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Contributions to the theory of the distribution of prime numbers in arithmetical progressions I

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1. In this paper we shall occupy ourselves with some questions concerning the distribution of prime numbers in arithmetical progressions of the form

$$l,\, l+k,\, l+2k,\, \ldots \tag{1.1}$$

where $0 < l < k$, $(l, k) = 1$.

Let us write, as is usual,

$$\psi(x, k, l) = \sum_{\substack{n \equiv l \,(\mathrm{mod}\, k) \\ n \leqslant x}} \Lambda(n),$$

where $\Lambda(n)$ is the familiar Dirichlet symbol;

$$\Lambda(n) = \begin{cases} \log p & \text{if} \quad n = p^a,\ a = 1, 2, \ldots;\ p \text{ prime number}, \\ 0 & \text{otherwise}. \end{cases}$$

It has been found that

$$\psi(x, k, l) \sim \frac{x}{\varphi(k)} \qquad (x \to \infty,\ \varphi(k) \text{ is Euler's function}) \tag{1.2}$$

for every fixed k, and in fact (1.2) is known as the prime number theorem for the progression (1.1).

Let us introduce error term in the asymptotic formula (1.2)

$$R(x, k, l) = \psi(x, k, l) - \frac{x}{\varphi(k)}$$

and ask about the difference in orders of magnitude of the expressions

$$\max_{1 \leqslant x \leqslant T} |R(x, k, l_1)|, \quad \max_{1 \leqslant x \leqslant T} |R(x, k, l_2)|$$

$$(l_1 \neq l_2,\ 0 < l_i < k,\ (l_i, k) = 1,\ i = 1, 2)$$

as T is large enough. Some partial results in this direction have been obtained in [1], [2]. Namely it has been shown there that

$$(1.3)\qquad \max_{1\leqslant x\leqslant T}|R(x,k,l)|\leqslant T^{\delta(T)}\exp\left(4\frac{\log T}{\sqrt{\log\log T}}\right)\max_{1\leqslant x\leqslant T}|R(x,k,1)|$$

for (¹) $T\geqslant\max(c_0,\exp k^2)$, where $\delta(T)$ was a certain function tending to zero with $T\to\infty$.

Essential thing for establishing (1.3) was the inequality (implicitly contained in [2])

$$(1.4)\qquad \max_{1\leqslant x\leqslant T}|R(x,k,1)|\geqslant T^{\beta(T,k)}\exp\left(-4\frac{\log T}{\sqrt{\log\log T}}\right),$$

holding for all $T\geqslant\max(c_1,\exp k^2)$, with $\beta(T,k)$ being the real part of an arbitrary zero $\varrho=\beta+i\gamma$ of some arbitrarily taken Dirichlet L-function mod k. It follows that in order to get an extension of (1.3) with two arbitrary numbers l_1, l_2, one should seek for an analogue of (1.4) valid for general error term $R(x,k,l)$. So far so good, but the inevitable difficulty arises now. Namely, the proof of (1.4) based on the following lemma of P. Turán (see [5], p. 52).

Let $z_1, z_2, \ldots, z_M$ be complex numbers such that

$$|z_1|\geqslant|z_2|\geqslant\ldots\geqslant|z_M|,\qquad |z_1|\geqslant 1,$$

and let $b_1, b_2, \ldots, b_M$ be any complex numbers. Then, if m is positive and $N\geqslant M$, there exists an integer r such that

$$m\leqslant r\leqslant m+N,$$

$$(1.5)\qquad |b_1z_1^r+b_2z_2^r+\ldots+b_Mz_M^r|\geqslant\left(\frac{1}{48e^2}\cdot\frac{N}{2N+m}\right)^N\min_{1\leqslant j\leqslant M}|b_1+b_2+\ldots+b_j|.$$

This lemma was then applied with

$$(1.6)\qquad b_1=b_2=\ldots=b_M=1,$$

which was due to the fact that all characters take value 1 at $l=1$. As the latter obviously breaks down in the case of arbitrary l, we should obtain in general a troublesome factor at the right-hand side of (1.5) (in fact the actual b_i-numbers are of form $1/\chi(l)$, χ a character mod k). Therefore it is evident that in the first instance we ought to find some improvement of (1.5), that is to say, to obtain some more comfortable

(¹) Throughout this paper $c_0, c_1, c_2, \ldots$ stand always for positive numerical constants.

factor in place of $\min_{1\leqslant j\leqslant M}|b_1+b_2+\ldots+b_j|$. This sort of result has been stated in Lemma 1. To be sure, even having the latter I did not succeed in proving the desired inequality for $R(x,k,l)$ without any hypothesis. Nevertheless I have been able (Theorem 1) to deduce the estimate

$$(1.7)\qquad \int_X^T\frac{|R(x,k,l)|}{x}\,dx>T^{1/2}\exp\left(-2\frac{\log T}{\log\log T}\right),$$

where

$$X=T\exp\left(-(\log T)^{3/4}\right)$$

(and so *a fortiori* $\max_{X\leqslant x\leqslant T}|R(x,k,l)|>T^{1/2}\exp\left(-3\frac{\log T}{\log\log T}\right)$), holding for all $T\geqslant\max(c_2,\exp k^{40})$, from the following conjecture

(1.8) *In the rectangle $0<\sigma<1$, $|t|\leqslant\max(c_3,k^7)$, $s=\sigma+it$, L-functions* mod k *may vanish only at points of the line $\sigma=\frac{1}{2}$. The numerical constant c_3 is supposed to be sufficiently large and can be explicitly calculated.*

It would be desirable to have at the right-hand side of (1.7) T^{β_0} (in place of $T^{1/2}$), with $\beta_0\geqslant\frac{1}{2}$ being real part of an arbitrarily fixed zero of $L(s,\chi)$ mod k. Such inequality would facilitate working out comparison-theorems in the distribution of primes (not only prime powers!) in two arithmetical progressions. This, however, is no longer possible by the method employed. In fact, following the previous way one could not even assert that any one term of the form

$$\frac{1}{\varphi(k)}\cdot\frac{1}{\chi(l)}\cdot D^{\varrho}\cdot\left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2,\qquad |I_\varrho|\leqslant|\gamma_1|-1$$

would occur in the sum $b_1+b_2+\ldots+b_j$, so that the lower estimation of $\min_{h\leqslant j\leqslant N}|b_1+b_2+\ldots+b_j|$ would evidently break down.

Natural question is, of course, what lower evidence for

$$\int_X^T\frac{|R(x,k,l)|}{x}\,dx$$

can be supplied without any hypothesis. The method which I have used when (1.8) assumed true, is still, to a certain extent, applicable now. I have, in fact, been able (Theorem 2) to conclude

$$(1.9)\qquad \int_X^T \frac{|R(x,k,l)|}{x}\,dx > T^{1/4}, \qquad \text{with} \qquad X = T\exp\left(-(\log T)^{0.9}\right)$$

(and also $\max_{X\leqslant x\leqslant T}|R(x,k,l)| > T^{1/4}$) for $T \geqslant \max(c_4, \exp k^{30L_0})$, where L_0 is the constant of Linnik, i.e. such number that to arbitrarily given l, k, $0 < l < k$, $(l,k) = 1$ there always exists prime number $P \equiv l \,(\mathrm{mod}\, k)$, $k < P \leqslant k^{L_0}$ (²).

