Let $\mathfrak{U}_0$ be the class of all separable Banach spaces having an unconditional basis (see [3], Chapt. IV, § 4). Suppose that we have defined the classes $\mathfrak{U}_\beta$ for all $\beta < \alpha$ $(\alpha < \omega_1)$. We define $\mathfrak{U}_\alpha$ as the class of all separable Banach spaces X which have the following properties:

(a) $X \notin \mathfrak{U}_\beta$ for each $\beta < \alpha$;

(b) there exist sequences (X_n) of subspaces of X and (β_n) of ordinal numbers $< \alpha$ such that $X_n \in \mathfrak{U}_{\beta_n}$ $(n = 1, 2, \ldots)$ and every element $x \in X$ may be uniquely represented as a sum of an unconditionally convergent series $x = \sum_{n=1}^{\infty} x_n$, where $x_n \in X_n$ for $n = 1, 2, \ldots$

We say that the separable Banach space *belongs to the class* $\mathfrak{U}_{\omega_1}$ if $X \in \mathfrak{U}_\alpha$ for no $\alpha < \omega_1$.

Questions:

1. Are all classes $\mathfrak{U}_\alpha$ (for $\alpha \leqslant \omega_1$) non-empty?

2. Does there exist for every $0 \leqslant \alpha \leqslant \omega_1$ a compact metric space Q such that $C(Q) \in \mathfrak{U}_\alpha$?

We know only that $C^\omega \in \mathfrak{U}_0$, $C^{\omega^\omega} \in \mathfrak{U}_1$, $C(Q) \in \mathfrak{U}_{\omega_1}$ for uncountable Q.

5.5. Let X be a Banach space with the conjugate space X^* isomorphic to l. Does there exist an ordinal α such that $X \sim C^\alpha$?

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States of operator algebras

by

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§ 0. Introduction & summary. Let $\mathcal{H}$ be a fixed Hilbert space and denote by $B = B(\mathcal{H})$ the algebra of all bounded endomorphisms of $\mathcal{H}$. B is a complete normed algebra with an involution which carries $T \in B$ into its adjoint T^*; this algebra is non-commutative unless $\mathcal{H}$ is one-dimensional. If $\mathcal{A}$ is a self-adjoint (i. e. stable under *) subalgebra of B, we follow Segal [5] in extending the customary language of statistical quantum mechanics by applying the name "state of $\mathcal{A}$" to any positive-definite linear form f on $\mathcal{A}$, i. e. a linear form f on $\mathcal{A}$ such that $f(T^*) = \overline{f(T)}$ and $f(T^*T) \geqslant 0$ for arbitrary $T \in \mathcal{A}$. These correspond to the "mixed states" of a quantum mechanical assemblage and are therefore thought of as being compounded in some way from the "pure states"

$$f_x: T \to (Tx, x), \tag{0.1}$$

where x is an arbitrary element of $\mathcal{H}$. The main aim of this paper is to discover more precisely how some at least of these mixed states are obtained from the pure ones.

When $\mathcal{A} = B$, von Neumann gives one answer to this problem, at least for those states which are weakly continuous. On the other hand, Segal [5] discusses a fairly general type of algebra $\mathcal{A}$ and shows that there exist always sufficiently many pure or minimal states to make plausible the possibility of expressing a wide class of states in terms of these. However, Segal does not concern himself with any explicit representation of this kind. von Neumann's approach ([6], Chapter IV) for $\mathcal{A} = B$ is very direct and leads to a representation in terms of the trace. Unfortunately his approach is not adaptable in any obvious way to states initially defined only on some subalgebra $\mathcal{A}$ of B. This is one reason for seeking an apparently different representation.

The proposed alternative is a representation in terms of positive integral combinations of pure states:

$$f(T) = \int_\Sigma (Tx, x)\, dm(x), \tag{A}$$

where Σ is the unit ball in $\mathfrak{X}$ and m is a positive Radon measure on Σ. As will be shown in § 7, this type of representation is equivalent to von Neumann's trace representation in case $\mathcal{A} = B$.

The one type of algebra for which a completely satisfactory decomposition theorem is known is the CC^*-algebra (= commutative C^*-algebra in Segal's terminology; [5], p. 75), though here the basic components are quite different from the pure states (0.1). Since we shall often appeal to this theory, we recall in § 1 the requisite results, amounting essentially to the abstract form of the Bochner Theorem for CC^*-algebras.

