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On some problems of the arithmetical theory of continued fractions

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§ 1. For a given quadratic surd ξ let us denote by

$$[b_0, b_1, \ldots, b_{h-1}, \overline{b_h, b_{h+1}, \ldots, b_{h+k-1}}]$$

its expansion into an arithmetical continued fraction, by $\operatorname{lp}\xi$—the length of the shortest period of this expansion, by $\operatorname{lap}\xi$—the number of terms before the period. For some polynomials $f(n)$ assuming only integral values (so-called *integer-valued* polynomials) there are known formulae for the expansion of $\sqrt{f(n)}$ into continued fractions such that the partial quotients are also integer-valued polynomials and $\operatorname{lp}\sqrt{f(n)}$ is independent of n (cf. [3], [5]). Recently H. Schmidt has proved ([3], Satz 10) that

If h is an integer $\neq 0, \pm 1, \pm 2, \pm 4$, then for each n_0 the set of all integers $\geqslant n_0$ cannot be decomposed into a finite number of classes, so that the relation

$$\sqrt{n^2+h} = [p_0(n), \overline{p_1(n), \ldots, p_k(n)}], \quad n \geqslant n_0, \quad n \in K,$$

holds for each class K (p_ν are polynomials assuming integral values for $n \in K$, k depends only upon K).

This theorem suggests the following problem P.

P. *Decide for a given integer-valued polynomial $f(n)$ whether*

$$\overline{\lim}\operatorname{lp}\sqrt{f(n)} < \infty.$$

An investigation of this problem is the main aim of the present paper.

In § 2 we investigate the relation between $\operatorname{lp}\xi$ and $\operatorname{lp}\big((p\xi+r)/(q\xi+s)\big)$, where p, q, r, s are integers.

In § 3 we give a negative solution of the problem P for polynomials of odd degree and for a large class of polynomials of even degree.

In § 4 after more accurate study of the behaviour of the function $\operatorname{lp}\sqrt{n^2+h}$ and on the base of results of § 2 we give a complete solution of the problem P for polynomials of the second degree.

We shall use the following notation; ξ, ξ', ξ'' will denote either rational numbers or quadratic surds; in the latter case η, η', η'' will be corresponding conjugate numbers. Putting

$$\xi = [b_0, b_1, b_2, \ldots]$$

we shall assume simultaneously

$$\begin{aligned} &A_{-1} = 1, \quad A_0 = b_0, \quad A_\nu = b_\nu A_{\nu-1} + A_{\nu-2}, \\ &B_{-1} = 0, \quad B_0 = 1, \quad B_\nu = b_\nu B_{\nu-1} + B_{\nu-2}, \end{aligned} \tag{1}$$

(whence $[b_0, b_1, \ldots, b_n] = A_n/B_n$) and

$$\xi_\nu = [b_\nu, b_{\nu+1}, b_{\nu+2}, \ldots] \quad \text{(cf. [2], p. 24 and 34).}$$

For rational ξ we put $\operatorname{lp}\xi = 0$ and

$$\operatorname{lap}\xi = \begin{cases} 1 & \text{if} \quad \xi \text{ is an integer,} \\ h & \text{if} \quad \xi = [b_0, b_1, \ldots, b_{h-1}],\ b_{h-1} > 1 \end{cases}$$

(the so-called *normal* expansion).

§ 2. Lemma 1. *Let $b > 1$, h and s be positive integers. If*

$$\xi = [b_0, b_1, \ldots, b_{h-1}] \tag{2'}$$

or

$$\xi = [b_0, b_1, \ldots, b_{h-1}, \overline{b_h, b_{h+1}, \ldots, b_{h+k-1}}], \tag{2''}$$

where

$$b_i < b \quad (1 \leqslant i \leqslant h-1), \tag{3}$$

then

$$i \leqslant B_i \leqslant b^i \quad (0 \leqslant i \leqslant h-1). \tag{4}$$

Moreover, if for some integers p and r

$$\xi' = (p\xi + r)/s, \tag{5}$$

then

$$\operatorname{lap}\xi' < 2sb^h. \tag{6}$$

Proof. Formula (4) follows by easy induction from (1) and (2). Hence for rational ξ we immediately get the remaining part of the lemma.

In fact, putting

$$\xi' = [b'_0, b'_1, \ldots, b'_{h'-1}] = \frac{A'_{h'-1}}{B'_{h'-1}},$$

we have in view of (5)

$$\frac{A'_{h'-1}}{B'_{h'-1}} = \frac{pA_{h-1} + rB_{h-1}}{sB_{h-1}};$$

then $\operatorname{lap}\xi' = h' \leqslant B'_{h'-1} \leqslant sB_{h-1} \leqslant sb^{h-1} < 2sb^h$.

In the case $p = 0$ we have likewise

$$\frac{A'_{h'-1}}{B'_{h'-1}} = \frac{r}{s}, \quad \text{whence} \quad \operatorname{lap}\xi' = h' \leqslant B'_{h'-1} \leqslant s < 2sb^h.$$

One can therefore assume that ξ is irrational and $p \neq 0$. It follows from (2'') that

$$\xi = [b_0, b_1, \ldots, b_{h-1}, \xi_h];$$

ξ_h, which has a pure period in its expansion, is by a well-known theorem, a reduced surd, i.e.

$$\xi_h > 1, \quad 0 > \eta_h > -1. \tag{7}$$

On the basis of well-known formulae (cf. [2], § 13, (7)) we have:

$$\xi = \frac{A_{h-1}}{B_{h-1}} + \frac{(-1)^{h-1}}{B_{h-1}(B_{h-1}\xi_h + B_{h-2})},$$

$$\eta = \frac{A_{h-1}}{B_{h-1}} + \frac{(-1)^{h-1}}{B_{h-1}(B_{h-1}\eta_h + B_{h-2})},$$

whence

$$|\xi - \eta| = \frac{|1 - \eta_h/\xi_h|}{|B_{h-1} + B_{h-2}/\xi_h| \cdot |B_{h-1}\eta_h + B_{h-2}|}.$$

