We add here a numerical example of the splitting formula in the simplest case where $F = \Omega$ is rational number field and $\mathfrak{p} = 2$. Let Ω_2^* be the multiplicative group of non-zero elements of the 2-adic number field Ω_2. Then, for every representative of Ω_2^*/Ω_2^{*2}, the value of $w_2(a)$ is given by

$$\begin{array}{rcl} a & = & 1,\ 5,\ -1,\ -5,\ 2,\ 10,\ -2,\ -10 \\ w_2(a) & = & 1,\ 1,\ \ i,\ \ i,\ 1,\ -1,\ \ i,\ -i. \end{array}$$

This gives, for example,

$$\left(\frac{10,\ -2}{2}\right) = \frac{-1 \cdot i}{i} = -1.$$

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Reçu par la Rédaction le 6. 4. 1960

ACTA ARITHMETICA
VI (1961)

On the existence of primes in short arithmetical progressions

by

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Introduction. In 1944 Linnik (see [4]) proved the existence of an absolute constant $c > 0$ such that the smallest prime in any arithmetical progression $ku+l$, $(k, l) = 1$, $u = 0, 1, 2, \ldots$ does not exceed k^c. In 1954 Rodosskiĭ (see [6]) gave a shorter proof in which a fundamental lemma of Linnik was replaced by a weaker result (see further (10)). Introducing a new parameter in Rodosskiĭ's proof in 1955 I proved (see [2]) the existence of an absolute constant $c > 0$ such that there is at least one prime $p \equiv l \pmod{k}$, $(k, l) = 1$, in the interval

$$(1) \qquad (x, xk^c) \quad \text{for all} \quad x \geqslant 1$$

and I proved that there are other absolute constants c_1, c_2 $(c_2 > c_1 > 0)$ such that

$$(2) \qquad \pi(x; k, l) > xk^{-c_1} \quad \text{for all} \quad x \in (k^{c_2}, k^{k^2}),$$

if $(k, l) = 1$ and $\pi(x; k, l)$ denotes the number of primes $p \equiv l \pmod{k}$ not exceeding x.

The estimates (1) and (2) are of some importance for $x < \exp k^{\varepsilon_1}$, ε_1 denoting (throughout this paper) an arbitrarily small positive constant. In this case the uncertainty about the existence or nonexistence of the real exceptional zero of Dirichlet's function $L(s, \chi)$ with a real character χ modulo k is the reason why the existing estimates of $\pi(x; k, l)$ and estimates of the difference of consecutive primes $\equiv l \pmod{k}$ fail to give us any positive information. For $x \geqslant \exp k^{\varepsilon_1}$ and $k > k_0(\varepsilon_1)$ according to Tchudakoff ([3]) there is at least one prime $\equiv l \pmod{k}$ in the interval

$$(3) \qquad \left(x, x(1+x^{-1/4})\right),$$

and $\pi(x; k, l) > c_3(\varepsilon_1)x/\varphi(k)\log x$, where $\varphi(k)$ is Euler's function denoting the number of natural numbers $l \leqslant k$ with $(l, k) = 1$ (1).

(1) For these results see, for example, K. Prachar [5], IX Satz 2.2, IV Satz 8.2; IX Satz 3.2, IX Satz 4.2. (Roman numbers denoting the chapters, A the appendix).

It is the aim of this paper to improve the estimates (1) and (2) in such a way that the increasing of x should diminish the length of the interval in which there is at least one prime $p \equiv l\,(\mathrm{mod}\,k)$ and it should possibly increase the ratio $\varphi(k)\pi(x; k, l):x$. The principal result o this paper may be formulated as the following

THEOREM. *There are absolute constants $c > 0$, $c' > 0$ such that for any positive $\varepsilon \leqslant c$, for all $k \geqslant k_0(\varepsilon)$ and all*

$$x \geqslant k^{c'\log(c/\varepsilon)}$$

there is at least one prime $p \equiv l\,(\mathrm{mod}\,k)$, $(k, l) = 1$ in the interval

$$(x, xk^{\varepsilon}).$$

Actually there are $\geqslant x/\varphi(k)k^{2\varepsilon}$ primes $\equiv l\,(\mathrm{mod}\,k)$ for $x < \exp k^{\varepsilon}$, $k \geqslant k_1(\varepsilon)$.

By absolute constants we understand constants which are independent of k, l, ε.

The function $k_0(\varepsilon)$ of the theorem depends on Siegel's constant $c_5(\varepsilon)$ (see further (8)), no estimate of which is known at present.

COROLLARY. *We have*

$$\pi(x; k, l) > x/\varphi(k)k^{3\varepsilon} \tag{4}$$

for all $x \in (k^{c'\log(c/\varepsilon)+\varepsilon}, \exp k^{\varepsilon})$, $k \geqslant k_2(\varepsilon)$.

Using $\varepsilon = c$ in the theorem and the corollary we get (2) and the result concerning (1) as stated above.

For another consequence of the theorem see the note on functions of Liouville and Möbius at the end of this paper.

We shall prove the theorem by the method of Linnik-Rodosskiĭ supplied with two more parameters and applied to other dissection of the critical strip in regions of summation. Using this method for $x \geqslant \exp k^{\varepsilon_1}$ we can prove the existence of a prime $p \equiv l\,(\mathrm{mod}\,k)$ in the interval

$$\big(x, x(1+c_4)\big),$$

where c_4 is some absolute constant > 0 (see further (67)). In proving this we use a rather weak estimate of the number of L-functions having zeros in the neighbourhood of $s = 1$. Therefore we cannot get as good an estimate as (3), the proof of which uses a more profound knowledge about the distribution of the zeros of L-functions; nevertheless it cannot be applied to the case $x < \exp k^{\varepsilon_1}$.

A short note containing the main results of this paper has been sent to the Doklady Akad. Nauk SSSR.

Preliminary theorems. We take for granted the general properties of Dirichlet's functions $L(s, \chi)$ with characters χ modulo $k > 1$ to such an extent as is given in Prachar's book [5] and, with a few exceptions, we use the same notation.