Summary. § 2, 3 and 4 deal with some conditions which are always necessary, and for some algebras also sufficient, in order that the state f shall admit a representation (A). In § 5 we show that in general these necessary conditions are not fulfilled, and we exhibit states admitting no representation (A). § 6, which is somewhat of the nature of an "aside", shows how the theory of topological tensor products leads to similar but less precise results for continuous linear forms which are not necessarily positive-definite. Finally, in § 7, we consider the relationship between the preceding results and those of von Neumann for the full algebra $B(\mathfrak{X})$.

In order to avoid certain trivial exceptions to some of our assertions, *we exclude throughout the case in which $\mathcal{A}$ is reduced to the zero operator.*

1. The Bochner Theorem for CC^*-algebras. For the reader's convenience we give here a very brief resumé of the decomposition theorem as it applies to these algebras.

Let $\mathcal{A}$ be a CC^*-algebra, or more generally any commutative, complete normed algebra with an involution such that $\|T^*T\| = \|T\|^2$ for each $T \in \mathcal{A}$. Such algebras are discussed in [1], Section VIII, [2], p. 69-71, and [4], Chapter V; in any one of these references will be found an account of the key results we are about to summarise.

Associated with $\mathcal{A}$ is a locally compact space X, called the *character space of $\mathcal{A}$*, such that $\mathcal{A}$ is isomorphic and isometric with $C_0(X)$, the algebra of complex-valued continuous functions on X which tend to zero at infinity. The points of X are the (necessarily positive definite) characters χ of $\mathcal{A}$, and the said isomorphism associates with $T \in \mathcal{A}$ the function $\hat{T}: \chi \to \chi(T)$, an element of $C_0(X)$ termed the *Gelfand transform* of T.

At this point we interject a definition applying to a general algebra $\mathcal{A}$ with involution (not necessarily CC^*). If f is a state of $\mathcal{A}$ we define

$$N(f) = \operatorname{Sup}\{f(T^*T) : T \in \mathcal{A}, \|T\| \leqslant 1\};$$

$N(f)$ may be $+\infty$. Note that if $I \in \mathcal{A}$, then $N(f) = f(I)$.

Returning to the story of CC^*-algebras, the isomorphism $\mathcal{A} \approx C_0(X)$ leads very quickly to an abstract Bochner Theorem: if f is a state of $\mathcal{A}$ for which $N(f) < +\infty$, there exists a unique positive Radon measure μ on X of total mass $\mu(X) = N(f)$ such that

$$f(T) = \int_X \chi(T)\, d\mu(\chi) \tag{1.1}$$

for all $T \in \mathcal{A}$. The subnormalised states f, i. e. those for which $N(f) \leqslant 1$, form a weakly compact, convex subset of the dual of $\mathcal{A}$. The characters χ, together with 0, are the extreme points of this set; they are also the minimal states in the sense that they admit no decomposition into the sum of two states unless each summand is a scalar multiple of the original. In retrospect one sees (1.1) as an illustration of the Krein-Milman Theorem in action. It gives a perfect decomposition of the desired type.

§ 2. Some preliminaries. We turn now to the consideration of the formula (A). As has been said, Σ denotes the closed unit ball $\|x\| \leqslant 1$ in $\mathfrak{X}$. We shall denote by Σ_1 the "boundary" $\|x\| = 1$ of this ball. It would be natural to suppose that m is concentrated on Σ_1. However, m is usually not uniquely determined by f; and the proof that in some cases m may be chosen to be concentrated on Σ_1 is rather indirect (see § 4, Remark 2; § 7).

We shall always equip Σ with the topology induced by the weak topology of $\mathfrak{X}$; it is thus a compact space, and Σ_1 is an F_σ-set in Σ. It is essential in view of the results of § 5 to remember that m is to be a *Radon measure*, and not a general finitely-additive measure.