Since, in view of (7),

$$0 < -\eta_h/\xi_h, \quad 0 < B_{h-1} + B_{h-2}/\xi_h < B_{h-1} + B_{h-2} \leqslant B_h,$$
$$-B_{h-1} < B_{h-1}\eta_h + B_{h-2} < B_{h-2} < B_{h-1},$$

we get by (4)

$$|\xi - \eta| > 1/B_{h-1}B_h > 1/b^{2h-1}$$

and by (5)

$$|\xi' - \eta'| = \frac{|p|}{s}|\xi - \eta| > \frac{1}{sb^{2h-1}}.$$

If $\xi' > \eta'$, we assume $h' = 2sb^h - 2$. Therefore, in view of (4),

$$(8) \qquad B'_{h'-1}B'_{h'-2} \geqslant h'(h'-1) = (2sb^h-2)(2sb^h-3) \geqslant sb^{2h-1} > \frac{1}{|\xi'-\eta'|}.$$

We shall prove that $\xi'_{h'}$ is a reduced surd. It follows from the formula

$$\eta' = \frac{A'_{h'-1}}{B'_{h'-1}} + \frac{(-1)^{h'-1}}{B'_{h'-1}(B'_{h'-1}\eta'_{h'}+B'_{h'-2})}$$

that

$$B'_{h'-1}(B'_{h'-1}\eta'_{h'}+B'_{h'-2}) = \frac{(-1)^{h'}}{A'_{h'-1}/B'_{h'-1}-\eta'}.$$

Since $h' = 2sb^h-2$ is even, we have

$$A'_{h'-1}/B'_{h'-1}-\eta' > \xi'-\eta' > 0.$$

The last two formulae together give

$$\frac{1}{\xi'-\eta'} > (B'_{h'-1}\eta'_{h'}+B'_{h'-2})B'_{h'-1} > 0.$$

We then get, on the one hand,

$$0 < B'_{h'-1}\eta'_{h'}+B'_{h'-2}, \quad \text{whence} \quad \eta'_{h'} > -B'_{h'-2}/B'_{h'-1} > -1;$$

on the other hand, in view of (8),

$$B'_{h'-1}B'_{h'-2} > B'_{h'-1}(B'_{h'-1}\eta'_{h'}+B'_{h'-2}), \quad \text{whence} \quad \eta'_{h'} < 0.$$

Therefore $0 > \eta'_{h'} > -1$ and since $\xi'_{h'} > 1$, the surd $\xi'_{h'}$ is reduced (for $h' = 2sb^h-2$).

In the case $\eta' > \xi'$ we prove similarly that the surd $\xi'_{h'}$ is reduced for $h' = 2sb^h-1$. Since a reduced surd gives in its expansion a pure period, we have in both cases

$$\operatorname{lap}\xi' = h' < 2sb^h, \quad \text{q. e. d.}$$

Remark. Inequalities (4) and (6) can be greatly improved; however, it is without any importance for the applications intended.

In the following we shall profit by a theorem used in the investigation of Hurwitz's continued fractions and due to A. Hurwitz and A. Châtelet. We quote this theorem according to Perron's monograph ([2], Satz 4.1) with slight changes in his notation to avoid confusion with ours.

H. *Let $[b_0, b_1, b_2, \ldots]$ be the arithmetical continued fraction for a quadratic surd ξ_0, A_λ, B_λ—the numerators and denominators of its convergents, and ξ_λ—its complete quotients. Further, let*

$$\xi' = \frac{p_0\xi+r_0}{s_0} \quad (p_0, r_0, s_0\text{—integers},\ p_0>0,\ s_0>0,\ p_0s_0 = d>1).$$

For any index ν ($\geqslant 1$) the number

$$\frac{p_0[b_0, b_1, \ldots, b_{\nu-1}]+r_0}{s_0} = \frac{p_0A_{\nu-1}+r_0B_{\nu-1}}{s_0B_{\nu-1}}$$

can be developed in an arithmetical continued fraction $[d_0, d_1, \ldots, d_{\mu-1}]$ and besides the number of its terms can be chosen so that $\mu \equiv \nu \pmod 2$; let C_λ, D_λ be the numerators and denominators of its convergents, so that in particular

$$\frac{p_0A_{\nu-1}+r_0B_{\nu-1}}{s_0B_{\nu-1}} = \frac{C_{\mu-1}}{D_{\mu-1}}.$$

Then there exist three uniquely determined integers p_1, r_1, s_1 such that the formula

$$\begin{pmatrix} p_0 & r_0 \\ 0 & s_0\end{pmatrix}\begin{pmatrix} A_{\nu-1} & A_{\nu-2} \\ B_{\nu-1} & B_{\nu-2}\end{pmatrix} = \begin{pmatrix} C_{\mu-1} & C_{\mu-2} \\ D_{\mu-1} & D_{\mu-2}\end{pmatrix}\begin{pmatrix} p_1 & r_1 \\ 0 & s_1\end{pmatrix}$$

holds and besides

$$p_1>0, \quad s_1>0, \quad p_1s_1 = d, \quad -s_1 \leqslant r_1 \leqslant p_1,$$

$$\xi' = [d_0, d_1, \ldots, d_{\mu-1}, \xi'_\mu], \quad \textit{where} \quad \xi'_\mu = \frac{p_1\xi_\nu+r_1}{s_1}.$$

The theorem quoted obviously preserves its validity for $d=1$ as well as for rational ξ; in the latter case under the condition $\nu \leqslant \operatorname{lap}\xi$.

On the basis of Lemma 1 and theorem H we shall show

THEOREM 1. *For arbitrary positive integers m and d there exists a number $M = M(m, d)$ such that if $\operatorname{lap}\xi \leqslant m$ and*

$$(9) \qquad \xi' = \frac{p_0\xi+r_0}{s_0} \quad (p_0, r_0, s_0\text{—integers},\ p_0, s_0>0,\ p_0s_0 = d)$$

then $\operatorname{lap}\xi' \leqslant M$.

Proof. We shall prove it by induction with respect to m. For $m=1$ the theorem follows immediately from Lemma 1, whence after the substitution $b=2$, $h=1$ (assumption (3) being satisfied in emptiness), $p=p_0$, $r=r_0$, $s=s_0$ we get

$$\operatorname{lap}\xi' < 4s_0.$$

Assume now that the theorem is valid for $m = h-1$ $(h > 1)$; we shall show that it is valid for $m = h$.

By hypothesis there exists a number $M(h-1, d)$ such that if $\operatorname{lap}\xi \leqslant h-1$ and $\xi' = (p\xi+r)/s$ $(p, r, s$—integers, $p > 0$, $s > 0$, $ps = d)$, then

$$\operatorname{lap}\xi' \leqslant M(h-1, d).$$

Let $M = 2M(h-1, d)+2^{h+1}d^{h+1}$. The proof will be complete if we show that for any ξ such that $\operatorname{lap}\xi \leqslant h$ the number ξ' defined by (9) satisfies the inequality

$$\operatorname{lap}\xi' \leqslant M.$$

Since $M(h-1, d) < M$, we can assume that $\operatorname{lap}\xi = h$ and that ξ is given by one of the formulae (2).

