goodner [4], p. 93, we should have $c_0\in\mathfrak{P}$, which is not true because there is no projection from m onto its subspace c_0 (Philips [29]).

Now it suffices to apply result (a), which we have already proved, q. e. d.

Philips [29] has observed that $m\in\mathfrak{P}$. Thus, by Proposition 2, if a B-space X contains a subspace Y isomorphic to m, then Y is complemented in X. On the other hand, if X is a subspace complemented in $m\in\mathfrak{P}$, then, by Goodner's result quoted earlier, $X\in\mathfrak{P}$. Hence by Theorem 6 and Proposition 4 we obtain

COROLLARY 5. *Let* X *be a subspace complemented in* m. *Then if* X *is infinitely dimensional, then* X *is not separable, and if* X *contains a subspace* Y *isomorphic to* m, *then* X *is isomorphic to* m.

In particular by Theorem 6 (b) we obtain

COROLLARY 6. *Let* X *be a subspace complemented in* m *and let* X *be isomorphic to a conjugate space* Y^* *of a* B*-space* Y. *Then* X *is isomorphic to* m *or is of a finite dimension.*

COROLLARY 7. *Let* X *be a subspace of* m *and* $X\in\mathfrak{P}_1$. *Then* X *is isomorphic to* m *or* X *is of a finite dimension.*

Proof. According to the Nachbin-Kelley theorem (see [7], p. 95) if $X\in\mathfrak{P}_1$ then there is a compact Hausdorff space S extremally disconected such that X is isometrically isomorphic to $C(S)$. By a result given in [14] if S is an infinite extremally disconected compact Hausdorff space, then $C(S)$ contains a subspace isometrically isomorphic to m. Now it is sufficient to apply Corollary 5, q. e. d.

Corollary 7 implies that each infinitely dimensional B-space X, such that $X\in\mathfrak{P}_1$ and there is a sequence (x_n^*) in X^* such that $\sup_n\|x_n^*\| < \infty$ and $\|x\|\leqslant\sup_n|x_n^*(x)|$, $x\in X$, is isomorphic to m. In particular the

space L_∞ (see [28]), as well as the space M_{CB} of all real functions defined on $[0,1]$ and having the property of Baire [10] (Semadeni's thesis), are isomorphic to m.

It is interesting to compare Corollary 7 with the following example, which is due to Z. Semadeni.

Example. Let $\{\mathcal{I}_\alpha\}$, $\alpha\in\mathfrak{a}$ and $\overline{\overline{\mathfrak{a}}}=2^{\mathfrak{c}}=\mathfrak{f}$ ($\mathfrak{c}$ is the power of the continuum) be a family of intervals and let μ_α be the Lebesgue measure on $\mathcal{I}_\alpha$. By μ we denote the product measure of the measures μ_α defined on the Tychonoff cube $\mathcal{I}^{\mathfrak{f}}$. We consider the space $L_\infty(\mathcal{I}^{\mathfrak{f}},\mu)$ of all real functions on $\mathcal{I}^{\mathfrak{f}}$ measurable and essentially bounded with respect to the measure μ.

The space $L_\infty(\mathcal{I}^{\mathfrak{f}},\mu)\in\mathfrak{P}_1$, *but it is not isomorphic to a space* $m(A)$ *of all real bounded functions defined on a set* A.

Proof. Since $L_\infty(\mathcal{I}^{\mathfrak{f}},\mu)$ is a boundedly complete vector lattice (see [7], p. 106), $L_\infty(\mathcal{I}^{\mathfrak{f}},\mu)\in\mathfrak{P}_1$.

The space $L_\infty(\mathcal{I}^{\mathfrak{f}},\mu)$ is isomorphic to a strictly convex space, because we may introduce in $L_\infty(\mathcal{I}^{\mathfrak{f}},\mu)$ a new strictly convex norm $|||\cdot|||$ equivalent to the original one by

$$|||x||| = \operatorname{sup\,ess}|x(t)| + \Big(\int_{\mathcal{I}^{\mathfrak{f}}} x^2(t)\,d\mu\Big)^{1/2}.$$

Thus according to a result of Day [8] the space $L_\infty(\mathcal{I}^{\mathfrak{f}},\mu)$ is not isomorphic to a space $m(A)$ for $\overline{\overline{A}}>\aleph_0$. On the other hand, since $\overline{\overline{L_\infty(\mathcal{I}^{\mathfrak{f}},\mu)}}>\mathfrak{c}$, the space $L_\infty(\mathcal{I}^{\mathfrak{f}},\mu)$ is not isomorphic to a space $m(A)$ for $\overline{\overline{A}}\leqslant\aleph_0$.