The existence of the integral appearing in (A) for each $T \in B$ and each m is settled by the following considerations. It is not difficult to show that (Tx, x) is a continuous function of $x \in \Sigma$ if and only if T is compact (= completely continuous). Every positive selfadjoint $T \in B$ is the strong limit of an increasing directed family of compact positive self-adjoint operators (e. g. operators PT, where P is a finite-dimensional projector; if $\mathfrak{X}$ is separable, one needs only an increasing denumerable sequence). Consequently (Tx, x) is a lower semicontinuous function of $x \in \Sigma$, whenever $T \in B$ is positive self-adjoint. Any $T \in B$ is a linear combination of four positive self-adjoint operators, so that (Tx, x) is certainly Borel-measurable. It is obviously bounded for $x \in \Sigma$, hence is integrable for any Radon measure m on Σ.

§ 3. Properties of states defined by (A). The remarks of the last paragraph show that, m being any Radon measure on Σ, (A) is effective in defining a linear form f on B. It is furthermore trivial to verify that f

is a state of B, provided m is a positive measure. We proceed to enumerate certain properties common to all such states f.

Let us recall that a directed family $(T_i) \subset B$ is said *to converge weakly to* $T \in B$ if $(T_i x, y) \to (Tx, y)$ for fixed but arbitrary $x, y \in \mathcal{X}$.

THEOREM 1. *Let m be a positive Radon measure on Σ, having unit total mass, and let f be defined by (A). Then:*

(i) *f is a normalised state of B;*

(ii) *the inequality*

$$f(T) \leqslant \operatorname*{Sup}_{x \in \Sigma} (Tx, x) \tag{B}$$

holds for each self-adjoint $T \in B$;

(iii) *if a sequence (T_n) converges weakly to $T \in B$, then*

$$\lim_{n\to\infty} f(T_n) = f(T);$$

(iv) *if (T_i) is an increasing directed family of self-adjoint operators in B which converges weakly to $T \in B$, then $\lim_i f(T_i) = f(T)$.*

Proof. Both (i) and (ii) are immediate. As for (iii), we note that weak convergence of the sequence (T_n) implies (Banach-Steinhaus Theorem) that the T_n remain bounded-in-norm; so the Borel functions $(T_n x, x)$ on Σ are uniformly bounded and converge simply to (Tx, x), whence (iii). In (iv), there is no loss of generality in assuming the T_i to be positive self-adjoint. Then the functions $(T_i x, x)$ on Σ form an increasing directed family of positive, lower semicontinuous functions whose upper envelope is (Tx, x). Integration theory leads at once to the stated conclusion.

Remark 1. Property (iv) may be briefly described by saying that f is weakly order-continuous. Moreover, as we shall see in § 6 below, B may be identified with the dual of the projective tensor product $\mathcal{X} \hat{\otimes} \mathcal{X}'$, and it follows from this that on each norm-bounded subset of B convergence relative to the weak topology defined by this duality coincides with the weak convergence spoken of in Theorem 1. If $\mathcal{X}$ is separable, this induced topology is metrisable; and then (iii) in fact says that f is weakly continuous on B. Some direct consequences of such a restriction on f are noted in § 6.

§ 4. Converses of Theorem 1. To what extent are the properties listed in Theorem 1 sufficient to ensure that f admits a representation (A)? As we shall see, the inequality (B), holding for self-adjoint $T \in \mathcal{A}$, is decisive for the existence of a representation (A) for compact operators. The role of (iv) would appear to be the foundation upon which (A), if true for compact $T \in \mathcal{A}$, may be extended to more general operators.

THEOREM 2. *Let $\mathcal{A}$ be a self-adjoint subalgebra of $B(\mathcal{X})$, and let f be a state of $\mathcal{A}$ such that (B) holds for self-adjoint $T \in \mathcal{A}$. Then*

(a) *f is the weak limit of convex combinations of pure states,*

$$\sum_{1 \leqslant p \leqslant n} c_p f_{x_p},$$

where the $x_p \in \Sigma$ and where the numbers c_p satisfy $c_p \geqslant 0$ and $\sum_{1\leqslant p\leqslant n} c_p \leqslant 1$;

(b) *there exists a positive Radon measure m on Σ satisfying $m(\Sigma) \leqslant 1$ and such that (A) holds for each compact $T \in \mathcal{A}$.*

Proof. It is enough to look at the behaviour of f on the set $\mathcal{A}_0$ of self-adjoint elements of $\mathcal{A}$. $\mathcal{A}_0$ is a real vector space, and each state f of $\mathcal{A}$ may be considered as an element of the real dual of $\mathcal{A}_0$. With this viewpoint adopted, (B) and the Bipolar Theorem shows that f is weakly adherent to the convex envelope of the pure states $f_x (x \in \Sigma)$. This signifies that (a) is true.