$A, B, C, c_0, c_5, c_6, \ldots, k_3, k_4, \ldots, \alpha, \eta_0, \varepsilon, \varepsilon', a, b$ denote positive constants which may depend on each other but not on k, l (and not on any of the parameters $t, T, \lambda, \ldots$, which are used further on). The dependence on $\varepsilon, \varepsilon'$ or ε_1 is always marked in the usual way.

n denotes natural numbers, p—primes. The natural numbers k and l are always supposed to have the highest common divisor $(k, l) = 1$. (By (a, b) for real a, b we denote the interval $a < x < b$ as well, but there is no danger of a confusion arising from this ambiguity.)

$y \ll x$ or $y = O(x)$ for positive x has the meaning of the inequality $|y|:x < c_n$ for some n $(n > 5)$.

The complex variable is generally denoted by $s = \sigma + it$ $(\sigma = \mathrm{re}\,s$, $t = \mathrm{im}\,s)$; sometimes we use w or z as well.

We use $\exp z$ for the exponential function e^z, whenever it is more convenient for the print.

Further we shall use the following properties 1-9 of $L(s, \chi)$ or other functions.

1. *In the region*

$$\sigma \geqslant 1 - c_0/\log k(|t|+2) \geqslant \tfrac{3}{4} \tag{5}$$

for all characters χ modulo k we have $L(s, \chi) \neq 0$, with at most one exception corresponding to a function $L(s, \chi_1)$ with a real non-principal character χ_1; this function $L(s, \chi_1)$ may have in (5) a single real zero $\beta_1 < 1$ ([5], IV Satz 6.9).

2. *There is an A such that for $\delta_1 = 1 - \beta_1$,*

$$\delta_0 = \begin{cases} \delta_1 & \text{if} \quad \delta_1 \leqslant A/\log k, \\ A/\log k & \text{otherwise}, \end{cases} \tag{6}$$

$$\lambda_0 = A\log\frac{eA}{\delta_0\log k} \in [A, \tfrac{1}{2}\log k], \tag{7}$$

the rectangle $(1 - \lambda_0/\log k \leqslant \sigma \leqslant 1, |t| \leqslant k)$ contains no zeros ϱ of any function $L(s, \chi)$ with a character χ modulo k with probably one exception $\varrho = \beta_1$ ([5], X (4.8), (4.9), (4.10)).

3. (Siegel's theorem). *For any $\varepsilon > 0$ and any real character χ modulo k we have $L(\sigma, \chi) \neq 0$ in the region*

$$\sigma \geqslant 1 - c_5(\varepsilon)k^{-\varepsilon} \tag{8}$$

([5], IV Satz 8.2).

4. *Let $N_\chi(T)$ denote the number of zeros of $L(s,\chi)$ in the rectangle $(0 \leqslant \sigma \leqslant 1,\ |t| \leqslant T)$. Then for any $T \geqslant 2$ we have*

$$N_\chi(T) = \frac{1}{\pi} T\log T + a(k)T + O(\log kT), \tag{9}$$

where $a(k)$ is a real function $\ll \log 2k$, which does not depend on T ([5], VII Satz 3.4).

5. *Let $N(\delta, T) = N(\delta, T, k)$ denote the number of zeros of all functions $L(s,\chi)$ with characters modulo k in the rectangle $(\sigma \geqslant 1-\delta,\ |t| \leqslant T)$. Then there is a C such that for all $\lambda \in [0, \log k]$ we have*

$$N(\lambda/\log k,\ e^{\lambda}/\log k) < e^{C\lambda}. \tag{10}$$

(See [5], X Satz 2.2. This is a simple consequence of Rodosskiĭ's substitute for Linnik's fundamental lemma. The latter gives a similar estimate for the number of the functions $L(s,\chi)$ which have at least one zero in the rectangle $\sigma > 1-\lambda/\log k$, $|t| \leqslant \min(\lambda^{100}, \log^3 k)$.)

6. *Let $\lambda_1, \lambda_2, \ldots$ be a sequence of non-decreasing real numbers with $\lim \lambda_n = \infty$, and let a_n $(n = 1, 2, \ldots)$ denote arbitrary real or complex numbers. Then for any real or complex function $g(\xi)$ having a continuous derivative in the segment $\lambda_1 \leqslant \xi \leqslant x$ we have*

$$\sum_{\lambda_1 \leqslant \lambda_n \leqslant x} a_n g(\lambda_n) = A(x)g(x) - \int_{\lambda_1}^{x} A(\xi) g'(\xi)\, d\xi, \tag{11}$$

where

$$A(\xi) = \sum_{\lambda_1 \leqslant \lambda_n \leqslant \xi} a_n$$

([5], A, Satz 1.4.).

7. *Let $\Lambda(n)$ be $\log p$ if n is a positive power of the prime p and 0 otherwise. Then we have for all $x \geqslant k^4$*

$$\psi(x; k, l) = \sum_{\substack{n \leqslant x \\ n \equiv l \,(\mathrm{mod}\, k)}} \Lambda(n) \ll x/\varphi(k) \tag{12}$$

(See [5], VII, proof of Lemma 7.1).

8. *For any positive numbers m, y and any real number a we have*

$$\int_{a-i\infty}^{a+i\infty} m^{-z} e^{z^2 y}\, dz = i\sqrt{\frac{\pi}{y}}\, e^{-(\log m)^2/4y} \tag{13}$$

([5], A, Lemma 3.3).

9. *For any positive numbers x, y we have*

$$\sum_{n=2}^{\infty} \chi(n)\Lambda(n) n^{-s} \exp\left(-\frac{\log^2 n/x}{4y}\right) = i\sqrt{\frac{y}{\pi}} \int_{2-i\infty}^{2+i\infty} \frac{L'}{L}(w,\chi)\, x^{w-s} e^{(w-s)^2 y}\, dw. \tag{14}$$

To prove this we use in (13) $m = n/x$, $a = 2-\sigma$, multiply through by $\chi(n)\Lambda(n)n^{-s}$ and sum over all $n \geqslant 2$. Since

$$\sum_n \chi(n)\Lambda(n) n^{-z-s} = -\frac{L'}{L}(z+s, \chi),$$

we get (14) putting $z+s = w$. (See [5], A Satz 3.3.)