If for each $i < h$ is $b_i < 2d$, then putting in Lemma 1 $b = 2d$, $p = p_0$, $r = r_0$, $s = s_0$, we get

$$\operatorname{lap}\xi' \leqslant 2s_0(2d)^h \leqslant 2^{h+1}d^{h+1} \leqslant M.$$

It remains to consider the case where for some $\nu < h$: $b_\nu \geqslant 2d$. We then have

$$\xi = [b_0, b_1, \ldots, b_{\nu-1}, \xi_\nu], \qquad \xi_\nu \geqslant b_\nu \geqslant 2d. \tag{10}$$

In virtue of theorem H there exist integers p_1, r_1, s_1 such that

$$p_1 > 0, \quad s_1 > 0, \quad p_1 s_1 = d, \quad -s_1 \leqslant r_1 \leqslant p_1, \tag{11}$$

$$\frac{p_0[b_0, b_1, \ldots, b_{\nu-1}]+r_0}{s_0} = [d_0, d_1, \ldots, d_{\mu-1}], \tag{12}$$

$$\xi' = \frac{p_0\xi+r_0}{s_0} = [d_0, d_1, \ldots, d_{\mu-1}, \xi'_\mu], \qquad \xi'_\mu = \frac{p_1\xi_\nu+r_1}{s_1}. \tag{13}$$

From (10) and (11) we get

$$\xi'_\mu \geqslant \xi_\nu/s_1 - 1 \geqslant \xi_\nu/d - 1 \geqslant 1,$$

which together with formula (13) proves that numbers $d_0, d_1, \ldots, d_{\mu-1}$ are the initial partial quotients of the number ξ'. Hence

$$\operatorname{lap}\xi' \leqslant \mu + \operatorname{lap}\xi'_\mu. \tag{14}$$

Meanwhile, by (12)

$$\mu \leqslant 1+\operatorname{lap}\frac{p_0[b_0, b_1, \ldots, b_{\nu-1}]+r_0}{s_0}$$

and since $\operatorname{lap}[b_0, b_1, \ldots, b_{\nu-1}] \leqslant \nu < h$, we have in virtue of the inductive assumption

$$\mu \leqslant 1+M(h-1, d). \tag{15}$$

On the other hand, since $\operatorname{lap}\xi_\nu = \operatorname{lap}\xi - \nu < h$, we have

$$\operatorname{lap}\xi'_\mu \leqslant M(h-1, d) \tag{16}$$

and finally by (14), (15), (16) we get

$$\operatorname{lap}\xi' \leqslant 1+2M(h-1, d) \leqslant M, \quad \text{q. e. d.}$$

Corollary. *For any positive integer m and arbitrary integers d and q there exists a number $M = M(m, d, q)$ such that if*

$$\operatorname{lap}\xi \leqslant m, \qquad \xi' = \frac{p\xi+r}{q\xi+s} \qquad (p, r, s\text{—}integers,\ q\xi+s \neq 0)$$

and $ps-qr = d$, then $\operatorname{lap}\xi' \leqslant M$.

Proof. The case $d = 0$ is trivial; thus let $d \neq 0$. It is easy to verify the equality (cf. [2], p. 56):

$$-[b_0, b_1, b_2, b_3, \ldots] = \begin{cases} [-(b_0+1), 1, b_1-1, b_2, b_3, \ldots] & \text{for} \quad b_1 > 1, \\ [-(b_0+1), b_2+1, b_3, b_4, \ldots] & \text{for} \quad b_1 = 1, \end{cases}$$

whence

$$\operatorname{lap}(-\xi) \leqslant 3+\operatorname{lap}\xi. \tag{17}$$

If $q = 0$, then $s \neq 0$ and we have

$$\xi' = \frac{\operatorname{sgn}p}{\operatorname{sgn}s}\cdot\frac{|p|\,\xi+r\operatorname{sgn}p}{|s|};$$

the corollary follows therefore directly from Theorem 1 and formula (17).

If $q \neq 0$, then

$$\xi' = \operatorname{sgn}q\cdot\frac{\zeta^{-1}+p}{|q|}, \qquad \zeta = -\frac{\operatorname{sgn}q}{\operatorname{sgn}d}\cdot\frac{|q|\,\xi+s\cdot\operatorname{sgn}q}{|d|},$$

and we obtain the corollary applying Theorem 1 successively to the numbers ζ and ξ', using formula (17) and the obvious inequality

$$\operatorname{lap}\zeta^{-1} \leqslant 1+\operatorname{lap}\zeta.$$

Lemma 2. *Let b, k, p and s be positive integers. If ξ is given by (2″) and ξ' by (5) and if*

$$b_i < b \qquad (h \leqslant i \leqslant h+k-1), \tag{18}$$

then $\operatorname{lp}\xi' \leqslant 8(ps)^2 b^{2k}$.

Proof. It follows from (2'') that

$$\xi_h = [b_h, b_{h+1}, \ldots, b_{h+k-1}, \xi_h];$$

the number ξ_h satisfies therefore the equation

$$B_{k-1,h}x^2+(B_{k-2,h}-A_{k-1,h})x-A_{k-2,h} = 0, \tag{19}$$

where numbers $A_{\lambda,h}$ and $B_{\lambda,h}$ are respectively the numerator and the denominator of the λth convergent of $[b_h, b_{h+1}, \ldots]$.