Turning to (b), a typical approximative sum referred to in (a) can be written in the form of a vector-integral $\int_\Sigma f_x dm'(x)$, where m' is the measure

$$\sum_{1\leqslant p\leqslant n} c_p \cdot \varepsilon_{x_p}$$

and ε_x is the Dirac measure at x. It follows from (a) that there exists a directed family (m_i) of measures of this type for which

$$f(T) = \lim_i \int_\Sigma (Tx, x)\, dm_i(x)$$

for each $T \in \mathcal{A}_0$. The family (m_i) has a vague limiting point, say m, which is necessarily a positive Radon measure m on Σ of total mass $m(\Sigma) \leqslant 1$. Then, if T is compact (so that (Tx, x) is continuous), the directed family of numbers: $i \to \int_\Sigma (Tx, x)\, dm_i(x)$ admits $\int_\Sigma (Tx, x)\, dm(x)$ as a limiting point. (b) is thereby proved.

We append two supplementary remarks.

Remark 2. If (A) is known to hold for all T in a subset $\mathcal{F}$ of $\mathcal{A}$, and if f is known to be order continuous (weakly or strongly), then (A) must still hold for all $T \in \mathcal{F}'$, where $\mathcal{F}'$ comprises those operators expressible as finite linear combinations of positive self-adjoint operators $S \in \mathcal{A}$ which are limits (weakly or strongly) of increasing directed families of positive self-adjoint members of $\mathcal{F}$. (Cf. the proof of Theorem 1, (iv)).

Remark 3. If (A) is known to hold for some m satisfying $m(\Sigma) \leqslant 1$ and all T in a set $\mathcal{F} \subset \mathcal{A}$ of self-adjoint operators, and if it is known that

$$\operatorname{Sup}\{f(T)\colon T \in \mathcal{F},\ \|T\| \leqslant 1\{ \geqslant 1, \tag{4.1}$$

then in fact $m(\Sigma) = 1$ and m is concentrated on the boundary Σ_1. Indeed, given $\varepsilon > 0$, we choose $T \in \mathcal{F}$ for which $\|T\| \leqslant 1$ and $f(T) \geqslant 1 - \varepsilon$, i. e.

$$\int_{\Sigma} (Tx, x)\, dm(x) \geqslant 1 - \varepsilon.$$

This shows already that $m(\Sigma) \geqslant 1 - \varepsilon$, so that ($\varepsilon$ being freely chosen) $m(\Sigma)$ must be exactly 1. Moreover, if σ_r $(0 < r < 1)$ is the ball $\|x\| \leqslant r$, we see that $1 - \varepsilon \leqslant \int_{\Sigma} (Tx, x)\, dm(x) \leqslant r^2 m(\sigma_r) + (1 - m(\sigma_r))$ and so $m(\sigma_r) \leqslant (1 - r^2)^{-1}\varepsilon$. Letting $\varepsilon \to 0$, we conclude that $m(\sigma_r) = 0$ and so, by countable additivity of m, that $m(\Sigma - \Sigma_1) = 0$, as asserted.

In applying Theorem 2 the following auxiliary result is useful.

PROPOSITION 1. *Let $\mathcal{A}$ be a closed, self-adjoint subalgebra of $B(\mathfrak{X})$, $\mathcal{A}'$ its commutator. Let f be a state of $\mathcal{A}$ and suppose that both $N(f)$ and*

$$\text{(4.2)} \qquad \operatorname{Sup}\{f(T^*T) : T \in \mathcal{A} \cap \mathcal{A}', \|T\| \leqslant 1\}$$

have the value one. Then (B) holds for each self-adjoint $T \in \mathcal{A}$.