The substantial novelty in the proof of (1.9), not to mention the theorem of Linnik, is the use of the following density-theorem (see [4], Satz 1.1, p. 299 and p. 323).

Let $0 < a < 1$ and $N(a, T) = N(a, T, k)$ stand for the number of zeros of all L-functions mod k *in the rectangle*

$$a < \sigma < 1, \qquad |t| \leqslant T.$$

Then, if $T \geqslant k$

$$(1.10)\qquad N(a, T) < c_5(k^4\cdot T^{8/3})^{1-a}\log^8 T.$$

I did not take care to obtain the best possible exponent at the right-hand side of (1.9); $\frac{1}{4}$ seems to be the optimal simple one. The improvement of (1.10), known as the "density-hypothesis", had it been right, would have led to $\frac{1}{3}-\varepsilon$ (³).

By partial integration one could state inequalities corresponding to those of Theorems 1 and 2 for

$$\int_X^T \left|\prod(x,k,l) - \frac{1}{\varphi(k)}\operatorname{li} x\right| x^{-1}\,dx,$$

where

$$\prod(x,k,l) = \sum_{\substack{p^m\equiv l \,(\mathrm{mod}\, k)\\ p^m \leqslant x}} \frac{1}{m}, \qquad \operatorname{li} x = \int_2^x \frac{du}{\log u}.$$

(²) Strictly speaking Linnik's theorem asserts only the right-hand side inequality $P < k^{L_0}$. However, it has been shown implicitly (see e. g. [4], p. 369, the inequality (4.25)) that there are "large" primes $P \equiv l \,(\mathrm{mod}\, k)$, $P < k^{L_0}$, whence the left-hand side.

(³) See Remark to Theorem 2.

Similar problems are to be passed when investigating the distribution of prime numbers in two different progressions with the same modulus k. Again it is of interest to study the order of magnitude of

$$\max_{1\leqslant x\leqslant T}|\psi(x,k,l_1) - \psi(x,k,l_2)|.$$

I defer this and related questions to the forthcoming continuation of the present paper.

As to the conjecture (1.8), it probably might be established by means of computing for not too large numerical values of k. Note e.g. that in the case of the Riemann zeta-function, i.e. for $k = 1$, the following evidence has been checked (see [3]).

In the rectangle $0 < \sigma < 1$, $|t| \leqslant 10^5$, $s = \sigma + it$ all the zeros of the Riemann zeta-function lie on the line $\sigma = \frac{1}{2}$.

2. LEMMA 1. *Let m be a non-negative number and $z_1, z_2, \ldots, z_N$ complex numbers such that*

$$1 = |z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_h| \geqslant \ldots \geqslant |z_N|, \qquad |z_h| > 2\frac{N}{m+N}.$$

Then there exists an integer ν with $m \leqslant \nu \leqslant m+N$ such that

$$(2.1)\qquad \frac{|b_1z_1^\nu + b_2z_2^\nu + \ldots + b_Nz_N^\nu|}{(\frac{1}{2}|z_h|)^\nu} \geqslant \min_{h\leqslant j<h_1}|b_1+b_2+\ldots+b_j|\left(\frac{1}{24e}\cdot\frac{N}{2N+m}\right)^N,$$

where $h_1 \leqslant N$ is any integer for which $|z_{h_1}| < |z_h| - N/(m+N)$. In that case when there do not exist numbers h_1 satisfying the latter inequality, we put at the right-hand side of (2.1) $\min_{h\leqslant j\leqslant N}|b_1+b_2+\ldots+b_j|$ *instead* (⁴).

Proof. We can confine ourselves to an outline of proof as it does not essentially differ from that of Satz IX in [5]. First of all we assume that all z_i's are different numbers (in the general case we apply a simple limiting process) and find similarly to [5] that the inequality

$$(2.2)\qquad \prod_{j=1}^N |z - z_j| \geqslant \left(\frac{1}{6e}\cdot\frac{N}{2N+m}\right)^N$$

holds everywhere outside of some set which may be covered by no more than N circles having joint sum of diameters not exceeding $\frac{2}{3}N/(2N+m)$. Then there obviously exists an r_0 with

$$(2.3)\qquad |z_h| - \frac{2}{3}\cdot\frac{N}{2N+m} \leqslant r_0 \leqslant |z_h|$$

(⁴) Compare [5], Satz VII and Satz X. In this paper Lemma 1 will be used only in the particular case of $h_1 = N$. The general statement will be of importance in some further applications.

and such that (2.2) holds on the whole boundary $|z| = r_0$. Further, it follows that for every set of integers $j_1, j_2, \ldots, j_r$ with $(1 \leqslant) j_1 < j_2 < \ldots \ldots < j_r (\leqslant N)$ we have

$$\prod_{\mu=1}^{r} |z - z_{j_\mu}| \geqslant \left(\frac{1}{12e} \cdot \frac{N}{2N+m}\right)^N \quad \text{for} \quad |z| = r_0. \tag{2.4}$$

Case I.

$$|z_j| \geqslant r_0 \quad \text{for} \quad j = 1, 2, \ldots, N. \tag{2.5}$$

Then [5], Satz VII furnishes that there exists some integer ν with $m \leqslant \nu \leqslant m+N$ such that

$$|b_1 z_1^\nu + b_2 z_2^\nu + \ldots + b_N z_N^\nu| \geqslant |b_1 + b_2 + \ldots + b_N| r_0^\nu \left(\frac{N}{2e(N+m)}\right)^N$$

$$\geqslant \min_{h \leqslant j \leqslant N} |b_1 + b_2 + \ldots + b_j| \left(|z_h| - \frac{2N}{3(2N+m)}\right)^\nu \cdot \left(\frac{N}{2e(N+m)}\right)^N,$$

whence noting that

$$|z_h| - \frac{2N}{3(2N+m)} > \frac{|z_h|}{2} \tag{2.6}$$

and that owing to (2.3) and (2.5) there is no $h_1 \leqslant N$ with

$$|z_{h_1}| < |z_h| - \frac{N}{N+m}$$

we obtain the desired inequality (2.1).

Case II. There exists some integer l with

$$1 \leqslant l < N,$$

such that

$$1 = |z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_l| > r_0 > |z_{l+1}| \geqslant \ldots \geqslant |z_N|.$$

Note that

$$h \leqslant l < h_1. \tag{2.7}$$

Let us write

$$f_1(z) = \prod_{j=l+1}^{N} (z - z_j) = \sum_{j=0}^{N-l} c_j^{(1)} z^{N-l-j}.$$

It is evident that

$$|c_j^{(1)}| \leqslant \binom{N-l}{j}, \quad j = 1, 2, \ldots, (N-l). \tag{2.8}$$

Let $f_2(z)$ stand for the polynomial of degree $\leqslant l-1$, assuming at $z = z_1, z_2, \ldots, z_l$ the values

$$\frac{1}{z_1^{[m]+1} f_1(z_1)}, \quad \frac{1}{z_2^{[m]+1} f_1(z_2)}, \quad \ldots, \quad \frac{1}{z_l^{[m]+1} f_1(z_l)}.$$