Proof of the auxiliary inequality (46). We use in (14) $s = -\frac{1}{2}$, multiply through by $\bar{\chi}(l)$ (the complex conjugate of $\chi(l)$) and sum over all $\varphi(k)$ characters modulo k taking into account that

$$\frac{1}{\varphi(k)} \sum_\chi \chi(n)\bar{\chi}(l) = \begin{cases} 1 & \text{if} \quad n \equiv l \,(\mathrm{mod}\, k), \\ 0 & \text{otherwise} \end{cases}$$

([5], IV (2.11)). Moving the path of integration to the line $\mathrm{re}\, w = -\frac{1}{2}$ (which is legitimate, since the integrals and sum of residues are absolutely convergent) we get the identity

$$\begin{aligned} \Phi(x, y; k, l) &= \varphi(k) \sum_{\substack{n=2 \\ n \equiv l \,(\mathrm{mod}\, k)}}^{\infty} \Lambda(n)\sqrt{n}\, \exp\left(-\frac{\log^2 n/x}{4y}\right) \\ &= 2\sqrt{\pi y}\, x^{3/2} \exp\left(\tfrac{9}{4}y\right) - 2\sqrt{\pi y} \sum_{\chi, \varrho_\chi} \bar{\chi}(l) \operatorname*{Res}_{w=\varrho_\chi} \frac{L'}{L}(w,\chi)\, x^{w+1/2} \exp\{(w+\tfrac{1}{2})^2 y\} + \\ &\quad + \sum_\chi \bar{\chi}(l)\, i\sqrt{y/\pi} \int_{-1/2-i\infty}^{-1/2+i\infty} \frac{L'}{L}(w,\chi)\, x^{w+1/2} \exp\{(w+\tfrac{1}{2})^2 y\}\, dw, \end{aligned} \tag{15}$$

where $\Phi(x, y; k, l)$ denotes the function defined by the left side of (15). On the right side ϱ_χ runs over all zeros of $L(s,\chi)$ having the real parts $\geqslant 0$. Since

$$\frac{L'}{L}\left(-\tfrac{1}{2}+it, \chi\right) \ll \log k(|t|+2)$$

([5], VII Satz 4.3) and $k>1$, the integral in (15) is $\ll y^{-1/2}\log 2k$ (cf. [5], VII, proof of Satz 6.1). Hence, if β_1 denotes the probably existing real zero of $L(s,\chi_1)$ with the real exceptional character χ_1 and if we put

$$\delta_1 = 1-\beta_1, \qquad \varrho = 1-\delta+i\tau, \tag{16}$$

we have, by (15),

$$\Phi(x,y;k,l) \tag{17}$$
$$= 2\sqrt{\pi y}\,x^{3/2}\exp(\tfrac{9}{4}y)\{1-E_1x^{-\delta_1}\chi_1(l)\exp[-\delta_1(3-\delta_1)y]-S\}+O\big(\varphi(k)\log k\big),$$

where

$$E_1 = \begin{cases} 1 & \text{if } \beta_1 \text{ exists,} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$S = \sum_{\chi}\bar\chi(l)\sum_{\varrho_\chi(\neq\beta_1)} x^{-\delta}\exp\{[-\delta(3-\delta)-\tau^2+i\tau(3-2\delta)]y+i\tau\log x\}. \tag{18}$$

Let $f(\eta)$ be a real or complex function of the real variable η and B any natural number. By $I_Bf(\eta)$ we denote the integration B times repeated, of the function $f(\eta)$, the range of integration being $(\eta,\eta+1)$. For any constant a and any continuous functions $f(\eta)$, $g(\eta)$ we have

$$I_B1 = 1, \qquad I_Baf = aI_Bf, \qquad I_B(f+g) = I_Bf+I_Bg, \tag{19}$$

and

$$I_Be^{a\eta} = \left(\frac{e^a-1}{a}\right)^B e^{a\eta}. \tag{20}$$

Let $f(\eta)$, $f_1(\eta)$ be real continuous functions and let $f\leqslant f_1$ for all η. Then we have

$$I_Bf \leqslant I_Bf_1, \tag{21}$$

$$\min_{\eta\leqslant t\leqslant\eta+B} f(t) \leqslant I_Bf(\eta) \leqslant \max_{\eta\leqslant t\leqslant\eta+B} f(t). \tag{22}$$

The properties (19)-(22) may be proved by induction with respect to B.

Further we will use a fixed B satisfying the inequality

$$B \geqslant \max(2, C+1), \tag{23}$$

where C is defined by (10), and a constant $\eta_0 \ll 1$,

$$\eta_0 \geqslant \max(4, C), \tag{24}$$

other restrictions on which will follow. Suppose η_0 to be the initial value of η; the operation I_B increases it by B and hence we have the restriction on η:

$$\eta_0 \leqslant \eta \leqslant \eta_0+B. \tag{25}$$

We shall use in (17)

$$x = k^{\xi}, \qquad \xi \geqslant 0, \tag{26}$$

$$y = \frac{\eta}{\nu}\log k, \tag{27}$$

with η satisfying (25) and

$$1 \leqslant \nu \leqslant \log k, \qquad \nu \leqslant e^{a\xi} \tag{28}$$

for all $\xi\geqslant 0$, a satisfying the inequalities

$$0 < a \leqslant \min(\tfrac{1}{10}, \tfrac{1}{3}B^{-1}, A/3B), \tag{29}$$

where A and B are defined by (6), (7), (23). Let us divide (17) through by $2\sqrt{\pi y}\,x^{3/2}\exp(\frac{9}{4}y)$ and effect the operation I_B. Writing

$$I_B\frac{\Phi(x,y;k,l)}{2\sqrt{\pi y}\,x^{3/2}\exp(\frac{9}{4}y)} = U, \tag{30}$$

we get, by (17), (19),

$$U = 1-E_1\chi_1(l)x^{-\delta_1}I_Be^{-\delta_1(3-\delta_1)y}-I_BS+ \tag{31}$$
$$+I_By^{-1/2}\exp(-\tfrac{9}{4}y)\,O\big(x^{-3/2}\varphi(k)\log k\big).$$