Proof. Take any self-adjoint $T_1 \in \mathcal{A}$. Let $\mathcal{A}_1$ be the CC^*-algebra generated by $\{T_1\} \cup (\mathcal{A} \cap \mathcal{A}')$. Then $\mathcal{A}_1 \subset \mathcal{A}$. Let $f_1 = f|\mathcal{A}_1$: this is a state of $\mathcal{A}_1$ which, since $N(f)$ and (4.2) have the common value one, is normalised. Hence, by the results recalled in § 1, there is a positive Radon measure μ on the character space X of A_1 such that $\mu(X) = 1$ and

$$f_1(T) = \int_X \hat{T}\, d\mu \qquad (T \in \mathcal{A}_1).$$

This shows in particular that $f_1(T) \leqslant \|T\|$ for $T \in \mathcal{A}_1$. Furthermore, since $\mathcal{A}_1 \approx C_0(X)$, we may choose a positive self-adjoint $E \in \mathcal{A}_1$ such that $\|E\| \leqslant 1$ and $nE + T_1$ is positive self-adjoint, n being a suitably chosen real number. Also, since (4.2) has the value one, an "increase" in E may be made so as to arrange that $f_1(E) \geqslant 1 - \varepsilon/n$, ε being any preassigned positive number. Then, by what we have just proved (applied to $T = nE + T_1$), we have

$$nf_1(E) + f(T_1) = f_1(nE + T_1) \leqslant \|nE + T_1\|.$$

Since $nE + T_1$ is positive self-adjoint, the last term is equal to

$$\operatorname{Sup}_{x \in \Sigma} ((nE + T_1)x, x) \leqslant n + \operatorname{Sup}_{x \in \Sigma} (T_1 x, x).$$

Hence

$$f(T_1) \leqslant \varepsilon + \operatorname{Sup}_{x \in \Sigma} (T_1 x, x).$$

Letting $\varepsilon \to 0$, the desired result follows. This completes the proof.

By combining this result with Theorem 2 we obtain directly

THEOREM 3. *Let $\mathcal{A}$, $\mathcal{A}'$ and f be as in Proposition 1. Then there exists a positive Radon measure m on Σ such that $m(\Sigma) \leqslant 1$ and (A) holds for all compact $T \in \mathcal{A}$. Furthermore,* (a) *of Theorem 2 is valid.*

If we apply Theorem 3 to the case $\mathcal{A} = B(\mathfrak{X})$, then $\mathcal{A}'$ is $B(\mathfrak{X})$ or $\{I\}$ according as $\dim \mathfrak{X} = 1$ or $\dim \mathfrak{X} > 1$. The hypotheses if Theorem 3 reduce then to normalisation of f: $f(I) = 1$ — as indeed it does whenever $I \in \mathcal{A}$. Also, any positive self-adjoint $T \in B$ is the strong limit of an increasing directed family of compact operators (e. g. the operators TP, where P is a variable projector of finite rank). So, bearing in mind Remarks 2 and 3, we derive

THEOREM 4. *If f is a normalised, strongly order-continuous state of $B(\mathfrak{X})$, f admits a representation (A) in which the positive Radon measure m on Σ satisfies $m(\Sigma) = 1$ and $m(\Sigma - \Sigma_1) = 0$.*

Remark. 4 If $\mathfrak{X}$ is separable, strong ordercontinuity is required only in sequential form.

If $\mathcal{A}$ is a CC^*-algebra, $\mathcal{A}' \supset \mathcal{A}$, and we derive similarly

THEOREM 5. *If $\mathcal{A}$ is a CC^*-algebra and f a normalised state of $\mathcal{A}$, there exists a positive Radon m on Σ such that $m(\Sigma) \leqslant 1$ and (A) holds for all compact $T \in \mathcal{A}$. Once again,* (a) *of Theorem 2 is valid.*

To describe matters very briefly, the preceding results establish the existence of a representation (A) for any closed self-adjoint subalgebra $\mathcal{A}$ of $B(\mathfrak{X})$ and any order-continuous state f of $\mathcal{A}$, provided that (a) $\mathcal{A} \cap \mathcal{A}'$ is "sufficiently large" (cf. Theorem 3), and (b) $\mathcal{A}$ contains "sufficiently many" compact operators.