We may write $f_2(z)$ in the form

$$f_2(z) = c_0^{(2)} + c_1^{(2)}(z - z_1) + c_2^{(2)}(z - z_1)(z - z_2) + \ldots + c_{l-1}^{(2)}(z - z_1)\ldots(z - z_{l-1}),$$

and find as in [5]

$$c_j^{(2)} = \frac{1}{2\pi i} \int_{|w|=r_0} \frac{dw}{w^{[m]+1} f_1(w)(w - z_1)(w - z_2)\ldots(w - z_{j+1})},$$

$$j = 0, 1, 2, \ldots, (l-1).$$

In view of (2.4) and (2.6) we get then for $j = 0, 1, 2, \ldots, (l-1)$

$$|c_j^{(2)}| \leqslant \frac{1}{r_0^{[m]}} \cdot \frac{1}{\left(\frac{1}{12e} \cdot \frac{N}{2N+m}\right)^N} \leqslant \left|\frac{z_h}{2}\right|^{-[m]} \cdot \frac{1}{\left(\frac{1}{12e} \cdot \frac{N}{2N+m}\right)^N}. \tag{2.9}$$

Putting now $f_2(z)$ in the form

$$f_2(z) = \sum_{j=0}^{l-1} c_j^{(3)} z^j,$$

we obtain similarly to [5]

$$|c_j^{(3)}| \leqslant |c_j^{(2)}| + |c_{j+1}^{(2)}| \binom{j+1}{1} + |c_{j+2}^{(2)}| \binom{j+2}{2} + \ldots + |c_{l-1}^{(2)}| \binom{l-1}{l-j-1},$$

whence, by (2.9),

$$|c_j^{(3)}| \leqslant \left|\frac{z_h}{2}\right|^{-[m]} \frac{1}{\left(\frac{1}{12e} \cdot \frac{N}{2N+m}\right)^N} \cdot \binom{l}{j+1}, \quad j = 0, 1, 2, \ldots, (l-1). \tag{2.10}$$

We introduce still another polynomial

$$f_3(z) = z^{[m]+1} f_1(z) \cdot f_2(z) = \sum_{j=[m]+1}^{[m]+N} c_j^{(4)} z^j$$

and observe that

$$f_3(z_1) = f_3(z_2) = \ldots = f_3(z_l) = 1,$$

$$f_3(z_{l+1}) = f_3(z_{l+2}) = \ldots = f_3(z_N) = 0.$$

Hence

$$b_1+b_2+\ldots+b_l = \sum_{j=[m]+1}^{[m]+N} c_j^{(4)}(b_1 z_1^j+b_2 z_2^j+\ldots+b_N z_N^j)$$

and by (2.7)

$$(2.11)\qquad \min_{h\leqslant j<h_1} |b_1+b_2+\ldots+b_j| \leqslant \max_{[m]+1\leqslant \nu\leqslant [m]+N} |b_1 z_1^\nu+b_2 z_2^\nu+\ldots+b_N z_N^\nu| \sum_{j=[m]+1}^{[m]+N} |c_j^{(4)}|.$$

It remains to estimate the latter sum from above. We find that

$$\sum_{j=[m]+1}^{[m]+N} |c_j^{(4)}| \leqslant \Big(\sum_{j_1=0}^{N-1} |c_{j_1}^{(1)}|\Big)\Big(\sum_{j_2=0}^{l-1} |c_{j_2}^{(3)}|\Big),$$

whence by (2.8) and (2.10)

$$\sum_{j=[m]+1}^{[m]+N} |c_j^{(4)}| \leqslant \left|\frac{z_h}{2}\right|^{-[m]} \cdot \frac{1}{\left(\dfrac{1}{24e}\cdot\dfrac{N}{2N+m}\right)^N}.$$

This and (2.11) give (2.1).

3. In this section we give two further lemmas. Their statements differ only slightly and for brevity's sake we shall prove them at the same time.

Lemma 2. *Let $k\geqslant 3$, $0<l<k$, $(l,k)=1$. Suppose* (1.8) *satisfied. Then there exists a number D, $\frac{1}{2}\max(c_6, k^3)\leqslant D\leqslant \max(c_6, k^3)$, such that*

$$(3.1)\qquad \left|\frac{1}{\varphi(k)}\sum_{(\chi)}\frac{1}{\chi(l)}\sum_{\varrho(\chi)} D^\varrho\left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2\right| \geqslant c_7 D\log D,$$

where $\psi=1/3D$, χ runs through all characters $\bmod k$ and $\varrho=\varrho(\chi)$ through the zeros of $L(s,\chi)$ lying in the strip $0<\sigma<1$.

The other lemma asserts a little less but holds without any conjecture.

Lemma 3. *Let $k\geqslant 3$, $0<l<k$, $(l,k)=1$. Let L_0 be the constant of Linnik. There exists a number D_1, $\max(c_8, k)<D_1\leqslant \max(c_9, k^{L_0})$, such that*

$$(3.2)\qquad \left|\frac{1}{\varphi(k)}\sum_{(\chi)}\frac{1}{\chi(l)}\sum_{\varrho(\chi)} D_1^\varrho\left(\frac{e^{\psi_1\varrho}-e^{-\psi_1\varrho}}{2\psi_1\varrho}\right)^2\right| \geqslant c_{10} D_1\log D_1,$$

with $\psi_1=1/3D_1$ and χ, $\varrho(\chi)$ running as in Lemma 2.

Proof. First we shall confine ourselves to $k\geqslant k_0$, where k_0 is a sufficiently large constant. Consequently, as far as Lemma 2 is concerned, it can be taken that in the rectangle $0<\sigma<1$, $|t|\leqslant k^7$, L-functions $\bmod k$ have no zeros outside $\sigma=\frac{1}{2}$.

There certainly exists a prime or prime square D with $D\equiv l \pmod k$, $\frac{1}{2}k^3\leqslant D\leqslant k^3$. In fact, we have (see [4], p. 232, Satz 4.6)

$$\psi(k^3, k, l) = \frac{k^3}{\varphi(k)} - \frac{1}{\varphi(k)}\sum_{(\chi)}\frac{1}{\chi(l)}\sum_{|\Im\varrho|\leqslant k^3}\frac{k^{3\varrho}}{\varrho} + O(\log^2 k)$$

and

$$\psi(\tfrac{1}{2}k^3, k, l) = \frac{k^3}{2\varphi(k)} - \frac{1}{\varphi(k)}\sum_{(\chi)}\frac{1}{\chi(l)}\sum_{|\Im\varrho|\leqslant k^3/2}\frac{(\frac{1}{2}k^3)^\varrho}{\varrho} + O(\log^2 k),$$

whence and owing to (1.8) we obtain

$$\psi(k^3,k,l)-\psi(\tfrac{1}{2}k^3,k,l) = \frac{k^3}{2\varphi(k)} + O(k^{3/2}\log^2 k) \geqslant c_{11}k^2.$$

On the other hand we have obviously

$$\psi(k^3,k,l)-\psi(\tfrac{1}{2}k^3,k,l) = \sum_{\substack{p\equiv l \pmod k\\ k^3/2<p\leqslant k^3}}\log p + \sum_{\substack{p^2\equiv l \pmod k\\ k^3/2<p^2\leqslant k^3}}\log p + O(k\log k)$$

and the existence of D clearly follows.

Let χ be arbitrary non-principal character $\bmod k$ and χ^* the corresponding primitive character. The latter's modulus will be denoted by k^* $(\leqslant k)$. We have clearly $\chi(D)=\chi^*(D)=\chi(l)$.