Denoting by Δ the discriminant of the equation (19) we have

$$\Delta = (B_{k-2,h}-A_{k-1,h})^2+4B_{k-1,h}A_{k-2,h} = (A_{k-1,h}+B_{k-2,h})^2+4(-1)^{k-1},$$

and since from (18) easily follows

$$A_{k-1,h} < b^k, \quad B_{k-2,h} < b^k,$$

we get

$$\Delta \leqslant 4b^{2k}. \tag{20}$$

It follows from the formulae

$$\xi' = \frac{p\xi+r}{s}, \quad \xi = \frac{A_{h-1}\xi_h+A_{h-2}}{B_{h-1}\xi_h+B_{h-2}}$$

and from equation (19) for ξ_h that the number ξ_h' satisfies the equation

$$Ax^2+Bx+C = 0, \tag{21}$$

where integers A, B, C are defined by the formula

$$\begin{pmatrix}2A & B\\ B & 2C\end{pmatrix} = \begin{pmatrix}s & 0\\ -r & p\end{pmatrix}\begin{pmatrix}B_{h-2} & A_{h-2}\\ -B_{h-1} & A_{h-1}\end{pmatrix}\begin{pmatrix}2B_{k-1,h} & -A_{k-1,h}\\ B_{k-2,h}-A_{k-1,h} & -2A_{k-2,h}\end{pmatrix}\times$$
$$\times\begin{pmatrix}B_{h-2} & -B_{h-1}\\ -A_{h-2} & A_{h-1}\end{pmatrix}\begin{pmatrix}s & -r\\ 0 & p\end{pmatrix}. \tag{22}$$

On the other hand, as can easily be seen from Lagrange's proof of his well-known theorem about periodical expansions of quadratic surds (cf. [2], pp. 66-68), if ξ' is a root of equation (21), then

$$\operatorname{lp}\xi' \leqslant 2\Delta',$$

where Δ' is the discriminant of that very equation. But, as follows from (22),

$$\Delta' = -\begin{vmatrix}2A & B\\ B & 2C\end{vmatrix} = (ps)^2(A_{h-1}B_{h-2}-B_{h-1}A_{h-2})^2\Delta = (ps)^2\Delta.$$

The last two formulae together with (20) finally give

$$\operatorname{lp}\xi' \leqslant 8(ps)^2b^{2k}, \quad \text{q. e. d.}$$

THEOREM 2. *For arbitrary integers $n>0$ and d, there exists a number $N = N(n, d)$ such that if*

$$\operatorname{lp}\xi \leqslant n, \quad \xi' = \frac{p\xi+r}{q\xi+s} \quad (p, q, r, s\text{—integers}, \ q\xi+s \neq 0) \tag{23}$$
$$\text{and } ps-qr = d,$$

then $\operatorname{lp}\xi' \leqslant N$.

Proof. The case of ξ—rational or $d = 0$ is trivial; let ξ be a quadratic surd, $d \neq 0$. On the basis of Theorem 1 there exists a number $M(n, |d|)$ such that, if p, r, s—integers, $p>0$, $s>0$, $ps = |d|$ and $\operatorname{lap}\zeta \leqslant n$, then

$$\operatorname{lap}\frac{p\zeta+r}{s} < M(n, |d|).$$

Let $N = M(n, |d|)(|d|+1)^2+2^{2n+3}d^{2n+2}$. We shall show that if conditions (23) hold, then

$$\operatorname{lp}\xi' \leqslant N.$$

Put

$$\beta = q/(p, q), \quad \delta = -p/(p, q).$$

Since $(\beta, \delta) = 1$, there exist integers a and γ such that

$$a\delta-\beta\gamma = \operatorname{sgn} d. \tag{24}$$

Putting

$$\xi'' = \frac{a\xi'+\gamma}{\beta\xi'+\delta} \tag{25}$$

we get

$$\xi'' = \frac{p_0\xi+r_0}{s_0},$$

where the integers p_0, r_0, s_0 are defined by the formula

$$\begin{pmatrix}p_0 & r_0\\ 0 & s_0\end{pmatrix} = \begin{pmatrix}a & \gamma\\ \beta & \delta\end{pmatrix}\begin{pmatrix}p & r\\ q & s\end{pmatrix};$$

thus $p_0s_0 = (a\delta-\beta\gamma)(ps-qr) = d\operatorname{sgn} d = |d|$.

In view of formulae (24) and (25), the surds ξ' and ξ'' are equivalent, whence

$$\operatorname{lp}\xi' = \operatorname{lp}\xi'' = \operatorname{lp}\frac{p_0\xi+r_0}{s_0}.$$

Changing, if necessary, the signs of p_0, r_0, s_0 we can therefore assume that

$$\xi' = \frac{p_0\xi+r_0}{s_0}, \quad p_0>0, \quad s_0>0, \quad p_0s_0 = d>0.$$

Let ξ be given by formula (2''), where $k\leqslant n$. If for each i such that $h\leqslant i\leqslant h+k-1$ we have $b_i<2d$, then, putting in Lemma 2 $b=2d$, $p=p_0$, $s=s_0$, we get

$$\operatorname{lp}\xi' \leqslant 8(p_0s_0)^2(2d)^{2k} \leqslant 8d^2(2d)^{2n} = 2^{2n+3}d^{2n+2} \leqslant N.$$

It remains to consider the case, where for some $\nu\geqslant h$ holds $b_\nu\geqslant 2d$. We then have

$$\xi = [b_0, b_1, \ldots, b_{\nu-1}, \overline{b_\nu, b_{\nu+1}, \ldots, b_{\nu+k-1}}, b_\nu, b_{\nu+1}, \ldots, b_{\nu+k-1}, \ldots].$$

Using theorem H we get

$$\frac{p_0[b_0, b_1, \ldots, b_{\nu-1}]+r_0}{s_0} = [d_0, \ldots, d_{\mu_1-1}],$$

(26) $$\xi' = [d_0, \ldots, d_{\mu_1-1}, \xi'_{\mu_1}], \quad \xi'_{\mu_1} = \frac{p_1\xi_\nu+r_1}{s_1},$$

(27) $$p_1>0, \quad s_1>0, \quad p_1s_1 = d, \quad -s_1\leqslant r_1\leqslant p_1,$$

and for all $i\geqslant 1$

(28) $$\frac{p_i[b_\nu, \ldots, b_{\nu+k-1}]+r_i}{s_i} = [d_{\mu_i}, \ldots, d_{\mu_{i+1}-1}],$$

(29) $$\xi'_{\mu_i} = [d_{\mu_i}, \ldots, d_{\mu_{i+1}-1}, \xi'_{\mu_{i+1}}], \quad \xi'_{\mu_{i+1}} = \frac{p_{i+1}\xi_\nu+r_{i+1}}{s_{i+1}},$$

(30) $$p_{i+1}>0, \quad s_{i+1}>0, \quad p_{i+1}s_{i+1} = d, \quad -s_{i+1}\leqslant r_{i+1}\leqslant p_{i+1}.$$

In view of the inequality $\xi_\nu>b_\nu\geqslant 2d$, it follows from (27) and (30) that $\xi_{\mu_i}>1$ $(i=1,2,\ldots)$; the number ξ' has therefore the following expansion into an arithmetical continued fraction;

$$\xi' = [d_0, \ldots, d_{\mu_1-1}, d_{\mu_1}, \ldots, d_{\mu_2-1}, d_{\mu_2}, \ldots, d_{\mu_3-1}, d_{\mu_3}, \ldots].$$