Using (27), (21), (22), (28), (24), (29) we get the estimate

$$I_By^{-1/2}\exp(-\tfrac{9}{4}y) \leqslant \exp\left(-\tfrac{9}{4}\frac{\eta_0}{\nu}\log k\right)$$
$$\leqslant \begin{cases} 1 & \text{if } \xi>10, \\ \exp(-\tfrac{9}{4}\eta_0e^{-a\xi}\log k) \leqslant k^{-9e^{-a\xi}} \leqslant k^{-9/e} < k^{-3,2} & \text{if } 0\leqslant\xi\leqslant 10. \end{cases}$$

Hence for all $x\geqslant 1$ and $k>k_3$ we get the estimate

$$I_By^{-1/2}\exp(-\tfrac{9}{4}y)\,O\big(x^{-3/2}\varphi(k)\log k\big) < k^{-2}. \tag{32}$$

By (27), (21), (22) we have

$$I_Be^{-\delta_1(3-\delta_1)y} < \exp\left(-2\frac{\eta_0}{\nu}\,\delta_1\log k\right). \tag{33}$$

Using (18), (20), (27) we get the expression

$$(34)\qquad |I_B S| \leqslant (2\nu)^B \sum_{\varrho} x^{-\delta} \frac{\exp\left\{-\frac{\eta_0}{\nu}(2\delta+\tau^2)\log k\right\}}{|(2\delta+\tau^2+i\tau)\log k + i\tau\log x|^B} = T_1+T_2+T_3.$$

The summation is extended over all zeros $\varrho = 1-\delta+i\tau \neq \beta_1$ with $\delta \leqslant 1$ of all the functions $L(s,\chi)$ with characters χ modulo k. The critical strip $0 \leqslant \sigma \leqslant 1$ is cut in three regions G_1, G_2, G_3 as defined below, and T_1, T_2, T_3 denote the corresponding parts of the sum in (34).

Let G_1 be the region $(0 \leqslant \sigma \leqslant 1,\ |t| \geqslant \log k)$. Then we have

$$T_1 \ll \sum_{|\tau|\geqslant \log k} \exp\left(-\frac{\eta_0}{\nu}\tau^2\log k\right)$$

$$\ll \varphi(k)\int_{\log k}^{\infty} \exp\left(-\frac{\eta_0}{\nu}t^2\log k\right)\cdot t^2\log k\cdot \log kt\,dt$$

$$< k\log^2 k\int_{\log k}^{\infty}\exp\left(-\frac{\eta_0}{\nu}t^2\log k\right)\cdot t^2\log t\,dt$$

$$< k\log^2 k\int_{\log k}^{\infty}\exp\left(-\frac{\eta_0}{\nu}t^2\log k+3\log t\right)dt$$

$$< k\log^2 k\int_{\log k}^{\infty}\exp(-c_6 t^2)\,dt$$

$$< k\log^2 k\int_{\log k}^{\infty}\exp(-c_6 t\log k)\,dt = \frac{k\log^2 k}{c_6\log k}k^{-c_6\log k},$$

by (34), (28), (9), (11). Hence for all $k > k_4 \geqslant k_3$ we have

$$(35)\qquad T_1 < k^{-2}.$$

Let G_2 be the region of the points $s = 1-\lambda/\log k + i\gamma/\log k$ with

$$(36)\qquad \lambda_0 \leqslant \lambda \leqslant \log k,\quad |\gamma| \leqslant \gamma_1 = \gamma_1(\lambda) = \min(e^{\lambda}, \log^2 k),$$

λ_0 being defined by (7), and let us write the zeros $\varrho\notin G_1$ as follows:

$$(37)\qquad \varrho = 1-\lambda/\log k + i\gamma/\log k,\quad \lambda = \lambda_\varrho,\quad \gamma = \gamma_\varrho.$$

Using (34), (26), (37), (7), (11), (10), (29), we get the estimates

$$T_2 < (2\nu)^B \sum_{\varrho\in G_2} e^{-\lambda\xi}\frac{\exp(-2\lambda\eta_0/\nu)}{(2\lambda)^B} \leqslant \left(\frac{\nu}{A}\right)^B \sum_{\lambda_0\leqslant\lambda\leqslant\log k}\exp\{-(\xi+2\eta_0/\nu)\lambda\}$$

$$\ll \nu^B\left\{\int_{\lambda_0}^{\log k}(\xi+2\eta_0/\nu)\exp[-(\xi+2\eta_0/\nu)\lambda]e^{C\lambda}d\lambda + \exp[-(\xi+2\eta_0/\nu-C)\log k]\right\}$$

$$\ll \nu^B\exp\{-(\xi+2\eta_0/\nu-C)\lambda_0\} < \exp\{-(\tfrac{2}{3}\xi+2\eta_0/\nu-C)\lambda_0\}$$

$$< \begin{cases}\exp[-(\tfrac{1}{3}\xi+2\eta_0/\nu)\lambda_0] & \text{if}\quad \xi\geqslant 3C,\\ \exp[-(\tfrac{2}{3}\xi+\eta_0/\nu)\lambda_0] & \text{if}\quad \xi < 3C,\ \eta_0\geqslant Ce^{3aC}.\end{cases}$$

Hence we have

$$(38)\qquad T_2 < c_7\exp[-\lambda_0(\tfrac{1}{3}\xi+\eta_0/\nu)]\quad \text{for}\quad \eta_0 > Ce^{3aC}.$$