Let us start from the integral

$$I(\chi) = \frac{1}{2\pi i}\int_{(2)} D^s\left(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\right)^2\cdot\left(-\frac{L'}{L}(s,\chi^*)\right)ds$$

$$= \sum_{n=1}^{\infty}\chi^*(n)\Lambda(n)\frac{1}{2\pi i}\int_{(2)} D^s\left(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\right)^2\frac{ds}{n^s}$$

$$= \sum_{n=1}^{\infty}\chi^*(n)\Lambda(n)\cdot\frac{1}{2\pi i}\int_{(2)}\left(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\right)^2\left(\frac{D}{n}\right)^s ds.$$

It may be noted that

$$\int_{(2)}\left(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\right)^2\cdot\left(\frac{D}{n}\right)^s ds = 0 \quad\text{for}\quad n\geqslant De^{2\psi},$$

and also, if we move the line of integration to $\sigma = -1$,

$$\int_{(2)} \left(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\right)^2 \left(\frac{D}{n}\right)^s ds = 0 \quad \text{for} \quad n \leqslant De^{-2\psi}.$$

But

$$D-1 < D(1-2\psi) < De^{-2\psi} < D < De^{2\psi} < D(1+3\psi) = D+1,$$

so that

$$I(\chi) = \chi^*(D)\cdot\Lambda(D)\cdot\frac{1}{2\pi i}\int_{(2)} \frac{e^{2\psi s}-2+e^{-2\psi s}}{4\psi^2 s^2}\, ds$$
$$= \chi^*(D)\Lambda(D)\,\frac{\log(e^{2\psi})}{4\psi^2} = \chi(l)\cdot\Lambda(D)\cdot\frac{1}{2\psi}.$$

On the other hand by Cauchy's theorem of residues and [4] (Satz 4.3, p. 227)

$$I(\chi) = -\sum_{\varrho(\chi)} D^\varrho \left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2 - \nu_0(\chi^*) - \nu_{-1}(\chi^*)D^{-1}\left(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\right)_{s=-1} +$$
$$+\frac{1}{2\pi i}\int_{(-3/2)} D^s \left(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\right)^2 \left(-\frac{L'}{L}(s,\chi^*)\right) ds,$$

with $\nu_0(\chi^*)$, $\nu_{-1}(\chi^*)$ equal to 0 or 1.

Also by [4] (Satz 4.3, p. 227) we have

$$\left|\frac{1}{2\pi i}\int_{(-3/2)} D^s \left(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\right)^2 \left(-\frac{L'}{L}(s,\chi^*)\right) ds\right| < c_{12} D^{-3/2}\frac{\log k^*}{\psi^2}\int_2^\infty \frac{\log t}{t^2}\, dt$$
$$< c_{13} D^{1/2}\log D.$$

All in all we obtain

$$(3.3) \qquad -\frac{1}{\chi(l)}\sum_{\varrho(\chi)} D^\varrho \left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2 = \Lambda(D)\cdot\frac{1}{2\psi} + O(D^{1/2}\log D).$$

In the case of a principal character $\chi = \chi_0$ we similarly get

$$-\frac{1}{\chi_0(l)}\sum_{\varrho(\chi_0)} D^\varrho \left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2 = \Lambda(D)\cdot\frac{1}{2\psi} - D\left(\frac{e^{\psi}-e^{-\psi}}{2\psi}\right)^2 + O(D^{1/2}\log D),$$

i.e.

$$(3.4) \qquad -\frac{1}{\chi_0(l)}\sum_{\varrho(\chi_0)} D^\varrho \left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2 = \Lambda(D)\cdot\frac{1}{2\psi} - D + O(D^{1/2}\log D).$$

Multiplying all the relations (3.3) and (3.4) by $1/\varphi(k)$ and summing up we obtain

$$-\frac{1}{\varphi(k)}\sum_{(\chi)}\frac{1}{\chi(l)}\sum_{\varrho(\chi)} D^\varrho \left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2 = \Lambda(D)\cdot\frac{1}{2\psi} + O(D),$$

whence (3.1) by

$$\Lambda(D)\cdot\frac{1}{2\psi} > c_{14} D\log D.$$

It is easily seen that the case of $k \leqslant k_0$ provides no difficulty as we can always find then a prime $D \equiv l \pmod{k}$ with $\frac{1}{2}k_0^3 \leqslant D \leqslant k_0^3$.

Proof of Lemma 3 follows exactly the above lines if we note that in virtue of Linnik's theorem there exists a prime number D_1 with $k < D_1 \leqslant$ $\leqslant k^{L_0}$. Also the case of $k \leqslant k_0$ may be settled similarly.

4. Theorem 1. *Let* $k \geqslant 3$, $0 < l < k$, $(l, k) = 1$. *Suppose* (1.8) *satisfied. Then we have*

$$(4.1) \qquad \int_X^T \left|\frac{\psi(x,k,l)}{x} - \frac{1}{\varphi(k)}\right| dx > T^{1/2}\exp\left(-2\frac{\log T}{\log\log T}\right)$$

with

$$X = T\exp\left(-(\log T)^{3/4}\right)$$

for

$$(4.2) \qquad T \geqslant \max\left(c_{15}, \exp(k^{40})\right),$$

where c_{15} *is a calculable numerical constant.*

Proof. As before we shall prove Theorem 1—or rather deduce (4.1) from (1.8) taken with the rectangle $0 < \sigma < 1$, $|t| \leqslant k^7$—for $k \geqslant k_0$, k_0 being a certain sufficiently large numerical constant. We pass to the general case on putting in (1.8) $c_3 = k_0^7$. One may then easily prove Lemma 2 with $c_6 = k_0^3$ and Theorem 1 with $c_{15} = \exp(k_0^{40})$.

Put

$$T_1 = \frac{T}{D}e^{-2\psi} \qquad (D, \psi \text{ as in Lemma 2}),$$

$$A = \tfrac{3}{5}\log\log T_1, \qquad B = (\log T_1)^{-1/4}, \qquad m = \frac{\log T_1}{A+B} - \log^{3/5} T_1 (\log\log T_1)^2,$$

r an integer, to be fixed later, satisfying

$$(4.3) \qquad m \leqslant r \leqslant \frac{\log T_1}{A+B} \quad \left(< \frac{5}{3}\cdot\frac{\log T_1}{\log\log T_1}\right).$$

Further write

$$F_l(s) = -\frac{1}{\varphi(k)}\sum_{(\chi)}\frac{1}{\chi(l)}\cdot\frac{L'}{L}(s,\chi) - \frac{1}{\varphi(k)}\zeta(s),$$

where the sum $\sum_{(\chi)}$ is to be extended over all the characters mod k.