It follows from (27) and (30) that the number of all possible different systems (p_i, r_i, s_i) does not exceed $d(d+2)$. Thus, among the systems (p_i, r_i, s_i) $(i=1,2,\ldots,(d+1)^2)$ there must be at least two identical ones; there exist therefore positive integers $g<j\leqslant(d+1)^2$ such that

$$p_g = p_j, \quad r_g = r_j, \quad s_g = s_j.$$

On the basis of (26) and (29) it follows hence that

$$\xi'_{\mu_j} = \xi'_{\mu_g};$$

thus

$$\operatorname{lp}\xi' \leqslant \mu_j-\mu_g = \sum_{i=g}^{j-1}(\mu_{i+1}-\mu_i).$$

On the other hand, in virtue of formula (28), the definition of the number $M(n, |d|)$ and the condition $k\leqslant n$,

$$\mu_{i+1}-\mu_i \leqslant 1+\operatorname{lap}\frac{p_i[b_\nu, \ldots, b_{\nu+k-1}]+r_i}{s_i} \leqslant M(n, |d|).$$

In view of $j-g<(d+1)^2$, we thus get

$$\operatorname{lp}\xi' \leqslant (j-g)M(n, |d|) \leqslant (d+1)^2M(n, |d|) \leqslant N, \qquad \text{q. e. d.}$$

Remark. As can easily be seen, we use in the proof given above only a special case of Theorem 1. We proved it in full generality only for a more complete characterisation of the relation between continued fractions and rational homographic transformations.

§ 3. LEMMA 3. *If $\xi^{(n)}\to\zeta$ ($\xi^{(n)}$ are quadratic surds, ζ an arbitrary irrational number) and*

(31) $$\xi^{(n)} \neq \zeta,$$

then

(32) $$\lim(\operatorname{lap}\xi^{(n)}+\operatorname{lp}\xi^{(n)}) = \infty.$$

Proof. If formula (32) does not hold, the sequence $\xi^{(n)}$ contains a subsequence for which

(33) $$\operatorname{lap}\xi^{(n)}+\operatorname{lp}\xi^{(n)} \leqslant L<\infty.$$

Proving Lemma 3 by reduction to absurdity we can therefore assume at once that inequality (33) holds. Let

$$\xi^{(n)} = [b_0^{(n)}, b_1^{(n)}, \ldots, b_{h_n-1}^{(n)}, \overline{b_{h_n}^{(n)}, b_{h_n+1}^{(n)}, \ldots, b_{h_n+k_n-1}^{(n)}}],$$

$$h_n = \operatorname{lap}\xi^{(n)}, \quad k_n = \operatorname{lp}\xi^{(n)}, \quad \zeta = [b_0, b_1, \ldots].$$

Since ζ is irrational, we have
$$\lim b_i^{(n)} = b_i \quad (i = 0, 1, 2, \ldots);$$
thus for every i there exists an n_i such that
$$(34) \qquad b_i^{(n)} = b_i \quad (n \geqslant n_i).$$
By (33) we have $h_n + k_n \leqslant L$. Putting $K = L!$ we have, for every n, $h_n \leqslant L$, $k_n \mid K$, whence
$$(35) \qquad b_i^{(n)} = b_{i+Kt}^{(n)} \quad (i \geqslant L,\ n \geqslant 1,\ t \geqslant 0).$$
Let $M = \max(n_0, n_1, \ldots, n_{K+L-1})$. We shall show that, contrary to assumption (31), for $n \geqslant M$, $\xi^{(n)} = \zeta$.

In fact, by (34) we have for $n \geqslant M$
$$(36) \qquad b_i^{(n)} = b_i \quad (0 \leqslant i < L+K).$$
Assume now that $j \geqslant L+K$. We obviously have $j = tK+i$, where t is an integer $\geqslant 0$, $L < i < L+K$ and according to (35)
$$(37) \qquad b_j^{(n)} = b_i^{(n)} \quad (n \geqslant 1).$$
Put $m = \max(M, n_j)$. By (36) we have
$$(38) \qquad b_i^{(n)} = b_i^{(m)} \quad (0 \leqslant i < L+K,\ n \geqslant M).$$
Applying successively formulae (37), (38), (37) and (35) we get for $n \geqslant M$
$$b_j^{(n)} = b_i^{(n)} = b_i^{(m)} = b_j^{(m)} = b_j \quad (j \geqslant L+K),$$
whence by (36) it follows at last that for $n \geqslant M$
$$\xi^{(n)} = \zeta, \quad \text{q. e. d.}$$
Remark. One can easily deduce from the lemma proved above Satz 11 and Satz 12 of [3]. There is no inverse implication, but the argumentation given above is a direct generalization of the method used by Schmidt in his proofs.

THEOREM 3. *Let $f(x) = a_0x^p + a_1x^{p-1} + \ldots + a_p$ be an integer-valued polynomial with $a_0 > 0$. If*

1. $p \equiv 1 \pmod 2$ *or*
2. $p \equiv 0 \pmod 2$ *and a_0 is not a rational square,*

then
$$\overline{\lim} \operatorname{lp} \sqrt{f(n)} = \infty.$$
Proof. In view of Lemma 3 and the equality $\operatorname{lap}\sqrt{f(n)} = 1$, it is sufficient to show that the set F of all the residues mod 1 of numbers $\sqrt{f(n)}$, $n = 1, 2, \ldots$ has at least one irrational point of accumulation. We shall prove more: that the set F is dense in $(0, 1)$.

In case 1 put $p = 2m+1$, $m \geqslant 0$. As can easily be seen, we have in the environment of ∞
$$\frac{d^k\sqrt{f(x)}}{dx^k} \sim \sqrt{a_0}\binom{m+\frac{1}{2}}{k} k!\, x^{m-k+1/2}.$$
On the other hand, by a well-known theorem of the theory of finite differences (cf. [4], p. 229, th. 221), we have
$$\Delta^k g(x) = \Delta x^k g^{(k)}(x + \Theta k \Delta x), \quad 0 < \Theta < 1,$$
where $g(x)$ is an arbitrary real function with the kth derivative continuous in the interval $(x, x+k\Delta x)$. Putting
$$g(x) = \sqrt{f(x)}, \quad \Delta x = 1,$$
we obtain by a comparison of the preceding two formulae
$$\Delta^k \sqrt{f(x)} \sim \sqrt{a_0}\binom{m+\frac{1}{2}}{k} k!\, x^{m-k+1/2},$$
whence for sufficiently large x
$$\Delta^m \sqrt{f(x)} \sim \sqrt{a_0}\binom{m+\frac{1}{2}}{m} m!\, x^{1/2},$$
$$\Delta^{m+1} \sqrt{f(x)} \sim \sqrt{a_0}\binom{m+\frac{1}{2}}{m+1} (m+1)!\, x^{-1/2};$$
thus
$$\Delta^m \sqrt{f(x)} \to \infty, \quad \Delta^{m+1}\sqrt{f(x)} \to 0.$$
The density of the set F follows immediately in virtue of a theorem of Csillag ([1], p. 152).