Let G_3 be the remaining part of the rectangle $(0\leqslant\sigma\leqslant 1-\lambda_0/\log k$, $|t|\leqslant\log k)$. Supposing $\lambda_0 \leqslant 2\log\log k$, we have

$$T_3 \ll \nu^B\sum_{\varrho\in G_3}\exp(-\xi\lambda_0-2\lambda_0\eta_0/\nu)|\gamma|^{-B} = \nu^B\exp\{-\lambda_0(\xi+2\eta_0/\nu)\}\sum_{\varrho\in G_3}|\gamma|^{-B}$$

$$\ll \nu^B\exp\{-\lambda_0(\xi+2\eta_0/\nu)\}\left\{\int_{\lambda_0}^{2\log\log k} e^{-B\lambda}e^{C\lambda}d\lambda + e^{-(B-C)2\log\log k}\right\}$$

$$\ll \nu^B\exp\{-\lambda_0(\xi+2\eta_0/\nu+B-C)\},$$

by (34), (37), (26), (7), (11), (10), (36), (23'); for $\lambda_0 > 2\log\log k$ there is no G_3 and consequently $T_3 = 0$. Hence in the same manner as in (38) we get the estimate

$$(39)\qquad T_3 < c_8\exp\{-\lambda_0(\tfrac{1}{3}\xi+\eta_0/\nu)\}.$$

Now we have, by (31), (32), (34), (35), (38), (39),

$$(40)\qquad U \geqslant 1 - x^{-\delta_1}\exp[-(2\eta_0/\nu)\delta_1\log k] - 2k^{-2} - c_9\exp\{-\lambda_0(\tfrac{1}{3}\xi+\eta_0/\nu)\}.$$

Increasing η_0, if necessary, we may suppose the inequality

$$(41)\qquad 2\eta_0 A \geqslant 1$$

holds. Since we have $x \geqslant 1$, and $\delta_1 \geqslant \delta_0$, by (6), and $1-e^{-t} \geqslant \frac{1}{2}t$ for $0 \leqslant t \leqslant 1$, using (41) we get the estimate

$$1-x^{-\delta_1}\exp[-(2\eta_0/\nu)\,\delta_1\log k] \geqslant 1-\exp[-(2\eta_0/\nu)\,\delta_0\log k] \geqslant 1-\exp\{(A\nu)^{-1}\delta_0\log k\} > \frac{\delta_0\log k}{2A\nu}. \tag{42}$$

By (29), (23) we have $a < \frac{1}{3}A$. Therefore there are non-negative solutions of the system of inequalities

$$A\xi/3 > a\xi+\log 4c_9, \qquad A\xi/3 \geqslant 1. \tag{43}$$

Let us denote by ξ_1 the least non-negative solution of (43) ($\xi_1 > 0$). Then for all $\xi \geqslant \xi_1$ we have

$$e^{-A\xi/3} < \frac{1}{4c_9}e^{-a\xi}$$

and consequently

$$c_9 e^{-A\xi/3} < 1/4\nu,$$

by (28). Using this inequality and (7), (6), (43) we get the estimate

$$\begin{aligned} c_9\exp\{-\lambda_0(\tfrac{1}{3}\xi+\eta_0/\nu)\} &< c_9\exp(-\tfrac{1}{3}\xi\lambda_0) \\ &= c_9\exp\left\{-\tfrac{1}{3}\xi\left(A\log\frac{eA}{\delta_0\log k}\right)\right\} \\ &= c_9\left(\frac{\delta_0\log k}{eA}\right)^{A\xi/3} = \left(\frac{\delta_0\log k}{A}\right)^{A\xi/3}\cdot c_9e^{-A\xi/3} \\ &< \frac{\delta_0\log k}{4A\nu} \end{aligned} \tag{44}$$

for $\xi \geqslant \xi_1$.

Increasing η_0, if necessary, we may suppose that the inequalities

$$\eta_0 Ae^{-a\xi_1} > a\xi_1+\log 4c_9, \qquad \eta_0Ae^{-a\xi_1} \geqslant 1 \tag{45}$$

are satisfied. Then we have, by (28),

$$\exp\left(-\frac{\eta_0}{\nu}A\right) < \frac{1}{4c_9}e^{-a\xi_1}, \qquad \frac{\eta_0}{\nu}A \geqslant 1$$

for $0 \leqslant \xi < \xi_1$, and consequently

$$c_9e^{-A\eta_0/\nu} < 1/4\nu, \qquad A\eta_0/\nu \geqslant 1.$$

Using these inequalities and (7), (6) we get the estimate

$$\begin{aligned} c_9\exp\left\{-\lambda_0\left(\tfrac{1}{3}\xi+\frac{\eta_0}{\nu}\right)\right\} &\leqslant c_9\exp\left(-\frac{\eta_0}{\nu}\lambda_0\right) \\ &= \left(\frac{\delta_0\log k}{A}\right)^{A\eta_0/\nu}\cdot c_9e^{-A\eta_0/\nu} \leqslant \frac{\delta_0\log k}{4A\nu} \end{aligned}$$

(cf. (44)). This proves (44) for all $\xi \geqslant 0$.

For all $k > k_5 \geqslant k_4$ we have

$$2k^{-2} < \frac{\delta_0\log k}{8A\nu},$$

by (6), (8), (28). Using this inequality and (40), (42), (44) we get the estimate

$$U > \frac{c_{10}}{\nu}\,\delta_0\log k \tag{46}$$

for $c_{10} \geqslant 1/8A$ and some $\eta_0 \leqslant 1$ satisfying (24), (38), (41), (45). This is the required auxiliary inequality.