5. Bary [3] and Gelfand [12] have proved that in the space L_2 each unconditional basis [11] is equivalent (see Definition 1) to an orthogonal basis. In particular each unconditional basis (x_n) in L_2 satisfying

$$0<\inf_n\|x_n\|\leqslant\sup_n\|x_n\|<\infty \tag{$*$}$$

is equivalent to the unit vector basis in l_2.

In this section we show (Theorem 7) that the fact that in L_2 and l_2 all unconditional bases satisfying $(*)$ are equivalent characterizes the

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[10] The definition of the function having the property of Baire, see [23], p. 306. We adopt the norm in M_{CB},

$$\|f\| = \inf_{\Delta\in K}\sup_{t\in(0,1)-\Delta}|f(t)|,$$

where K is the family of all subsets of $[0,1]$ of the first category.

[11] For the definition and basic properties of unconditional bases see [7], p. 73-77.

space L_2 among the spaces L_p for $p > 1$ (l_2 — among the spaces l_p). A result similar to our Theorem 7 was anounced without proof by Gaposhkin [11], [32] ([12]).

We do not know whether in the space l (or c_0) there exist two non-equivalent unconditional bases satisfying (*). On the contrary, we show that in L there is no unconditional basis (Proposition 9).

Definition 1. Let (x_n) and (y_n) be bases in B-spaces X and Y respectively. The bases (x_n) and (y_n) are said to be *equivalent* if for each sequence (t_n) of scalars the series $\sum\limits_n t_n x_n$ converges if and only if the series $\sum\limits_n t_n y_n$ converges.

Bases (x_n) and (y_n) are said to be *commutatively equivalent* (*c-equivalent*) if there is a permutation (p_n) of indices such that the sequence (y_{p_n}) forms a basis in Y equivalent to the basis (x_n).

We observe that for an arbitrary permutation (p_n) the unit vector basis (e_n) in l_p $(1 \leqslant p < \infty)$ is equivalent to the basis (e_{p_n}). Hence for each basis (x_n), if the basis (x_n) and (e_n) are equivalent, then they are c-equivalent.

Theorem 7. *If $1 < p \neq 2$, then in each of the spaces L_p and l_p there are two non-c-equivalent unconditional bases satisfying* (*).

Proof of Theorem 7 for l_p. We shall show that for $1 < p \neq 2$ in l_p there exists an unconditional basis satisfying (*) which is not equivalent (and thus not c-equivalent) to the unit vector basis (e_n).

Let us consider the space X_p of all real sequences (t_n) such that

$$|||(t_n)||| = \Big(\sum_{r=1}^{\infty}\Big(\sum_{n=s_{r-1}+1}^{s_1} t_n^2\Big)^{p/2}\Big)^{1/p} < \infty \quad \text{where} \quad s_\nu = \frac{\nu(\nu+1)}{2} \quad (\nu = 0, 1, \ldots).$$

It is easily seen that X_p under the norm $|||\ |||$ is a B-space and the sequence (f_n) where $f_n = (0\,0\underbrace{\ldots 1}_{n\text{-th place}} 0 \ldots)$ forms in X_p an unconditional basis satisfying (*). Since for each $1 < p < 2$ there is a sequence $(t_n^{(p)})$ in X_p such that $\sum\limits_{n=1}^{\infty}|t_n^{(p)}|^p = \infty$ and for each $p > 2$ there is a sequence $(t_n^{(p)})$ such that $|||(t_n^{(p)})||| = \infty$ but $\sum\limits_{n=1}^{\infty}|t_n^{(p)}|^p < \infty$, it is easily seen that the bases (f_n) in X_p and (e_n) in l_p are not c-equivalent. To complete the proof it is sufficient to establish the next proposition.

Proposition 7. *The spaces X_p and l_p are isomorphic.*

([12]) Added in print. In [32] it is proved only that for $1 < p \neq 2$ the Haar system is not equivalent with certain its permutation.

Proof. According to Theorem 1 it is sufficient to show that there is a subspace Y_p in l_p such that

1° $Y_p \sim X_p$,

2° Y_p is complemented in l_p.