We start from the integral

$$J(l,k,r,T)=\frac{1}{2\pi i}\int_{(2)}D^s\left(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\right)^2\cdot\left(e^{As}\frac{e^{Bs}-e^{-Bs}}{2Bs}\right)^r F_l(s)\,ds$$

and have by a simple evaluation

(4.4) $J(l,k,r,T)$

$$=\sum_{n=1}^{\infty}\left\{a_n^{(l)}\Lambda(n)-\frac{1}{\varphi(k)}\right\}\cdot\frac{1}{2\pi i}\int_{(2)}D^s\left(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\right)^2\left(e^{As}\frac{e^{Bs}-e^{-Bs}}{2Bs}\right)^r\frac{ds}{n^s},$$

where

$$a_n^{(l)}=\begin{cases}0 & \text{if} \quad n\not\equiv l\,(\mathrm{mod}\,k),\\ 1 & \text{if} \quad n\equiv l\,(\mathrm{mod}\,k).\end{cases}$$

We note that the integrals at the right-hand side of (4.4) vanish for $n\geqslant De^{2\psi}\cdot e^{(A+B)r}=X_2$ and also — if we push the line of integration to, say, $\sigma=-1$ — for $n\leqslant De^{-2\psi}\cdot e^{(A-B)r}=X_1$. Hence we obtain

$J(l,k,r,T)$

$$=\sum_{X_1\leqslant n\leqslant X_2}\left\{a_n^{(l)}\Lambda(n)-\frac{1}{\varphi(k)}\right\}\cdot\frac{1}{2\pi i}\int_{(0)}D^s\left(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\right)^2\left(e^{As}\frac{e^{Bs}-e^{-Bs}}{2Bs}\right)^r\frac{ds}{n^s}.$$

This, if we write $\tilde{R}(x,k,l)=\psi(x,k,l)-[x]/\varphi(k)$, can be expressed by the Stieltjes integral

$J(l,k,r,T)$

$$=\int_{X_1}^{X_2}\left\{\frac{1}{2\pi i}\int_{(0)}D^s\left(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\right)^2\left(e^{As}\frac{e^{Bs}-e^{-Bs}}{2Bs}\right)^r\frac{ds}{x^s}\right\}d\tilde{R}(x,k,l)$$

$$=\left\{\tilde{R}(x,k,l)\cdot\frac{1}{2\pi i}\int_{(0)}D^s\left(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\right)^2\left(e^{As}\frac{e^{Bs}-e^{-Bs}}{2Bs}\right)^r\frac{ds}{x^s}\right\}_{X_1}^{X_2}-$$

$$-\int_{X_1}^{X_2}\tilde{R}(x,k,l)\,d\left(\frac{1}{2\pi i}\int_{(0)}D^s\left(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\right)^2\left(e^{As}\frac{e^{Bs}-e^{-Bs}}{2Bs}\right)^r\frac{ds}{x^s}\right)$$

$$=-\int_{X_1}^{X_2}\tilde{R}(x,k,l)\,d\left\{\frac{1}{\pi}\int_0^{\infty}\cos\big(t(\log D+Ar-\log x)\big)\left(\frac{\sin\psi t}{\psi t}\right)^2\left(\frac{\sin Bt}{Bt}\right)^r dt\right\}$$

$$=\int_{X_1}^{X_2}\tilde{R}(x,k,l)\left\{\frac{1}{\pi}\int_0^{\infty}\sin\big(t(\log D+Ar-\log x)\big)\left(-\frac{t}{x}\right)\left(\frac{\sin\psi t}{\psi t}\right)^2\left(\frac{\sin Bt}{Bt}\right)^r dt\right\}dx.$$

Therefore

$$|J(l,k,r,T)|\leqslant\frac{1}{\pi}\int_0^{\infty}t\left(\frac{\sin\psi t}{\psi t}\right)^2\left|\frac{\sin Bt}{Bt}\right|^r dt\cdot\int_{X_1}^{X_2}\frac{|\tilde{R}(x,k,l)|}{x}\,dx.$$

But

$$\int_0^{\infty}t\left(\frac{\sin\psi t}{\psi t}\right)^2\left|\frac{\sin Bt}{Bt}\right|^r dt\leqslant\int_0^{\infty}t\left|\frac{\sin Bt}{Bt}\right|^r dt=\frac{1}{B^2}\int_0^{\infty}\left|\frac{\sin u}{u}\right|^r u\,du$$

$$\leqslant\frac{1}{B^2}\left(\int_0^1 du+\int_1^{\infty}\frac{du}{u^{r-1}}\right)<\frac{2}{B^2}<2\sqrt{\log T},$$

so that

$$|J(l,k,r,T)|<\sqrt{\log T}\int_{X_1}^{X_2}\frac{|\tilde{R}(x,k,l)|}{x}\,dx.$$

Noting further by (4.3) that

$$X_2=De^{(A+B)r+2\psi}\leqslant De^{2\psi}\cdot T_1=T$$

and

$$X_1=De^{(A-B)r-2\psi}$$
$$\geqslant D\exp\left(-2\psi-2Br+\log T_1-(A+B)\log^{3/5}T_1(\log\log T_1)^2\right)$$
$$>T\exp\left(-4\psi-\frac{10}{3}\cdot\frac{\log T_1}{\log\log T_1}(\log T_1)^{-1/4}-(\log T_1)^{3/5}(\log\log T_1)^3\right)$$
$$>T\exp\left(-(\log T)^{3/4}\right)$$

we obtain

(4.5) $$|J(l,k,r,T)|<\sqrt{\log T}\int_X^T\frac{|\tilde{R}(x,k,l)|}{x}\,dx,$$

where

$$X=T\exp\left(-(\log T)^{3/4}\right).$$

Now, there exists an infinite connected broken line U, consisting of segments alternately parallel to the axes, all lying in the strip

$$\tfrac{1}{30}\leqslant\sigma\leqslant\tfrac{1}{20},$$

such that for all the characters mod k

$$\left|\frac{L'}{L}(s,\chi)\right|\leqslant c_{16}k\log^2\big(k(|t|+1)\big)$$

on U (comp. [1], Lemma 4).

Cauchy's residues theorem applied to $J(l,k,r,T)$ gives

$$(4.6)\quad J(l,k,r,T)=-\frac{1}{\varphi(k)}\sum_{(\chi)}\frac{1}{\chi(l)}\sum_{\varrho>U}D^{\varrho}\left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2\left(e^{A\varrho}\frac{e^{B\varrho}-e^{-B\varrho}}{2B\varrho}\right)^r+$$

$$+\frac{1}{2\pi i}\int_{(U)}D^s\left(\frac{e^{\psi s}-e^{-\psi s}}{2\psi s}\right)^2\left(e^{As}\frac{e^{Bs}-e^{-Bs}}{2Bs}\right)^r F_l(s)\,ds$$

($\varrho=\varrho(\chi)>U$ means that the ϱ's are to be taken to the right of U).

The integral over U is clearly

$$O\left(D^{1/20}\frac{e^{(A+B)r/20}}{B^r}k\cdot\int_{-\infty}^{+\infty}\frac{\log^2\{k(|t|+1)\}}{(t^2+\frac{1}{900})^{r/2}}\,dt\right)$$

$$=O\left(T^{1/20+5/12}\cdot k\int_0^{\infty}\frac{\log^2(k(|t|+1))}{(t^2+\frac{1}{900})^{r/2}}\,dt\right)$$

$$=O\left(T^{0.47}k\log^2k\cdot c_{17}^r\int_0^{\infty}\frac{\log^2 t}{t^r}\,dt\right)=O(T^{0.48}).$$

If we drop in (4.6) those terms for which $|\Im\varrho|>Y\equiv(\log T_1)^{3/5}$ we shall get an error

$$O\left(\frac{De^{r(A+B)}}{B^r}\sum_{n\geqslant Y-1}\frac{\log kn}{n^r}\right)=O\left(T^{1+5/12}\log k\int_{\frac12 Y}^{\infty}\frac{\log x}{x^r}\,dx\right)$$

$$=O\left(T^{17/12}\frac{\exp\left(2\dfrac{\log T}{\log\log T}\right)}{(\log T_1)^{3r/5}}\right)$$

$$=O\left(\frac{T^{17/12}}{T_1}\exp\left(3\frac{\log T}{\log\log T}+\frac{3}{5}\log T_1\left(\frac{5}{3}-\frac{\log\log T_1}{A+B}\right)\right)\right)=O(T^{0.48}).$$

Hence we obtain

$$(4.7)\quad J(l,k,r,T)=-\frac{1}{\varphi(k)}\sum_{(\chi)}\frac{1}{\chi(l)}\sum_{\substack{\varrho>U\\|\Im\varrho|\leqslant Y}}D^{\varrho}\left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2\left(e^{A\varrho}\frac{e^{B\varrho}-e^{-B\varrho}}{2B\varrho}\right)^r+$$

$$+O(T^{0.48}).$$

The sum at the right-hand side of (4.7) will be denoted by S; it is easily seen that the number of terms in S does not exceed $k\log^{3/5}T_1(\log\log T_1)^2$.