Proof of the main inequality (59). We introduce the number

$$z = k^{\xi+4\eta/\nu} = xe^{4y} \tag{47}$$

and divide the sum (15) up into the partial sums

$$\Phi(x,y;k,l) = S_0+\varphi(k)\sum_{\substack{x<p<z\\ p\equiv l\,(\mathrm{mod}\,k)}}\exp\left(-\frac{\log^2 p/x}{4y}\right)\cdot\sqrt{p}\log p + S_1+\varphi(k)S_2, \tag{48}$$

where

$$S_0 = \varphi(k)\sum_{\substack{p\leqslant x\\ p\equiv l\,(\mathrm{mod}\,k)}}\exp\left(-\frac{\log^2 p/x}{4y}\right)\cdot\sqrt{p}\log p,$$

$$S_1 = \varphi(k)\sum_{\substack{n\geqslant z\\ n\equiv l\,(\mathrm{mod}\,k)}}\Lambda(n)\sqrt{n}\exp\left(-\frac{\log^2 n/x}{4y}\right), \tag{49}$$

$$S_2 = \sum_{\substack{n=p^a\leqslant z\\ a\geqslant 2,\, n\equiv l\,(\mathrm{mod}\,k)}}\Lambda(n)\sqrt{n}\exp\left(-\frac{\log^2 n/x}{4y}\right).$$

Using (49), (11), (12), (47), we get

$$(50)\qquad S_1 < c_{11}\int\limits_z^{\infty}\left(\frac{\log t/x}{2y}-\frac{1}{2}\right)\exp\left(\tfrac{1}{2}\log t-\frac{\log^2 t/x}{4y}\right)dt$$

$$= c_{11}\int\limits_{\log z/y}^{\infty}\left(\frac{u}{2y}-\frac{1}{2}\right)\exp\left(\frac{3u+\log x}{2}-\frac{u^2}{4y}\right)du$$

$$= c_{11}x^{3/2}\int\limits_{\log z/y}^{\infty}\left(\frac{u}{2y}-\frac{1}{2}\right)\exp\left(\frac{3u}{2}-\frac{u^2}{4y}\right)du$$

$$< 3c_{11}x^{3/2}\int\limits_{\log z/y}^{\infty}\left(\frac{u}{2y}-\frac{3}{2}\right)\exp\left(\frac{3u}{2}-\frac{u^2}{4y}\right)du$$

$$= 3c_{11}x^{3/2}\exp\left(\tfrac{3}{2}\log z/x-\frac{\log^2 z/x}{4y}\right) = 3c_{11}z^{3/2}e^{-4y} = 3c_{11}x^{3/2}e^{2y}$$

(since $u/2y \geqslant (\log z/x)/2y = 2$, by (47)). In the same manner using (11), (12) (with $k=1$) we get for a fixed $a \geqslant 2$

$$(51)\qquad \sum_{p^a\leqslant z}\exp\left(-\frac{a^2\log^2 p/x^{1/a}}{4y}\right)p^{a/2}\log p$$

$$< c_{11}\int\limits_2^{z^{1/a}}\left(-\frac{a}{2}+\frac{a^2\log t/x^{1/a}}{2y}\right)\exp\left(\frac{a}{2}\log t-\frac{a^2\log^2 t/x^{1/a}}{4y}\right)dt+$$

$$+c_{11}\exp\left(\tfrac{1}{2}\log z-\frac{\log^2 z/x}{4y}+\log z^{1/a}\right)$$

$$< c_{11}x^{1/a+1/2}\int\limits_{y/a}^{(\log z/x)/a}\left(-\frac{a}{2}+\frac{a^2u}{2y}\right)\exp\left\{-\frac{a^2u^2}{4y}+\left(1+\frac{a}{2}\right)u\right\}du+$$

$$+c_{11}\exp\left(\log z-\frac{\log^2 z/x}{4y}\right).$$

(The lower limit of the integral is obtained by taking $a^2u/2y-a/2 \geqslant 0$.) Since

$$y/a < (3a^{-2}+a^{-1})y < 4y/a = (\log z/x)/a,$$

by (47), we have

$$(52)\qquad \int\limits_{y/a}^{(\log z/x)/a}\left(\frac{a^2u}{2y}-\frac{a}{2}\right)\exp\left\{\left(\frac{a}{2}+1\right)u-\frac{a^2u^2}{4y}\right\}du$$

$$\leqslant \exp\left\{\left(\frac{3}{a^2}+\frac{1}{a}\right)y\right\}\int\limits_{y/a}^{(3a^{-2}+a^{-1})y}\left(\frac{a^2u}{2y}-\frac{a}{2}\right)\exp\left(\frac{a}{2}u-\frac{a^2u^2}{4y}\right)du+$$

$$+3\int\limits_{(3a^{-2}+a^{-1})y}^{(\log z/x)/a}\left(\frac{a^2u}{2y}-\frac{a}{2}-1\right)\exp\left\{\left(\frac{a}{2}+1\right)u-\frac{a^2u^2}{4y}\right\}du < 4\exp\left(\tfrac{3}{2}y\right).$$

In the sum S_2 we have $a < 2\log z$, by (49). Hence, by (51), (52), (47), (26), (27),

$$(53)\qquad S_2 \leqslant \sum_{2\leqslant a<2\log z}\ \sum_{p^a<z}\exp\left(-\frac{\log^2 p/x}{4y}\right)p^{a/2}\log p$$

$$< c_{12}\left\{x\exp\left(\tfrac{3}{2}y\right)+\exp\left(\log z-\frac{\log^2 z/x}{4y}\right)\right\}\log z$$

$$< c_{13}(4+\nu\xi/\eta_0)xy\exp\left(\tfrac{3}{2}y\right).$$

Using (49), (12), we get the estimate

$$(54)\qquad S_0 \leqslant \begin{cases} k\sum\limits_{p\leqslant k^4}\sqrt{p}\log p+\varphi(k)\sqrt{x}\sum\limits_{\substack{k^4<p\leqslant x\\ p\equiv l\,(\mathrm{mod}\,k)}}\log p \ll k^7+x^{3/2} \ll x^{3/2} & \text{if } \xi \geqslant 5,\\ k\sum\limits_{p\leqslant x}\sqrt{p}\log p \ll kx^{3/2} \leqslant x^{3/2}e^{2y} & \text{if } 0 \leqslant \xi < 5.\end{cases}$$