Let $[\Psi_0, \Psi_1, \ldots, \Psi_n]$ be the subspace of L_p spanned on the Rademacher functions $\Psi_0, \Psi_1, \ldots, \Psi_n$ and let A_n be the subspace of L_p spanned on the characteristic functions $\chi_{[(k-1)2^{-n}, k2^{-n}]}$ $(k = 1, 2, \ldots, 2^n, n = 0, 1, \ldots)$. Since A_n is isometrically isomorphic to l_p, the space $[\Psi_0, \Psi_1, \ldots, \Psi_n] \subset A_n$ is isometrically isomorphic to a subspace R_n of l_p, and finally the space $Y_p = (R_0 \oplus R_1 \oplus \ldots)_{l_p}$ is a subspace of the space $(l_p^2 \oplus l_p^{2^2} \oplus \ldots) \overset{1}{\sim} l_p$. Accordingly to inequality (8) there is a constant K_p such that $R_{n-1} \overset{K_p}{\sim} l_2^n$ $(n = 1, 2, \ldots)$. Hence, by Proposition 3, the spaces Y_p and $(l_2^1 \oplus l_2^2 \oplus \ldots)_{l_p}$ are isomorphic. On the other hand, it is clear that $X_p \overset{1}{\sim} (l_2^1 \oplus l_2^2 \oplus \ldots)_{l_p}$ and thus $Y_p \sim X_p$.

Let us put

$$Q_n = \sum_{i=0}^{n-1} \int_0^1 x(t)\,\psi_i(t)\,dt \cdot \psi_i, \qquad x \in L_p \qquad (p > 1,\ n = 1, 2, \ldots).$$

It is clear that Q_n is a projection from L_p onto $[\Psi_0, \Psi_1, \ldots, \Psi_{n-1}]$. By well-known properties of Rademacher functions (see [18], p. 245) there exists a constant C_p such that $\|Q_n\| \leqslant C_p$ $(n = 0, 1, \ldots)$. Since $A_n \supset [\Psi_0, \Psi_1, \ldots, \Psi_{n-1}]$ there is a projection P_n' from A_n onto $[\Psi_0, \Psi_1, \ldots, \Psi_{n-1}$, and finally there exists a projection P_n from l_p onto R_n with the norm $\|P_n\| \leqslant C_p$ $(n = 0, 1, \ldots)$. Thus, by Proposition 3, Y_p is complemented in l_p.

Proof of Theorem 7 for L_p. Denote by (χ_n) the Haar orthogonal system normalized in L_p. That is

$$\chi_0(t) = 1, \quad t \in [0, 1],$$

$$\chi_n(t) = \begin{cases} 2^{r/p} & \text{for} \quad 2^{-r}k < t < 2^{-r-1}(2k+1), \\ -2^{r/p} & \text{for} \quad 2^{-r-1}(2k+1) < t < 2^{-r}(k+1), \\ 0 & \text{elsewhere}, \end{cases}$$

where $r = [\log_2 n]$ and $k = n - 2^r$ $(n = 1, 2, \ldots)$.

According to results of Marcinkiewicz [24] and Gaposhkin [10] the sequence (χ_n) forms in L_p an unconditional basis satisfying (*). Now let us consider in the space $L_p \oplus l_2$ a sequence (w_n) such that

$$w_{2k+1} = (\chi_k, 0) \qquad (k = 0, 1, \ldots),$$

$$w_{2k} = (0, e_k) \qquad (k = 1, 2, \ldots),$$

where e_k is the k-th unit vector in l_2.

It is easily seen that the sequence (w_n) forms in $L_p \oplus l_2$ an unconditional basis satisfying $(*)$.

Since the relation of c-equivalence of bases is an isomorphic invariant, to complete the proof it is sufficient to show that

1° the spaces L_p and $L_p \oplus l_2$ are isomorphic,

2° for $1 < p \neq 2$ the bases (χ_n) in L_p and (w_n) in $L_p \oplus l_2$ are not c-equivalent.

1° From Proposition 5 if follows that there is a B-space X such that $L_p \sim X \oplus l_2$. Since $l_2 \oplus l_2 \sim l_2$, we obtain

$$L_p \sim X \oplus l_2 \sim X \oplus (l_2 \oplus l_2) \sim L_p \oplus l_2 .$$

2° Suppose that the bases (w_n) and (χ_n) are c-equivalent. Then there is a sequence (n_k) such that the basis (χ_{n_k}) in the space $[\chi_{n_k}]$ is equivalent to the unit vector basis (e_k) in l_2. Thus according to a result of [27] the spaces $[\chi_{n_k}]$ and l_2 are isomorphic. But this fact contradicts the following

PROPOSITION 8. *For no sequence of indices $n_1 < n_2 < \ldots$ the spaces $[\chi_{n_k}]$ and l_2 are isomorphic.*

LEMMA 4. *Let (Δ_k) be a sequence of different intervals such that $\Delta_k \subset [0,1]$ $(k = 1, 2, \ldots)$ and if $k_1 < k_2$ and $\Delta_{k_1} \cap \Delta_{k_2} \neq \emptyset$ then $\Delta_{k_1} \subset \Delta_{k_2}$. Then there exists a subsequence (Δ_{k_ν}) such that either* (α) *if $\nu \neq \mu$, then $\Delta_{k_\nu} \cap \Delta_{k_\mu} = \emptyset$ or* (β) *$\Delta_{k_1} \supset \Delta_{k_2} \supset \ldots$ ($\emptyset$ denotes the empty set).*