In case 2, we have, as can easily be seen
$$(39') \qquad f(x) = u^2(x) + v(x),$$
where u and v are polynomials with coefficients from $K(\sqrt{a_0})$ and
$$(39'') \qquad \operatorname{degree} v < \operatorname{degree} u = \tfrac{1}{2}\operatorname{degree} f, \quad u(\infty) = \infty.$$
Putting $p = 2m$, $u(x) = \alpha_0 x^m + \alpha_1 x^{m-1} + \ldots + \alpha_m$, we find from formulae (39) that $\alpha_0^2 = a_0$, whence according to the assumption about

a_0 it follows that a_0 is irrational. In virtue of a well-known theorem of Weyl, the set of all the residues mod 1 of numbers $u(n)$ $(n = 1, 2, \ldots)$ is dense in $(0, 1)$. Since, in view of (39)

$$\lim\left(\sqrt{f(x)}-u(x)\right) = 0,$$

the set F has the same property, q. e. d.

Remark. In both cases, 1 and 2, it is easy to give examples of polynomials $f(x)$ such that

$$\varliminf \operatorname{lp} \sqrt{f(n)} < \infty. \tag{40}$$

It suffices to assume $f_1(x) = x$, $f_2(x) = 2x^2$. The proof of inequality (40) for the polynomial $f_1(x)$ is immediate; for the polynomial $f_2(x)$ we use the fact that for an infinite sequence of positive integers x_k is $f_2(x_k) = y_k^2+1$ (y_k—integers), whence in view of the expansion

$$\sqrt{y^2+1} = (y, \overline{2y})$$

it follows that

$$\operatorname{lp} \sqrt{f_2(x_k)} = 1.$$

§ 4. Lemma 4. *Let $f(n)$ be an integer-valued polynomial and let*

$$\sqrt{f(n)} = u_0(n)+\frac{1|}{|u_1(n)}+\frac{1|}{|u_2(n)}+\ldots+\frac{1|}{|u_j(n)}+\frac{1|}{|w(n)} \tag{41}$$

where u_i are polynomials of a positive degree with rational coefficients and

$$\lim_{n\to\infty} w(n) = \infty. \tag{42}$$

Put

$$\begin{aligned} &T_{-1}(n) = 1, \quad T_0(n) = u_0(n), \quad T_\nu(n) = u_\nu(n)T_{\nu-1}(n)+T_{\nu-2}(n),\\ &U_{-1}(n) = 0, \quad U_0(n) = 1, \quad U_\nu(n) = u_\nu(n)U_{\nu-1}(n)+U_{\nu-2}(n), \end{aligned} \tag{43}$$

$$\sqrt{f(n)} = \xi = [b_0, b_1, b_2, \ldots], \quad b_i\text{—positive integers.} \tag{44}$$

Then, for every j and $n > n_0(j)$, there exists a $k = k(j, n)$ such that

$$\frac{A_k}{B_k} = \frac{T_j(n)}{U_j(n)}, \tag{45}$$

$$\xi_{k+1}(n) = (-1)^{j-k}\frac{U_j^2(n)}{B_k^2}w(n)+\frac{(-1)^{j-k}U_j(n)U_{j-1}(n)-B_kB_{k-1}}{B_k^2}. \tag{46}$$

Proof. Since the polynomials u_i have rational coefficients, there exists a positive integer m such that

$$T_j(n) = P(n)/m, \qquad U_j(n) = Q(n)/m, \tag{47}$$

where $P(n)$, $Q(n)$ are polynomials with integral coefficients.

From formulae (41) and (43) we get

$$\sqrt{f(n)} = \frac{T_j(n)}{U_j(n)}+\frac{(-1)^j}{U_j(n)\left(U_j(n)w(n)+U_{j-1}(n)\right)}, \tag{48}$$

whence in view of (47)

$$\left|\sqrt{f(n)}-\frac{P(n)}{Q(n)}\right| = \frac{1}{Q^2(n)}\left|\frac{m^2}{w(n)+U_{j-1}(n)/U_j(n)}\right|.$$

Since in view of (42) and (43)

$$w(n)+U_{j-1}(n)/U_j(n) \to \infty,$$

we have for sufficiently large n

$$\left|\sqrt{f(n)}-\frac{P(n)}{Q(n)}\right| < \frac{1}{2Q^2(n)}.$$

In virtue of a well-known theorem (cf. [2], Satz 2.14), $P(n)/Q(n)$ is therefore equal to some convergent of expansion (44). Then, for some k, equality (45) holds and since

$$\sqrt{f(n)} = \frac{A_k}{B_k}+\frac{(-1)^k}{B_k[B_k\xi_{k+1}+B_{k-1}]},$$

we get also (46) in view of (48).

Definition. For a given prime p and a given rational number $r \neq 0$ we shall denote by $\exp(p, r)$ the exponent with which p comes into the canonical expansion of r.

Lemma 5. *Suppose we are given a prime p and integers n and h, both $\neq 0$. Let then*

$$\begin{aligned} &P_{-1} = h, \quad P_0 = n, \quad P_\nu = 2nP_{\nu-1}+hP_{\nu-2},\\ &Q_{-1} = 0, \quad Q_0 = 1, \quad Q_\nu = 2nQ_{\nu-1}+hQ_{\nu-2}. \end{aligned} \tag{49}$$

If $\exp(p, h) > 2\exp(p, 2n)$, then for every integer $\nu \geqslant 0$

$$\begin{aligned} \exp(p, P_\nu) &= \exp(p, n)+\nu\exp(p, 2n),\\ \exp(p, Q_\nu) &= \nu\exp(p, 2n). \end{aligned} \tag{50}$$

Proof by induction with respect to ν. For $\nu = 0$ the lemma follows directly from formulae (49).