By (48), (54), (50), (53), we have

$$(55)\qquad \varphi(k)\sum_{x<p<z}\exp\left(-\frac{\log^2 p/x}{4y}\right)\sqrt{p}\log p$$

$$\geqslant \Phi(x,y;k,l)-c_{14}x^{3/2}e^{2y}-c_{13}\varphi(k)(4+\nu\xi/\eta_0)xy\exp\left(\tfrac{3}{2}y\right)$$

$$\geqslant \Phi(x,y;k,l)-\mu 2\sqrt{\pi y}\,x^{3/2}e^{2y}$$

where

$$(56)\qquad 0 < \mu < c_{15}\{y^{-1/2}+\varphi(k)(4+\nu\xi/\eta_0)\sqrt{y/x}\,e^{-y/2}\}.$$

For $\xi < 3$ and $k > k_6 \geqslant k_5$ we have

$$k(4+\nu\xi)\sqrt{y} \ll k\sqrt{\log k} < k^{1,4} < k^{4/2e^{0,3}} \leqslant k^{\eta/2\nu} \leqslant x^{1/2}e^{y/2},$$

whereas for $\xi \geqslant 3$ and all $k > k_7 \geqslant k_6$

$$k(4+\nu\xi)\sqrt{y} \ll k\sqrt{\log k}\, e^{a\xi} < k^{1,2+0,1\xi} \leqslant k^{\xi/2} < x^{1/2}e^{y/2}.$$

Using this we get, by (56),

$$\mu < c_{16}$$

since $y^{-1/2} < 1$. Hence, by (21), (22), (19), (27), (56), (25),

$$(57)\qquad I_B \mu e^{-y/4} < c_{16}e^{-(\eta_0/4\nu)\log k}.$$

Now let us divide (55) through by $2\sqrt{\pi y}\,x^{3/2}\exp(\frac{9}{4}y)$ and effect the operation I_B. Writing

$$(58)\qquad I_B[2\sqrt{\pi y}\,x^{3/2}\exp(\tfrac{9}{4}y)]^{-1}\varphi(k)\sum_{\substack{x<p<z\\ p\equiv l(\mathrm{mod}\,k)}}\exp\left(-\frac{\log^2 p/x}{4y}\right)\sqrt{p}\log p = V,$$

we have, by (19), (30), (55), (57), (58), (46),

$$(59)\qquad V \geqslant U - c_{16}e^{-(\eta_0/4\nu)\log k} > c_{10}\nu^{-1}\delta_0\log k - c_{16}e^{-(\eta_0/4\nu)\log k}.$$

This is the required inequality.

Proof of the theorem. Suppose that under some circumstances

$$(60)\qquad c_{16}e^{-(\eta_0/4\nu)\log k} \leqslant (c_{10}/2\nu)\,\delta_0\log k.$$

Then we have, by (59), (6), (8),

$$(61)\qquad V > (c_{10}/2\nu)\,\delta_0\log k > c_{17}(\varepsilon')k^{-\varepsilon'}$$

for all $\varepsilon' > 0$ and $k > k_8(\varepsilon') \geqslant k_7$. If there is a ν satisfying (28) and (60), then, by (61), (58), (47), (25), there is at least one prime $p \equiv l\,(\mathrm{mod}\,k)$ in the interval

$$(62)\qquad (x, xe^{4y}) = (x, xk^{4\eta/\nu}) \subset (x, xk^{4(\eta_0+B)/\nu}) = (x, xk^{\varepsilon}),$$

where we put $4(\eta_0+B)/\nu = \varepsilon$ or

$$(63)\qquad \nu = 4(\eta_0+B)/\varepsilon.$$

Take

$$(64)\qquad \varepsilon' = \varepsilon/16(\eta_0+B).$$

By (6), (8), we have $\delta_0 > c_{18}(\varepsilon')k^{-\varepsilon'/2}$ for all $k > k_9(\varepsilon') = k_{10}(\varepsilon) \geqslant k_8$. Therefore (60) would follow from the inequality

$$c_{16}e^{-(\eta_0/4\nu)\log k} < \frac{c_{10}}{2\nu}c_{18}k^{-\varepsilon'/2}\log k = c_{16}k^{\varepsilon'/2}\frac{c_{10}c_{18}\log k}{2c_{16}\nu}k^{-\varepsilon'}.$$

Since

$$k^{\varepsilon'/2}\frac{c_{10}c_{18}\log k}{2c_{16}\nu} > 1$$

(supposing that k_9 is large enough), it is sufficient to get the inequality

$$(65)\qquad e^{-(\eta_0/4\nu)\log k} \leqslant k^{-\varepsilon'},$$

whence (60) would follow as well. And we have (65) in consequence of (63), (64), (24).

It remains to prove that ν, as given by (63), satisfies the two conditions of (28):

$$1 \leqslant \nu \leqslant \log k, \qquad \nu \leqslant e^{a\xi}.$$

The inequality $\nu \geqslant 1$ is a consequence of the restriction $\varepsilon \leqslant 4(\eta_0+B) = c$. And $\nu \leqslant \log k$ for all $k \geqslant k_{11}(\varepsilon) \geqslant k_{10}$, by (63). From the second condition of (28) and from (26), (63) we get the restriction

$$\frac{4(\eta_0+B)}{\varepsilon} \leqslant e^{a\log x/\log k}$$

or

$$\frac{\log x}{\log k} \geqslant a^{-1}\log\frac{4(\eta_0+B)}{\varepsilon},$$

whence

$$x \geqslant k^{c'\log(c/\varepsilon)}, \qquad c' = 1/a, \qquad c = 4(\eta_0+B).$$

This proves the main part of the theorem.