§ 5. Existence of states admitting no representation (A). We aim to show that in general there exist states f of $\mathcal{A}$ which are normalisable $(N(f) < +\infty)$, which satisfy (B) for self-adjoint $T \in \mathcal{A}$, and which even so do not admit a representation (A), the failure to do so being attributable to their lack of order-continuity. This state of affairs presents itself whenever $\mathcal{A}$ fulfils the following condition:

(C) *$\mathcal{A}$ contains a sequence (T_n) of positive self-adjoint operators which is monotone decreasing, converges weakly to 0, but does not converge uniformly (i. e. in norm) to 0.*

Assuming the existence of such a sequence, consider the functions $\varphi_n(x) = (T_n x, x)$ on Σ. Then $\varphi_n \downarrow 0$ simply, but the convergence is not uniform. (Dini's Theorem informs us, incidentally, that the T_n cannot all be compact). From the general Proposition 2 to follow we may conclude that there exists a positive, bounded, finitely-additive measure p on Σ

such that

$$\operatorname*{Inf}_{n\geqslant 1}\int_{\Sigma}\varphi_n(x)\,dp(x) > 0.$$

This measure p, which of course fails to be a Radon measure, serves to define a state f of $\mathscr{A}$ via the formula

$$f(T) = \int_{\Sigma}(Tx, x)\,dp(x)$$

Thanks to the choice of p, $\operatorname*{Inf}_{n\geqslant 1} f(T_n) > 0$. Thus f is not order-continuous and therefore (§ 3) cannot admit a representation (A) (wherein m is assumed to be a Radon measure).

It remains to establish

PROPOSITION 2. *Let Σ be an arbitrary set, and let (φ_n) be a monotone decreasing sequence of bounded functions on Σ such that $\varphi_n \downarrow 0$ simply but not uniformly on Σ. Then there exists a positive, bounded, finitely-additive measure p on Σ such that*

$$\operatorname*{Inf}_{n\geqslant 1}\int_{\Sigma}\varphi_n(x)\,dp(x) > 0.$$

Proof. Denote by E the Banach space of bounded real-valued functions on Σ, equipped with the "sup" norm. It is known (see e. g. [1], Section VII) that the dual E' of E may be identified with the of all bounded, finitely-additive measures on Σ, the duality being established by integration:

$$\langle\psi, w\rangle = \int_{\Sigma}\psi\,dw \qquad (\psi\in E,\ w\in E').$$

Our hypotheses ensure that $\operatorname*{Inf}_{n\geqslant 1}\|\varphi_n\| = c > 0$. Let K be the convex envelope in E of the φ_n. Each $\psi\in K$ admits an expression

$$\psi = \sum_{1\leqslant i\leqslant k} a_i\varphi_{n_i},$$

where $a_i \geqslant 0$ and $\sum_{1\leqslant i\leqslant k} a_i = 1$. It follows at once that $\|\psi\| \geqslant c$ for each $\psi\in K$, hence in particular that 0 is not adherent to K. But then (Hahn-Banach Theorem) 0 is not even weakly adherent to K. So there exists some $w\in E'$ for which the sequence $\int_{\Sigma}\varphi_n\,dw$ does not converge to 0 as $n\to\infty$. The same must be true, therefore, if w is replaced by a suitable one of its positive and negative parts, say p.

Then

$$\operatorname*{Inf}_{n\geqslant 1}\int_{\Sigma}\varphi_n\,dp > 0,$$

as asserted. This completes the proof of Proposition 2.

Remark 5. Naturally, (C) fails to hold if $\mathscr{A}$ contains only compact operators. On the other hand, there is no difficulty in exhibiting a class of algebras $\mathscr{A}$ which satisfy condition (C).

Suppose for example that $\mathfrak{X}$ is separable, and that $\mathscr{A}$ is closed and contains at least one self-adjoint operator $\mathscr{A}$ whose spectrum is uncountable: we claim that then A satisfies (C). To see this, form the CC^*-algebra $\mathscr{A}_1\subset\mathscr{A}$ containing A, and let X be its character space. X is homeomorphic with the spectrum of A, hence is uncountable. Now with each point χ of X is associetad a "spectral projector" P_χ, the P_χ are orthogonal, and $P_\chi\neq 0$ if and only if the one-point set $\{\chi\}\subset X$ has positive $\mu_{x,x}$-measure for some $x\in\mathfrak{X}$ (cf. [2], p. 70-71). On the other hand, since $\mathfrak{X}$ is separable, at most countably many P_χ can be non-zero. Therefore, since X is uncountable, there must exist a point χ of X such that $\mu_{x,x}(\{\chi\}) = 0$ for all $x\in\mathfrak{X}$. In this case it is easy to construct a sequence $(h_n)\subset C_0(X)$ of positive functions which decrease monotonely to zero at all points of X other than χ, whilst $h_n(\chi) = 1$ for all n. To h_n corresponds then a self-adjoint $T_n\in\mathscr{A}_1\subset\mathscr{A}$, these T_n decrease monotonely, they converge strongly to 0 (since the $\hat{T}_n = h_n$ converge boundedly to 0 a. e. for each "spectral measure" $\mu_{x,x}$), and yet $\|T_n\| = \|h_n\| \geqslant 1$ for all n. Thus (C) is satisfied.