For $\nu = 1$ we have $P_1 = 2n^2+h$, $Q_1 = 2n$; thus $\exp(p, Q_1) = \exp(p, 2n)$. Since, by hypothesis,

$$\exp(p, h) > 2\exp(p, 2n) \geqslant \exp(p, 2n^2),$$

it follows that

$$\exp(p, P_1) = \exp(p, 2n^2) = \exp(p, n)+\exp(p, 2n)$$

and formulae (50) hold also for $\nu = 1$.

Assume now that the lemma is right for the numbers $\nu-2$ and $\nu-1$ ($\nu \geqslant 2$); we shall show its validity for ν.

It follows easily from the inductive assumption that

$$e_1 = \exp(p, 2nP_{\nu-1}) = \exp(p, n)+\nu\exp(p, 2n),$$

$$e_2 = \exp(p, hP_{\nu-2}) = \exp(p, n)+(\nu-2)\exp(p, 2n)+\exp(p, h),$$

$$e_3 = \exp(p, 2nQ_{\nu-1}) = \nu\exp(p, 2n),$$

$$e_4 = \exp(p, hQ_{\nu-2}) = (\nu-2)\exp(p, 2n)+\exp(p, h).$$

In view of the inequality $\exp(p, h) > 2\exp(p, 2n)$ we therefore have $e_1 < e_2$, $e_3 < e_4$, whence it follows by (49) that

$$\exp(p, P_\nu) = e_1 = \exp(p, n)+\nu\exp(p, 2n),$$

$$\exp(p, Q_\nu) = e_3 = \nu\exp(p, 2n), \qquad \text{q. e. d.}$$

THEOREM 4. *Suppose we are given an integer $h \neq 0$. Denote by E the set of all integers n such that $h|4n^2$. We have*

$$\lim_{\substack{n\to\infty \\ n\notin E}} \operatorname{lp}\sqrt{n^2+h} = \infty, \tag{51}$$

$$\varlimsup_{\substack{n\to\infty \\ n\in E}} \operatorname{lp}\sqrt{n^2+h} < \infty. \tag{52}$$

Proof. We begin with a proof of equality (51). Choose an arbitrary g; we shall show that for sufficiently large $n \notin E$

$$\operatorname{lp}\sqrt{n^2+h} > g.$$

It is easy to verify the identity

$$\sqrt{n^2+h} = n+\frac{1|}{|2n/h}+\frac{1|}{|n+\sqrt{n^2+h}},$$

from which we immediately obtain

$$\sqrt{n^2+h} = n+\frac{1|}{|2n/h}+\frac{1|}{|2n}+\ldots+\frac{1|}{|2n/h}+\frac{1|}{|n+\sqrt{n^2+h}}. \tag{53}$$

Put in Lemma 4

$$u_\nu = \begin{cases} 2n/h & (\nu \text{ odd} \leqslant 2g-1), \\ 2n & (\nu \text{ even} < 2g-1). \end{cases}$$

Comparing polynomials T_ν, U_ν determined for these u_ν by formulae (42) and polynomials P_ν, Q_ν defined by (49), we find by an easy induction

$$T_\nu = P_\nu h^{-[(\nu+1)/2]}, \qquad U_\nu = Q_\nu h^{-[(\nu+1)/2]}, \tag{54}$$

whence

$$\frac{T_\nu}{U_\nu} = \frac{P_\nu}{Q_\nu}. \tag{55}$$

Assume now that $n \notin E$, n so large that $\sqrt{n^2+h}$ is irrational, and

$$\sqrt{n^2+h} = \xi = [b_0, b_1, \ldots]. \tag{56}$$

In virtue of Lemma 4 for sufficiently large n for each $i \leqslant g$ there exists a k_i such that

$$\frac{A_{k_i}}{B_{k_i}} = \frac{T_{2i-1}}{U_{2i-1}}, \tag{57}$$

$$\xi_{k_i+1} = (-1)^{k_i-1}\frac{U_{2i-1}^2}{B_{k_i}^2}(n+\sqrt{n^2+h})+\frac{(-1)^{k_i-1}U_{2i-1}U_{2i-2}-B_{k_i}B_{k_i-1}}{B_{k_i}^2}. \tag{58}$$

Since $n \notin E$, there exists a prime p such that

$$\exp(p, h) > 2\exp(p, 2n) \tag{59}$$

and in virtue of Lemma 5

$$\exp(p, P_{2i-1}) = \exp(p, n)+(2i-1)\exp(p, 2n),$$

$$\exp(p, Q_{2i-1}) = (2i-1)\exp(p, 2n). \tag{60}$$

In view of (55) and (57), we therefore have

$$\exp\left(p, \frac{A_{k_i}}{B_{k_i}}\right) = \exp\left(p, \frac{P_{2i-1}}{Q_{2i-1}}\right) = \exp(p, n)$$

and since the fraction A_{k_i}/B_{k_i} is irreducible, it follows that

$$\exp(p, B_{k_i}) = 0.$$

On the basis of (54) and (60) we get hence

$$\exp\left(p, \frac{U_{2i-1}}{B_{k_i}}\right) = \exp(p, U_{2i-1}) = (2i-1)\exp(p, 2n)$$
$$-i\exp(p, h) = -\exp(p, 2n) - i\big(\exp(p, h) - 2\exp(p, 2n)\big).$$

Then, in view of inequality (59), the numbers $\exp(p, U_{2i-1}/B_{k_i})$ are for $i = 1, 2, \ldots, g$ all different; since $\sqrt{n^2+h}$ is irrational and (58) holds, the numbers ξ_{k_i+1} have the same property. Since $k_i+1 \geqslant 1 = \operatorname{lap}\sqrt{n^2+h}$, at least g different complete quotients occur in the period of expansion (56); we then have $\operatorname{lp}\xi \geqslant g$, which completes the proof of (51).

In order to prove formula (52) we shall use Theorem 2. From that theorem follows the existence of a number $N = N(h)$ such that if for positive integers D_1, D_2 and l

$$\sqrt{D_2} = \tfrac{1}{2}l\sqrt{D_1}, \quad 0 < l \leqslant |h| \quad \text{and} \quad \operatorname{lp}\sqrt{D_1} \leqslant 12,$$

then $\operatorname{lp}\sqrt{D_2} \leqslant N$.