Let $\varrho_1=\varrho_1(\chi_1)=\frac12+i\gamma_1$ be that zero from $0<\sigma<1$, $|t|\leqslant k^{6.5}$ which has the greatest absolute imaginary part. It may be noted that $|\Im\varrho|\leqslant|\Im\varrho_1|-1$, where $\varrho=\frac12+i\gamma$ are the zeros of $L(s,\chi)\bmod k$, implies

$$(4.8)\qquad \left|\frac{e^{B\varrho}-e^{-B\varrho}}{2B\varrho}\right|\geqslant\left|\frac{e^{B\varrho_1}-e^{-B\varrho_1}}{2B\varrho_1}\right|.$$

In fact, the left-hand side of (4.8) squared is

$$|1+\tfrac16B^2\varrho^2+O(B^4|\varrho|^4)|^2=|1+\tfrac16B^2\varrho^2|^2+O(B^4|\varrho|^4)$$

$$=\left(1+\tfrac16B^2(\tfrac14-\gamma^2)\right)^2+\frac{B^4\gamma^2}{36}+O(B^4|\varrho|^4)=1+\tfrac13B^2(\tfrac14-\gamma^2)+O(B^4|\varrho|^4)$$

and the right-hand side is similarly

$$1+\tfrac13B^2(\tfrac14-\gamma_1^2)+O(B^4|\varrho_1|^4)$$

so that (4.8) follows by (4.2).

Let, next, $\varrho_2=\varrho_2(\chi_2)=\beta_2+i\gamma_2$ be that zero from $0<\sigma<1$, $|t|\leqslant Y$ at which $\left|e^{A\varrho}\dfrac{e^{B\varrho}-e^{-B\varrho}}{2B\varrho}\right|$ is maximal. It is obvious that $\beta_2\geqslant\frac12$. Put S in the form

$$S=\left(\frac{e^{A\varrho_2}(e^{B\varrho_2}-e^{-B\varrho_2})}{2B\varrho_2}\right)^r\times$$

$$\times\sum_{(\chi)}\sum_{\substack{|\Im\varrho|\leqslant Y\\ \varrho>U}}\frac{1}{\varphi(k)}\cdot\frac{1}{\chi(l)}D^{\varrho}\left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2\cdot\left(e^{A(\varrho-\varrho_2)}\frac{e^{B\varrho}-e^{-B\varrho}}{e^{B\varrho_2}-e^{-B\varrho_2}}\cdot\frac{\varrho_2}{\varrho}\right)^r$$

and introduce the notation:

$$b_j=\frac{1}{\varphi(k)}\cdot\frac{1}{\chi(l)}\cdot D^{\varrho}\left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2,\qquad z_j=e^{A(\varrho-\varrho_2)}\cdot\frac{e^{B\varrho}-e^{-B\varrho}}{e^{B\varrho_2}-e^{-B\varrho_2}}\cdot\frac{\varrho_2}{\varrho},$$

$$z_h=e^{A(\varrho_1-\varrho_2)}\cdot\frac{e^{B\varrho_1}-e^{-B\varrho_1}}{e^{B\varrho_2}-e^{-B\varrho_2}}\cdot\frac{\varrho_2}{\varrho_1}.$$

We shall find the lower bound for S by (2.1) Lemma 1 taken with $\min\limits_{h\leqslant j\leqslant N}|b_1+b_2+\ldots+b_j|$, where $N=[k\cdot\log^{3/5}T_1(\log\log T_1)^2]$. If N is greater than the actual number of ϱ's in the considered domain, we put $z_j=b_j=0$ for the missing ones.

First of all we have

$$|z_h|=e^{A(1/2-\beta_2)}\cdot\left|\frac{e^{B\varrho_1}-e^{-B\varrho_1}}{2B\varrho_1}\right|\cdot\left|\frac{2B\varrho_2}{e^{B\varrho_2}-e^{-B\varrho_2}}\right|\geqslant c_{18}e^{-A/2}=\frac{c_{18}}{(\log T_1)^{0.3}}$$

and also

$$\frac{2N}{N+m} < \frac{2N}{m} < \frac{2k\log^{3/5}T_1(\log\log T_1)^2}{(\log T_1)/\log\log T_1} = \frac{2(\log\log T_1)^3 k}{(\log T_1)^{0.4}} < \frac{2(\log\log T_1)^3}{(\log T_1)^{0.35}}$$

so that

$$|z_h| > \frac{2N}{N+m}.$$

Hence, with a suitable r,

$$(4.9)\quad |S| \geqslant \left|\frac{z_h}{2}\right|^r \cdot \left|e^{A\varrho_2}\frac{e^{B\varrho_2}-e^{-B\varrho_2}}{2B\varrho_2}\right|^r \cdot \left(\frac{1}{24e}\cdot\frac{N}{2N+m}\right)^N \cdot \min_{h\leqslant j\leqslant N}|b_1+b_2+\ldots+b_j|.$$

Owing to (4.8)

$$b_1+b_2+\ldots+b_j$$

$$= \frac{1}{\varphi(k)}\sum_{(\chi)}\frac{1}{\chi(l)}\sum_{|\Im\varrho|\leqslant|\gamma_1|-1} D^\varrho\left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2 + O\left(\sum_{n\geqslant|\gamma_1|-2}\frac{D}{\psi^2}\cdot\frac{\log kn}{n^2}\right)$$

$$= \frac{1}{\varphi(k)}\sum_{(\chi)}\frac{1}{\chi(l)}\sum_{\varrho(\chi)} D^\varrho\left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2 + O\left(k^9\log k\int_{k^{6.5}/2}^{\infty}\frac{\log x}{x^2}\,dx\right)$$

$$= \frac{1}{\varphi(k)}\sum_{(\chi)}\frac{1}{\chi(l)}\sum_{\varrho(\chi)} D^\varrho\left(\frac{e^{\psi\varrho}-e^{-\psi\varrho}}{2\psi\varrho}\right)^2 + O(k^{5/2}\log^2 k)$$

and by Lemma 2, (3.1)

$$(4.10)\qquad \min_{h\leqslant j\leqslant N}|b_1+b_2+\ldots+b_j| > c_{19}k^3\log k.$$