§ 6. An analogue of Theorem 2. It is perhaps worth noting that continuous linear forms on $B = B(\mathfrak{X})$, and therefore states of B in particular, may be expressed in terms of tensor products, and that this viewpoint leads to an analogue of Theorem 2 in which precision is in some measure exchanged for generality.

It is necessary to make some identifications. In the first place we use the fact that $\mathfrak{X}$ is reflexive; more precisely we introduce the standard isometric mapping $J: x\to x'$ of $\mathfrak{X}$ onto $\mathfrak{X}'$ such that $\langle x, y'\rangle = (x, y)$. This allows one to set up an isometric isomorphism between $B(\mathfrak{X})$ and $B(\mathfrak{X}')$ by associating with $u\in B(\mathfrak{X}')$ the operator $T = (J^{-1}uJ)^*$. Secondly, we use the customary identification of bounded hermitian forms and bounded endomorphisms. When all this is done the construction ([3], p. 28) of the projective tensor product $\mathfrak{X}\hat{\otimes}\mathfrak{X}'$ ensures that it admits $B(\mathfrak{X})$ as its dual, the duality being such that $\langle x\otimes y', T\rangle = (Tx, y)$. Using the general principle of injecting a normed vector space into its bidual, $\mathfrak{X}\hat{\oplus}\mathfrak{X}'$ appears finally as a strongly closed vector subspace of $B(\mathfrak{X}')$. The pure state f_x defined by (0.1) is thereby identified with $x\oplus x'$. The decomposition (A) is at the same time expressed in the form

$$\text{(A}') \qquad f = \int_{\Sigma} (x \otimes x')\, dm(x);$$

the integral on the right of (A′) is interpreted in the "weak" sense, the mapping $x \to x \oplus x'$ of Σ into $B(\mathfrak{X})'$ being bounded and weakly Borel-measurable (cf. § 2).

A fundamental theorem about tensor products of Fréchet spaces leads at once to an analogue of Theorem 2 applying to arbitrary elements of $B(\mathfrak{X})'$.

THEOREM 6. *Let $\mathcal{A}$ be a vector subspace of $B(\mathfrak{X})$, and let f be any continuous linear form on $\mathcal{A}$. Then*

(a) *f is the weak limit of finite linear combinations*

$$\sum_{1 \leqslant p \leqslant n} \gamma_p (x_p \otimes y_p')$$

where $x_p, y_p \in \Sigma$ and the numbers γ_p satisfy $\sum_{1 \leqslant p \leqslant n} |\gamma_p| \leqslant \|f\| + \varepsilon$ ($\varepsilon > 0$ chosen freely in advance);

(b) *there exists a Radon measure M on $\Sigma \times \Sigma$ of total mass $|M|\,(\Sigma \times \Sigma) \leqslant \|f\| + \varepsilon$ such that*

$$f(T) = \int_{\Sigma \times \Sigma} \langle x \otimes y', T \rangle\, dM(x, y) = \int_{\Sigma \times \Sigma} (Tx, y)\, dM(x, y)$$

for all compact $T \in \mathcal{A}$.

Proof. By the Hahn-Banach Theorem we may as well assume that $\mathcal{A} = B(\mathfrak{X})$. Nor is there any loss of generality in assuming that $\|f\| = 1$. The Bipolar Theorem shows that the unit ball of any normed space is weakly dense in the unit ball of its bidual. In particular, therefore, f is the weak limit of elements of the unit ball in $\mathfrak{X} \hat{\otimes} \mathfrak{X}'$. On the other hand it is known ([3], p. 51, Théorème 1) that, given any $\varepsilon > 0$, each element of the unit ball of $\mathfrak{X} \hat{\otimes} \mathfrak{X}'$ can be written in the form

$$\sum_{p=1}^{\infty} \gamma_p (x_p \otimes y_p'),$$

where $\sum_{p=1}^{\infty} |\gamma_p| \leqslant 1 + \varepsilon$ and $x_p, y_p \in \Sigma$. Each such infinite sum is the strong limit of its finite partial sums. Whence assertion (a) follows at once.

