whence for a $K^s_{r,q}$ ($s \geqslant 3$) by virtue of (12) at most $D(\sigma, r, q, K)$ may be zero. Hence, being $D(t, r, q, K) = q^r + \ldots + q + 1 - N_t$, if $t \leqslant \sigma$, and by virtue of second result of the present § 6, we obtain that:

Given any $K^s_{r,q}$ ($s \geqslant 3$), *only two cases may occur*:

(I) $D(\sigma, r, q, K) > 0$, *and then* $K^s_{r,q}$ *is properly contained in at least one cap of kind* h, *where* $\sigma \leqslant h \leqslant s$.

(II) $D(\sigma, r, q, K) = 0$, *and then* $K^s_{r,q}$ *is not properly contained in any cap of kind* $h = \sigma, \ldots, s$ (*and so* $s \neq 4$, *cf. first result* § 2); *but, if* $s \geqslant 5$, $K^s_{r,q}$ *is contained in a cap of kind* $\sigma - 1$ (*because then* $\sigma - 1 \geqslant 2$ *and* $D(\sigma-1, r, q, K) > 0$).

If $a^t_{r,q}$ denotes the least positive integer x satisfying

$$\binom{x}{t}(q-1)^{t-1} + \ldots + \binom{x}{2}(q-1) + x - (q^r + \ldots + q + 1) \geqslant 0 ;$$

since for a complete $K^s_{r,q}$ both (11) and (12) must hold, we have

$$a^s_{r,q} \leqslant K \leqslant a^\sigma_{r,q} . \tag{13}$$

We have—as it can be easily proved—that

$$\lim_{q\to\infty} \frac{(a^t_{r,q})^t}{t!\, q^{r-t+1}} = 1 . \tag{14}$$

Then from (13) we see that, for any complete $K^s_{r,q}$ having q sufficiently large with respect to r, we have

$$\sqrt[s]{s!-1}\, q^{(r-s+1)/s} \leqslant K \leqslant \sqrt[\sigma]{\sigma!+1}\, q^{(r-\sigma+1)/\sigma} . \tag{15}$$

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ACTA ARITHMETICA
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Unitary products of arithmetical functions

by

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1. Introduction. In this paper n and r will represent positive integers. The *unitary product* (convolution) $q(n)$ of two arithmetical functions $f(n), g(n)$ was defined in an earlier paper [1] by

$$q(n) = \sum_{\substack{d\delta=n \\ (d,\delta)=1}} f(d) g(\delta) , \tag{1.1}$$

where the summation is over all relatively prime pairs d, δ such that $d\delta = n$, that is, over all complementary pairs d, δ of unitary divisors of n. If the condition, $(d, \delta) = 1$, is removed, the summation in (1.1) becomes the ordinary *Dirichlet* (or *direct*) *product* of the functions $f(n), g(n)$.

In [1] the unitary product was used in treating several asymptotic problems in elementary number theory. It is the purpose of the present paper to apply this method to additional problems involving the distribution of sets of integers. We shall use a generalized unitary inversion formula proved in § 3 (Theorem 2.3).

Let n have distinct prime factors $p_1, \ldots, p_t$, and place

$$n = p_1^{e_1} \ldots p_t^{e_t} , \tag{1.2}$$

so that $t = 0$ in case $n = 1$. Suppose a and b to be positive integers. We denote by $S_{a,b}$ the set of integers n in (1.2) such that each e_i is divisible by either a or b, and by $S^*_{a,b}$ the set of n such that each e_i is divisible by one of the integers a, b, but not by both ($i = 1, \ldots, t$). For real x, $S_{a,b}(x)$ and $S^*_{a,b}(x)$ will denote the number of $n \leqslant x$ contained in $S_{a,b}$ and $S^*_{a,b}$, respectively. Asymptotic representations of $S_{a,b}(x)$ and $S^*_{a,b}(x)$ are deduced in § 4 under certain natural conditions on a and b.

Our investigation of the distribution of $S_{a,b}$ and $S^*_{a,b}$ involves the consideration of two divisor functions, $\tau_{a,b}(n)$ and $\tau^*_{a,b}(n)$, defined as follows: $\tau_{a,b}(n)$ is the number of decompositions of n in the form $n = d^a\delta^b$, while $\tau^*_{a,b}(n)$ denotes the number of such decompositions, under the restriction, $(d, \delta) = 1$. In particular, we require estimates for the sums,

$$T_{a,b}(x, r) = \sum_{\substack{n \leqslant x \\ (n,r)=1}} \tau_{a,b}(n), \quad T^*_{a,b}(x, r) = \sum_{\substack{n \leqslant x \\ (n,r)=1}} \tau^*_{a,b}(n). \tag{1.3}$$

Asymptotic expressions for these sums with $a \neq b$ are obtained in § 3.