Further

$$\left|\frac{z_h}{2}\right|^r \cdot \left|e^{A\varrho_2}\frac{e^{B\varrho_2}-e^{-B\varrho_2}}{2B\varrho_2}\right|^r \cdot \left(\frac{1}{24e}\cdot\frac{N}{2N+m}\right)^N$$

$$= \frac{e^{Ar/2}}{2^r}\cdot\left|\frac{e^{B\varrho_1}-e^{-B\varrho_1}}{2B\varrho_1}\right|^r\cdot\left(\frac{1}{24e}\cdot\frac{N}{2N+m}\right)^N$$

$$\geqslant \frac{e^{Ar/2}}{2^r}(1-c_{20}B^2k^{13})^r\cdot\left(\frac{1}{\log T}\right)^{k\log^{3/5}T(\log\log T)^2}$$

$$> \frac{e^{Ar/2}}{2^r}\exp(-c_{21}B^2k^{13}r)\cdot\exp\left(-k\log^{3/5}T(\log\log T)^3\right)$$

$$> \frac{1}{2^r}\exp\left(\frac{1}{2}(A+B)r-(\log T)^{5/6}-(\log T)^{4/5}\cdot(\log\log T)^3\right)$$

$$> T^{1/2}\exp\left(-\frac{11}{6}\cdot\frac{\log T}{\log\log T}\right).$$

This, (4.7), (4.9) and (4.10) give

$$|J(l,k,r,T)| > T^{1/2}\exp\left(-\frac{23}{12}\cdot\frac{\log T}{\log\log T}\right),$$

whence by (4.5)

$$\int_X^T \frac{|R(x,k,l)|}{x}\,dx > T^{1/2}\exp\left(-2\frac{\log T}{\log\log T}\right).$$

5. THEOREM 2. *Let $k\geqslant 3$, $0<l<k$, $(l,k)=1$. We have*

$$(5.1)\qquad \int_X^T\left|\frac{\psi(x,k,l)}{x}-\frac{1}{\varphi(k)}\right|dx > T^{1/4}$$

with

$$X = T\exp\left(-(\log T)^{0.9}\right)$$

for

$$(5.2)\qquad T\geqslant \max(c_{22}, \exp k^{30L_0}),$$

where L_0 is Linnik's constant and c_{22} is numerically calculable.

Remark. The reason why we cannot obtain in the exponent more than $\frac{1}{3}$, on the density hypothesis, is the following. Proceeding as in the present proof we consider the rectangle

$$\delta\leqslant\sigma<1,\quad |t|\leqslant D^{2.5},$$

and are led similarly to an asymptotic formula for $b_1+b_2+\ldots+b_j$ with an error term $O\left(D_1^\delta\cdot N(1-\delta,D_1)\right) = O(D_1^{\delta+(2+\varepsilon)\delta})$. Since, owing to Lemma 3, the exponent $\delta+(2+\varepsilon)\delta$ cannot exceed 1, δ must consequently be less than $\frac{1}{3}$.

Proof. This proof has much in common with the previous one. Therefore for the greater part it will be enough if we content ourselves with sketchy explanations. We put, similarly as before,

$$T_2 = \frac{T}{D_1}e^{-2\psi_1},\quad (D_1,\psi_1 \text{ from Lemma 3}),\quad A_1 = \frac{2}{5}\log\log T_2,$$

$$B_1 = (\log T_2)^{-0.1},\quad m_1 = \frac{\log T_2}{A_1+B_1}-\log^{0.4}T_2(\log\log T_2)^2,$$

integer r_1 with

$$m_1\leqslant r_1\leqslant\frac{\log T_2}{A_1+B_1}\quad\left(<\frac{5}{2}\cdot\frac{\log T_2}{\log\log T_2}\right),$$

and consider the integral

$$J_1 = \frac{1}{2\pi i}\int_{(2)} D_1^s\left(\frac{e^{\psi_1 s}-e^{-\psi_1 s}}{2\psi_1 s}\right)^2\left(e^{A_1 s}\frac{e^{B_1 s}-e^{-B_1 s}}{2B_1 s}\right)^{r_1} F_l(s)\,ds.$$

Arguments similar to those used in the preceding section lead to

$$(5.3)\qquad |J_1| < \sqrt{\log T}\int_X^T \frac{|\tilde{R}(x,k,l)|}{x}\,dx,$$

where

$$X = T\cdot\exp\left(-(\log T)^{0.9}\right).$$

Again we have a connected broken line U_1, with segments alternately parallel to the axes, all lying in the strip

$$\tfrac{1}{300}\leqslant \sigma \leqslant \tfrac{1}{200},$$

such that for all the characters $\bmod k$

$$\left|\frac{L'}{L}(s,\chi)\right| \leqslant c_{23}k\log^2\{k(|t|+1)\}$$

on U_1.

Similarly as before we arrive at

$$(5.4)\qquad J_1 = -\frac{1}{\varphi(k)}\sum_{(\chi)}\frac{1}{\chi(l)}\sum_{\substack{\varrho>U_1\\ |\Im\varrho|\leqslant Y_1}} D_1^\varrho\left(\frac{e^{\psi_1\varrho}-e^{-\psi_1\varrho}}{2\psi_1\varrho}\right)^2\left(e^{A_1\varrho}\frac{e^{B_1\varrho}-e^{-B_1\varrho}}{2B_1\varrho}\right)^{r_1}+ O(T^{1/4+1/150})$$

with $Y_1 = (\log T_2)^{0.4}$.

The sum $\sum\limits_{(\chi)}\sum\limits_{\substack{\varrho>U_1\\ |\Im\varrho|\leqslant Y_1}}$ in (5.4), which will be denoted by S_1, has at most $N_1 \doteq [k\log^{0.4}T_2(\log\log T_2)^2]$ terms. Let $\varrho_3 = \varrho_3(\chi_3) = \beta_3+i\gamma_3$ be that zero from $0<\sigma<1$, $|t|\leqslant Y_1$, at which $\left|e^{A_1\varrho}\frac{e^{B_1\varrho}-e^{-B_1\varrho}}{2B_1\varrho}\right|$ is maximal.

Let, further, $\varrho_4 = \varrho_4(\chi_4) = \beta_4+i\gamma_4$ be that zero from the rectangle

$$\tfrac{7}{27}\leqslant\sigma<1,\qquad |t|\leqslant D_1^{2.5},$$

at which $\left|e^{A_1\varrho}\frac{e^{B_1\varrho}-e^{-B_1\varrho}}{2B_1\varrho}\right|$ is minimal.

If we write S_1 in the form

$$S_1 = \left\{\frac{e^{A_1\varrho_3}(e^{B_1\varrho_3}-e^{-B_1\varrho_3})}{2B_1\varrho_3}\right\}^{r_1}\times$$
$$\times\sum_{(\chi)}\sum_{\substack{\varrho>U_1\\ |\Im\varrho|\leqslant Y_1}}\frac{1}{\varphi(k)}\cdot\frac{1}{\chi(l)}D_1^\varrho\left(\frac{e^{\psi_1\varrho}-e^{-\psi_1\varrho}}{2\psi_1\varrho}\right)^2\left(\frac{e^{A_1(\varrho-\varrho_3)}(e^{B_1\varrho}-e^{-B_1\varrho})\varrho_3}{(e^{B_1\varrho_3}-e^{-B_1\varrho_3})\varrho}\right)^{r_1},$$

further put

$$b_j = \frac{1}{\varphi(k)}\cdot\frac{1}{\chi(l)}\cdot D_1^\varrho\cdot\left(\frac{e^{\psi_1\varrho}-e^{-\psi_1\varrho}}{2\psi_1\varrho}\right)^2,\qquad z_j = e^{A_1(\varrho-\varrho_3)}\frac{e^{B_1\varrho}-e^{-B_1\varrho}}{e^{B_1\varrho_3}-e^{-B_1\varrho_3}}\cdot\frac{\varrho_3}{\varrho},$$