Using (58), (61), (27), we get

$$I_B\sum_{\substack{x<p<z\\ p\equiv l(\mathrm{mod}\,k)}}\sqrt{\frac{\nu}{\eta\log k}}\exp\left(-\frac{\log^2 p/x}{4y} - \frac{9\eta}{4\nu}\log k\right)\sqrt{p}\log p > c_{17}(\varepsilon')x^{3/2}/\varphi(k)k^{\varepsilon'},$$

whence, by (26), (21), (62),

$$(66)\qquad \sum_{\substack{k^{\xi}<p<k^{\xi+\varepsilon}\\ p\equiv l(\mathrm{mod}\,k)}}\sqrt{p}\log p > c_{19}(\varepsilon')k^{3\xi/2}/\varphi(k)k^{\varepsilon'}.$$

Therefore the number of primes $p \equiv l \,(\mathrm{mod}\, k)$ in the interval (62) is at least

$$c_{19}(\varepsilon')k^{3\xi/2}/\varphi(k)k^{\varepsilon'+(\xi+\varepsilon)/2}\log k^{\xi+\varepsilon} > x/\varphi(k)k^{\varepsilon'+7\varepsilon/4}$$

for $x < \exp k^{\varepsilon}$, $k > k_{12}(\varepsilon) \geqslant k_{11}$. From this and (64), (24) we get the remaining part of the theorem.

Replacing ξ by $\xi-\varepsilon$ in (66), we get the inequality

$$\sum_{\substack{k^{\xi-\varepsilon}<p<k^{\xi}\\ p\equiv l(\mathrm{mod}\,k)}} \sqrt{p}\log p > c_{20}(\varepsilon')k^{3(\xi-\varepsilon)/2-\varepsilon'}/\varphi(k) \qquad (\xi > \varepsilon),$$

whence

$$\pi(k^{\xi};k,l)k^{\xi/2}\log k^{\xi} > c_{20}(\varepsilon')k^{3\xi/2-3\varepsilon/2-\varepsilon'}/\varphi(k)$$

or

$$\pi(x;k,l) > c_{20}x/\varphi(k)k^{5\varepsilon/2+\varepsilon'} \quad \text{for} \quad x = k^{\xi} < \exp k^{\varepsilon}.$$

From this and (64), (24) we get (4).

Note. Now suppose $x \geqslant \exp k^{\varepsilon_1}$. Then we have, by (8),

$$x^{-\delta_1} \leqslant \exp\{k^{\varepsilon_1}[-c_{18}(\varepsilon_1)k^{-\varepsilon_1/2}]\} = \exp\{-c_{18}(\varepsilon_1)k^{\varepsilon_1/2}\} < \tfrac{1}{8}$$

for all $k > k_{13}(\varepsilon_1) \geqslant k_8$. Therefore, by (40),

$$U > \tfrac{7}{8} - 2k^{-2} - c_9 e^{-\lambda_0(\xi/3+\eta_0/\nu)}.$$

Hence, by (59),

$$V \geqslant \tfrac{7}{8} - 2k^{-2} - c_9 e^{-\lambda_0(\xi/3+\eta_0/\nu)} - c_{16}e^{-(\eta_0/4\nu)\log k} > \tfrac{1}{2}$$

for $\nu = c_{21}^{-1}\log k$ with appropriate $c_{21} > 2$ (such that $c_{16}e^{-c_{21}} < \frac{1}{8}$), and for all $k > k_{14}(\varepsilon_1) \geqslant k_{13}$. From this and (58) we deduce that there is at least one prime $p \equiv l \,(\mathrm{mod}\, k)$ in the interval

$$(x,\, xk^{4(\eta_0+B)/\nu}) = (x,\, xc_{22}) \tag{67}$$

(cf. (62)), where

$$c_{22} = e^{4c_{21}(\eta_0+B)} > 1.$$

Note on the functions of Liouville and Möbius. Let $\lambda(n)$ and $\mu(n)$ be the functions of Liouville and Möbius, defined as follows: $\lambda(n) = (-1)^{v}$, v being the total number of prime factors of n, where multiple factors are counted a multiple number of times; $\mu(n) = \lambda(n)$, if n contains no square factor >1, and $=0$ otherwise.

As a simple consequence of the theorem we can prove that *there are absolute constants $a > 0$, $b > 0$ such that for any positive $\varepsilon \leqslant b$, for all*

$k > k_{15}(\varepsilon)$ and for all

$$x \geqslant k^{a\log(b/\varepsilon)} \tag{68}$$

the functions $\lambda(m)$, $\mu(m)$ change their signs at least once if m runs through the integers $\equiv l \,(\mathrm{mod}\, k)$ of the interval

$$(x,\, xk^{\varepsilon}). \tag{69}$$

To prove this, we take in (68) $a = 2c'$, $b = 2c$. Then, by the theorem, there are primes p, p', p'' such that

$$p \equiv l \,(\mathrm{mod}\, k), \qquad x < p < xk^{\varepsilon}$$

and

$$p' \equiv l \,(\mathrm{mod}\, k), \qquad \sqrt{x} < p' < \sqrt{x}k^{\varepsilon/2};$$

$$p'' \equiv 1 \,(\mathrm{mod}\, k), \qquad \sqrt{x} < p'' < \sqrt{x}k^{\varepsilon/2}, \qquad p'' \neq p',$$

whence $p'p'' \equiv l \,(\mathrm{mod}\, k)$, $x < p'p'' < xk^{\varepsilon}$; and we have $\lambda(p) = \mu(p) = -1$, $\lambda(p'p'') = \mu(p'p'') = 1$.

In 1948 I proved ([1]) that the functions $\Lambda(m)$, $\lambda(m)$, $\mu(m)$ ($m \equiv l \,(\mathrm{mod}\, k)$) keep their average values in the interval $(x, x+h)$ for some positive $h < x^{2/3}$ and $x \geqslant \exp k^{\varepsilon_1}$. In the present paper we have proved a weak analogy for $x < \exp k^{\varepsilon_1}$, namely the existence of the constants a, b such that for all x satisfying (68), in the interval (69) there is at least one prime $\equiv l \,(\mathrm{mod}\, k)$ and for m running through the numbers $\equiv l \,(\mathrm{mod}\, k)$ of (69) the functions $\lambda(m)$, $\mu(m)$ change their signs at least once.

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Reçu par la Rédaction le 7. 4. 1960