We shall show that for sufficiently large $n \in E$

$$\operatorname{lp}\sqrt{n^2+h} \leqslant N. \tag{61}$$

In fact, since $n \in E$, $h|4n^2$, there exist—as can easily be seen—integers α, $\beta \neq 0$ and positive integer x such that

$$2n = \alpha\beta x, \qquad h = \alpha\beta^2.$$

We obviously have

$$\sqrt{n^2+h} = \tfrac{1}{2}|\beta|\sqrt{(\alpha x)^2+4\alpha}, \qquad |\beta| \leqslant |h|. \tag{62}$$

On the other hand, as can be verified, the following expansions hold for $x \geqslant 5$:

$\alpha > 0$, x—even

$$\sqrt{(\alpha x)^2+4\alpha} = [\alpha x, \overline{\tfrac{1}{2}x, 2\alpha x}];$$

$\alpha > 0$ even, x—odd

$$\sqrt{(\alpha x)^2+4\alpha} = [\alpha x, \overline{\tfrac{1}{2}(x-1), 1, 1, \tfrac{1}{2}(\alpha x-2), 1, 1, \tfrac{1}{2}(x-1), 2\alpha x}];$$

$\alpha > 0$ odd, x—odd

$$\sqrt{(\alpha x)^2+4\alpha}$$
$$= [\alpha x, \overline{\tfrac{1}{2}(x-1), 1, 1, \tfrac{1}{2}(\alpha x-1), 2x, \tfrac{1}{2}(\alpha x-1), 1, 1, \tfrac{1}{2}(x-1), 2\alpha x}],$$

$\alpha < 0$, x—even

$$\sqrt{(\alpha x)^2+4\alpha} = [|\alpha|x-1, \overline{1, \tfrac{1}{2}(x-4), 1, 2|\alpha|x-2}];$$

$\alpha < 0$ even, x—odd

$$\sqrt{(\alpha x)^2+4\alpha}$$
$$= [|\alpha|x-1, \overline{1, \tfrac{1}{2}(x-3), 2, \tfrac{1}{2}(|\alpha|x-2), 2, \tfrac{1}{2}(x-3), 1, 2|\alpha|x-2}];$$

$\alpha < 0$ odd, x—odd

$$\sqrt{(x)^2+4\alpha}$$
$$= [|\alpha|x-1, \overline{1, \tfrac{1}{2}(x-3), 2, \tfrac{1}{2}(|\alpha|x-3), 1, 2x-2, 1, \tfrac{1}{2}(|\alpha|x-3), 2,}$$
$$\overline{\tfrac{1}{2}(x-3), 1, 2|\alpha|x-2}].$$

Thus, we always have $\operatorname{lp}\sqrt{(\alpha x)^2+4\alpha} \leqslant 12$ and formula (61) follows immediately from (62) and the definition of N.

THEOREM 5. *Let $f(n) = a^2n^2+bn+c$, a, b, c—integers, $a > 0$, $\Delta = b^2-4a^2c \neq 0$. The inequality*

$$\overline{\lim}\operatorname{lp}\sqrt{f(n)} < \infty \tag{63}$$

holds if and only if

$$\Delta \,|\, 4(2a^2, b)^2. \tag{64}$$

Proof. We obviously have

$$\sqrt{f(n)} = \frac{1}{2a}\sqrt{(2a^2n+b)^2-\Delta}$$

and in virtue of Theorem 2 inequality (63) is equivalent to the following

$$\overline{\lim}\operatorname{lp}\sqrt{(2a^2n+b)^2-\Delta} < \infty.$$

But in virtue of Theorem 4 the last inequality holds if and only if for some n_0

$$\Delta \,|\, 4(2a^2n+b)^2 \quad \text{for} \quad n > n_0. \tag{65}$$

We have

$$4(2a^2n+b)^2 = 4(2a^2, b)^2\left(\frac{2a^2}{(2a^2, b)}n + \frac{b}{(2a^2, b)}\right)^2.$$

Since the arithmetical progression

$$\frac{2a^2}{(2a^2,b)}n+\frac{b}{(2a^2,b)} \qquad (n=0,1,\ldots)$$

whose first term and difference are relatively prime, contains infinitely many numbers coprime with Δ, divisibility (64) is a necessary and sufficient condition of (65) and therefore also of (63), q. e. d.

Theorems 3 and 5 give together a complete solution of the problem P for polynomials of the second degree (the case $\Delta=0$ is trivial).

In order to obtain by a similar method a complete solution for polynomials of higher degree, it would be necessary to have for $\sqrt{f(n)}$ an expansion analogous to (53), i.e. an expansion of form (41) and then to know whether it is periodical.

Now, for polynomials $f(n)$ of the form

$$a^2n^{2m}+a_1n^{2m-1}+\ldots+a_{2m}$$

an expansion of form (41) is uniquely determined. In fact, let

$$\sqrt{f(n)}=u_0(n)+\frac{1|}{|u_1(n)}+\frac{1|}{|u_2(n)}+\ldots+\frac{1|}{|u_j(n)}+\frac{1|}{|w_{j+1}(n)}$$

and put for $i\leqslant j$

$$w_i(n)=u_i(n)+\frac{1|}{|u_{i+1}(n)}+\ldots+\frac{1|}{|u_j(n)}+\frac{1|}{|w_{j+1}(n)}.$$

Since $u_0(n)$ is the unique polynomial g such that $\lim\limits_{n\to\infty}\left(\sqrt{f(n)}-g(n)\right)=0$, we have $u_0(n)=u(n)$, where $u(n)$ is defined by formulae (39). Suppose now that we have determined polynomials $u_0,\ldots,u_{i-1}(n)$; we easily find

$$w_i(n)=\frac{\sqrt{f(n)}+p_i(n)}{q_i(n)}$$

where p_i, q_i are polynomials with rational coefficients, and then $u_i(n)$ is uniquely determined by the conditions

$$u(n)+p_i(n)=q_i(n)u_i(n)+r_i(n), \qquad \text{degree } r_i<\text{degree } q_i.$$

The construction of the sequence $u_i(n)$ is therefore easy, but the decision whether the sequence thus determined u_i is periodical presents a considerable difficulty even for polynomials of degree 4.

Thus, the investigation of the problem P has led us to the following problem P_1:

P_1. *To decide whether for a given polynomial $f(n)$ of the form*

$$a^2n^{2m}+a_1n^{2m-1}+\ldots+a_{2m} \quad (m,\ a,\ a_i \text{ are integers},\ m\geqslant 2,\ a\neq 0)$$

there exist polynomials u_i of positive degree with rational coefficients such that

$$\sqrt{f(n)}=u_0(n)+\overline{\frac{1|}{|u_1(n)}+\frac{1|}{|u_2(n)}+\ldots+\frac{1|}{|u_k(n)}}$$

(the dash denotes period).

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