$$z_h = e^{A_1(\varrho_4-\varrho_3)}\frac{e^{B_1\varrho_4}-e^{-B_1\varrho_4}}{e^{B_1\varrho_3}-e^{-B_1\varrho_3}}\cdot\frac{\varrho_3}{\varrho_4},$$

then, in view of

$$|z_h| = e^{A_1(\beta_4-\beta_3)}\cdot\left|\frac{e^{B_1\varrho_4}-e^{-B_1\varrho_4}}{2B_1\varrho_4}\right|\cdot\left|\frac{2B_1\varrho_3}{e^{B_1\varrho_3}-e^{-B_1\varrho_3}}\right| > e^{-\frac{20}{27}A_1}\cdot c_{24}$$

$$> c_{24}\cdot(\log T_2)^{-1/3} > \frac{k(\log\log T_2)^3}{(\log T_2)^{0.6}} > \frac{2N_1}{m_1} > \frac{2N_1}{N_1+m_1},$$

and putting, as previously, $z_j = b_j = 0$ for the eventual remaining j's, we can use Lemma 1, (2.1).

Hence

$$|S_1| \geqslant \left|\frac{z_h}{2}\right|^{r_1}\cdot\left|\frac{e^{A_1\varrho_3}(e^{B_1\varrho_3}-e^{-B_1\varrho_3})}{2B_1\varrho_3}\right|^{r_1}\cdot\left(\frac{1}{24e}\cdot\frac{N_1}{2N_1+m}\right)^{N_1}\times$$
$$\times\min_{h\leqslant j\leqslant N_1}|b_1+b_2+\ldots+b_j|$$

and we obtain similarly as in § 4

$$|S_1| \geqslant T^{7/27}\exp\left(-3\frac{\log T}{\log\log T}\right)\cdot\min_{h\leqslant j\leqslant N_1}|b_1+b_2+\ldots+b_j|.$$

Now the last factor! We have clearly (with $h\leqslant j\leqslant N_1$)

$$b_1+b_2+\ldots+b_j = \frac{1}{\varphi(k)}\sum_{(\chi)}\frac{1}{\chi(l)}\sum_{\varrho(\chi)}D_1^\varrho\left(\frac{e^{\psi_1\varrho}-e^{-\psi_1\varrho}}{2\psi_1\varrho}\right)^2+$$
$$+O\Big(\sum_{n\geqslant D_1^{2.5}-1}\frac{\log kn}{n^2}\cdot D_1^3\Big)+O\Big(\frac{1}{\varphi(k)}\sum_{(\chi)}\sum_{\substack{\varrho(\chi)\\ \Re\varrho<7/27}}|D_1^\varrho|\cdot\left|\frac{e^{\psi_1\varrho}-e^{-\psi_1\varrho}}{2\psi_1\varrho}\right|^2\Big).$$

The first error term is

$$O\Big(\sum_{n\geqslant D_1^{2.5}-1}\frac{\log kn}{n^2}\cdot D_1^3\Big) = O\Big(D_1^3\log k\int_{1/2D_1^{2.5}}^{\infty}\frac{\log x}{x^2}\,dx\Big)$$
$$= O\Big(D_1^3\log^2 D_1\cdot\frac{1}{D_1^{2/5}}\Big) = O(D_1^{1/2}\log^2 D_1);$$

in order to estimate the second one we use (1.10). This gives

$$\frac{1}{\varphi(k)}\sum_{(\chi)}\sum_{\substack{\varrho(\chi)\\ \Re\varrho<7/27}}|D_1^{\varrho}|\cdot\left|\frac{e^{\psi_1\varrho}-e^{-\psi_1\varrho}}{2\psi_1\varrho}\right|^2=\frac{1}{\varphi(k)}\sum_{(\chi)}\Bigg(\sum_{\substack{|\Im\varrho|<D_1\\ \Re\varrho<7/27}}+\sum_{\substack{|\Im\varrho|\geqslant D_1\\ \Re\varrho<7/27}}\Bigg)$$

$$\leqslant c_{25}\left(\frac{D_1^{7/27}}{\varphi(k)}\cdot N\left(\tfrac{20}{27},D_1\right)+\frac{D_1^{7/27}}{\varphi(k)}\int\limits_{D_1}^{\infty}\frac{1}{\psi_1^2}\cdot\frac{dN\left(\frac{20}{27},x\right)}{x^2}\right)$$

$$\leqslant c_{26}\left(\frac{D_1^{7/27}}{\varphi(k)}\cdot k^{28/27}\cdot D_1^{56/81}\log^8 D_1+\frac{D_1^{2+7/27}}{\varphi(k)}\int\limits_{D_1}^{\infty}\frac{N\left(\frac{20}{27},x\right)}{x^3}\,dx\right)$$

$$\leqslant c_{27}\left(D_1^{80/81}\log^9 D_1+D_1^{2+7/27}\log D_1\int\limits_{D_1}^{\infty}\frac{\log^8 x}{x^{187/81}}\,dx\right)\leqslant c_{28}D_1,$$

so that by Lemma 3, (3.2)

$$\min_{h\leqslant j\leqslant N_1}|b_1+b_2+\ldots+b_j|>c_{29}D_1\log D_1,$$

and finally

$$|S_1|>T^{7/27}\cdot\exp\left(-3\frac{\log T}{\log\log T}\right).$$

Hence by (5.3) and (5.4) we obtain *a fortiori* (5.1).

References

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Über die Umkehrung eines Satzes von Ingham

by

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1. Ich versuche in dieser Arbeit, zwei Fragen zu beantworten, die Herr Professor P. Turán mir gestellt hat. Die erste Frage betrifft die Möglichkeit der Umkehrung eines bekannten Satzes von A. E. Ingham, die zweite Frage betrifft den Einfluß den das Erdös-Selbergsche Restglied im Primzahlsatz auf die Nullstellenfreiheit der Riemannschen ζ-Funktion ausüben kann. Ich habe eine teilweise Umkehrung des Inghamschen Satzes gefunden. Es hat sich gezeigt, daß man den Umkehrungssatz sogar so weit ausziehen kann, daß er insbesondere eine positive Antwort auf die zweite Frage liefert. Denn das Erdös-Selbergsche Restglied liegt eigentlich außerhalb der Anwendungsmöglichkeit des Satzes von Ingham.

A. E. Ingham hat folgenen Satz bewiesen ([2], S. 60-65):

Wenn die Riemannsche ζ-Funktion keine Nullstellen in dem Gebiet

$$\sigma>1-\eta(|t|) \tag{1.1}$$

besitzt, wo $\eta(t)$ für $t\geqslant 0$ eine abnehmende Funktion ist, die eine stetige Ableitung $\eta'(t)$ hat und folgende Bedingungen erfüllt:

$$0<\eta(t)\leqslant\tfrac{1}{2}, \tag{1.2}$$

$$\eta'(t)\to 0,\qquad t\to\infty \tag{1.3}$$

$$\frac{1}{\eta(t)}=O(\log t),\qquad t\to\infty \tag{1.4}$$

und wenn a eine Zahl aus dem Intervall $0<a<1$ bezeichnet, dann gilt die folgende Abschätzung des Restgliedes im Primzahlsatz

$$\Delta(x)=\sum_{n\leqslant x}\Lambda(n)-x=O(xe^{-\frac{1}{2}a\,\omega(x)}) \tag{1.5}$$