Assertion (b) follows from (a) exactly as in the proof of Theorem 2.

Remark 6. If we assume that f is weakly continuous, we may conclude that f is actually equal to the infinite sum $\sum_{p=1}^{\infty} \gamma_p (x_p \otimes y_p')$ for suitably chosen γ_p, x_p and y_p. To check weak continuity it suffices to verify weak continuity of f when restricted to bounded subsets of $B(\mathfrak{X})$; and if $\mathfrak{X}$ is separable, weak sequential continuity suffices.

§ 7. Connections with von Neumann's trace-states. Our states of $B = B(\mathfrak{X})$ have precisely the formal properties of the expectation values used for the statistical description of assemblages of quantum mechanical systems ([6], Chapter IV). Although von Neumann's expectation values are defined only for self-adjoint operators, they may be extended in an obvious manner into states as defined in § 1. Moreover, although von Neumann does not explicitly delimit any algebra specifically, it is clear from what is said (loc. cit. p. 313) that all self-adjoint operators are to be included; thus B is the only possible choice within the framework of this paper. (Physically one would demand the inclusion even of certain unbounded operators, but neither von Neumann's arguments nor ours cover such an extension.)

For this case our results are incorporated in Theorem 4. On the other hand, von Neumann's arguments lead to the conclusion that each weakly continuous state f of B admits a representation of the type

$$\text{(N)} \qquad f(T) = \mathrm{Tr}\, HT,$$

where "Tr" signifies trace, and where H is a positive self-adjoint operator depending only on f. We must leave aside infinite-valued expectation values (cf. [6], p. 310) and so assume that $\mathrm{Tr}\, H < +\infty$. With this limitation we speak of the states (N) as trace-states. The uniquely determined operator H serves to describe completely the behaviour of the assemblage in question; it (or rather its matrix representative) is the so-called "statistical matrix" of the assemblage.

Von Neumann shows that the minimal trace-states are precisely the pure states (0.1) (corresponding to the case in which H is a one-dimensional projector or a multiple thereof), and that there are no trace-states which are at the same time characters (dispersion-free states).

It is easily seen that the decompositions (A) and (N) are in a sense equivalent. On the one hand, if $H_x = x' \otimes x$ is the operator defined by $H_x y = (y, x)x$, so that (N) with $H = H_x$ gives the pure state (0.1), then the state f defined by (A) may be written as the trace-state corresponding to the operator

$$\text{(7.1)} \qquad H = \int_{\Sigma} H_x\, dm(x) = \int_{\Sigma} (x' \otimes x)\, dm(x);$$

the integral here exists in the weak sense and represents a positive self-adjoint operator with trace

$$\operatorname{Tr} H = \int_{\Sigma} (\operatorname{Tr} H_x)\, dm(x) = \int_{\Sigma} \|x\|^2 dm(x),$$

which is clearly finite. On the other hand, if H is positive self-adjoint and has finite trace, it is necessarily compact and we can choose a complete orthonormal base (e_n) for $\mathcal{X}$ and scalars $\lambda_n \geqslant 0$ such that

$$\operatorname{Tr} H = \sum_n \lambda_n < +\infty$$

and

$$H = \sum_n \lambda_n (e_n' \otimes e_n)$$

Then (N) takes the form

$$\operatorname{Tr} HT = \sum_n \lambda_n (Te_n, e_k) \qquad (T \epsilon B),$$

so that (A) holds with

$$m = \sum_n \lambda_n \cdot \varepsilon_{e_n}$$

where ε_x is the Dirac measure at x.

It appears from this argument that the measure m appearing in (A) is not uniquely determined (if it exists) by f, and this even in the most favourable case in which $\mathcal{A} = B$; cf. [6], p. 330-332.

Remark 7. As has been said in § 1, there are obvious difficulties in the path of extending von Neumann's direct proof of a representation (N) from the case of $B(\mathcal{X})$ to a subalgebra $\mathcal{A}$ thereof. Whilst the theory of CC^*-algebras may help in this direction (at least in respect of weakly closed algebras $\mathcal{A}$ and compact operators $T \epsilon \mathcal{A}$), all the simplicity of von Neumann's approach is lost, and there seems little to recommend this method in preference to those adopted in § 4 above.

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