Proof. Suppose that no subsequence of (Δ_k) satisfies (β). Then for every index k there is an index $\varphi(k)$ such that $\Delta_{\varphi(k)} \subset \Delta_k$ and $\Delta_{\varphi(k)}$ contains no intervals Δ_n different from itself $(n = 1, 2, \ldots)$. Since there are infinitely many intervals $\Delta_1, \Delta_2, \ldots$, the set $Z = \{\varphi(1), \varphi(2), \ldots\}$ is infinite. From the definition of Z it follows that if $k_1 \neq k_2$ and k_1 and k_2 belong to Z, then $\Delta_{k_\nu} \cap \Delta_{k_\mu} = \emptyset$. Hence the subsequence consisting of all elements of Z satisfies (α).

Proof of Proposition 8. According to theorem 7 in [1], p. 205, it is sufficient to show that the space $[\chi_{n_k}]$ contains a subspace Y isomorphic to l_p. To prove it we note that if (x_n) is a sequence in L_p such that

(**) *$x_n \neq 0$ and $x_n(t) x_m(t) = 0$ for almost all t $(n \neq m;\ n, m = 1, 2, \ldots)$ then the space $[x_n]$ is isometrically isomorphic to l_p* (we omit the simple proof of this fact).

Let us put

$$\Delta_k = \overline{\{t \in [0,1]: \chi_{n_k}(t) \neq 0\}} \quad (k = 1, 2, \ldots).$$

It is easily seen that the sequence (Δ_k) fulfils the assumptions of Lemma 4. Hence there is a subsequence (Δ_{k_ν}) satisfying either (α) or (β).

If (Δ_{k_ν}) satisfies (α), then the sequence $(\chi_{n'_\nu})$ (where $n'_\nu = n_{k_\nu}$) satisfies (**). Thus $[\chi_{n'_\nu}] \sim l_p$.

If (Δ_{k_ν}) satisfies (α), then we may define by induction the numbers λ_ν $(\nu = 1, 2, \ldots)$ in such a way that the sequence $(\chi_{n'_{2\nu-1}} - \lambda_\nu \chi_{n'_{2\nu}})$ satisfies (**), q. e. d.

We complete theorem 7 by the following

PROPOSITION 9. *In the space L there is no unconditional basis.*

Proof. Suppose that the sequence (x_n) is an unconditional basis in L. Since L is weakly complete, the unconditional basis (x_n) is boundedly complete [7], p. 74. Thus by Lemma 2 of [7], p. 70, L is isomorphic to a conjugate space Y^* of a B-space Y. But this fact contradicts a result of Gelfand given in [11], p. 265, q. e. d.

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On the theory of non-linear operator equations on conjugately similar spaces

by

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1. Introduction. It is the purpose of this paper to consider an eigenvalue problem for some operators F which map a Banach space R into the conjugate space $\bar{R}$. For this purpose, we take, as the Banach space R, a special kind of vector lattice, a *conjugately similar space* which has been introduced by Nakano [9]. Roughly speaking, this is a Banach space R such that a one-to-one correspondence T exists between R and $\bar{R}$. This correspondence T enables us to define a proper value λ and a proper element $a \in R$ of the operator F from R into $\bar{R}$ by the following equation:

$$Fa = \lambda Ta.$$

In the case of L_p-spaces ($p > 1$), this definition agrees with that of E. S. Citlanadze [4].

The definitions and elementary properties of the conjugately similar spaces will be given in § 2. In the next section we will prove a theorem of L. A. Ljusternik in its special form. The simple proof may be interesting. In § 4 we will consider the eigenvalue problem of a non-linear operator. The last section contains an application.

We express here our hearty thanks to Dr. Musielak for his valuable remarks on various points in this paper.

2. Conjugately similar spaces. Let R be a vector lattice which satisfies the following condition: for any system of positive elements x_λ ($\lambda \in \Lambda$) there exists an "infimum" element $\bigcap_{\lambda\in\Lambda} x_\lambda$. The conjugate space $\bar{R}$ of R is the totality of all linear (additive and homogeneous) functionals $\bar{x}$ on R which satisfy the following condition: if $x_\lambda \downarrow_{\lambda\in\Lambda} 0$ (1), then

$$\inf_{\lambda\in\Lambda} |\bar{x}(x_\lambda)| = 0.$$

(1) We write $x_\lambda \downarrow_{\lambda\in\Lambda} 0$ when $\{x_\lambda\ (\lambda\in\Lambda)\}$ is a non-increasing directed system and $\bigcap_{\lambda\in\Lambda} x_\lambda = 0$.