The precise approach is as follows. On the basis of the unitary product, the treatment of $S^*_{a,b}(x)$ is reduced to that of $S_{a,b}(x)$, whose treatment in a similar manner, is reduced to that of $T^*_{a,b}(x, r)$. The consideration of $T^*_{a,b}(x, r)$ is then, by means of direct products, made to depend upon that of $T_{a,b}(x, r)$. Finally, $T_{a,b}(x, r)$ is evaluated by applying the generalization (3.6) of a well known result concerning the sum of powers of consecutive integers. The method of the paper does not require the use of generating functions.

The paper is independent of [1] in all essential respects. The O-constants in the results of §§ 3 and 4 are not uniform in the parameters a and b.

2. Inversion formulas. The simplest approach to the topic of unitary inversion is by elementary abstract algebra. Let R denote the binary system consisting of all (complex-valued) arithmetical functions with respect to the unitary product and the ordinary function sum. For convenience, the unitary product (1.1) may be written $q = f \cdot g$.

Define

$$\varepsilon(n) = \begin{cases} 1 & (n = 1), \\ 0 & (n > 1); \end{cases} \tag{2.1}$$

also place $I(n) \equiv 1$ for all n, and let $\mu^*(n) = (-1)^t$, where t is the number of distinct prime divisors of n.

Remark 2.1. The functions $\varepsilon(n)$, $\mu^*(n)$, and $I(n)$ are multiplicative.

THEOREM 2.1. *The system R is a commutative ring with identity element $\varepsilon(n)$. The multiplicative functions of R form a semigroup relative to the unitary product.*

Proof. Clearly

$$(f \cdot g) \cdot h = \sum_{\substack{De=n \\ (D,e)=1}} h(e) \Big(\sum_{\substack{d\delta=D \\ (d,\delta)=1}} f(d) g(\delta) \Big) = \sum_{\substack{d\delta e=n \\ (d,\delta)=(d,e)=(\delta,e)=1}} f(d) g(\delta) h(e),$$

so that $(f \cdot g) \cdot h = f \cdot (g \cdot h)$, which establishes the associativity of the unitary product. The other ring properties are easily verified. The second half of the theorem is a restatement of [1], Lemma 6.1.

Let $\xi(n)$ and $\eta(n)$ denote functions satisfying

$$\sum_{\substack{d\delta=n \\ (d,\delta)=1}} \xi(d) \eta(\delta) = \varepsilon(n); \tag{2.2}$$

that is, $\xi(n)$ is an invertible element of R with inverse $\eta(n)$.

THEOREM 2.2. *If $\xi(n)$ and $\eta(n)$ satisfy* (2.2), *then*

$$f(n) = \sum_{\substack{d\delta=n \\ (d,\delta)=1}} \xi(d) g(\delta) \rightleftarrows g(n) = \sum_{\substack{d\delta=n \\ (d,\delta)=1}} \eta(d) f(\delta). \tag{2.3}$$

Proof. By (2.2), ξ has the multiplicative inverse $\xi^{-1} = \eta$ in the ring R; similarly, $\eta^{-1} = \xi$. Hence $f = \xi \cdot g \to g = \xi^{-1} \cdot f = \eta \cdot f$, and conversely, thus proving (2.3).

Next define $\nu_k(n)$, for positive integers k, to be 1 or 0 according as n is or is not a k-th power. Also let $\mu^*_k(n) = \mu^*(m)$ or 0 according as n is or is not of the form m^k.

LEMMA 2.1.

$$\sum_{\substack{d\delta=n \\ (d,\delta)=1}} \nu_k(d) \mu^*_k(\delta) = \varepsilon(n). \tag{2.4}$$

Proof. Evidently the left member of (2.4) is 0 unless n is a k-th power. Placing $n = m^k$, it suffices then to show that

$$\lambda(m) \equiv \sum_{\substack{d\delta=m \\ (d,\delta)=1}} \mu^*(\delta) = \varepsilon(m), \tag{2.5}$$

a fact proved in [1], Corollary 2.1.2. To see (2.5) in another way, note that the left member of (2.5) is $\lambda = \mu^* \cdot I$. By Remark 2.1 and the second half of Theorem 2.1, it therefore suffices to prove (2.5) in case m is of the form, $m = p^e$, p prime. But $\lambda(p^e) = 1$ or 0 according as $e = 0$ or $e \neq 0$, thus proving the lemma.

THEOREM 2.3.

$$f(n) = \sum_{\substack{d^k\delta=n \\ (d,\delta)=1}} g(\delta) \rightleftarrows g(n) = \sum_{\substack{d^k\delta=n \\ (d,\delta)=1}} \mu^*(d) f(\delta). \tag{2.6}$$

Proof. By (2.4), this result is a consequence of Theorem 2.2, with $\xi(n) = \nu_k(n)$ and $\eta(n) = \mu^*_k(n)$.

3. Estimates for $T_{a,b}(x, r)$ and $T^*_{a,b}(x, r)$. We first introduce some notation and recall several known facts that will be useful for the later discussion. The Möbius function $\mu(n)$ has the characteristic property,

$$\sum_{d\delta=n} \mu(d) = \varepsilon(n), \tag{3.1}$$

so that $\mu(n)$ and $\mu^*(n)$ play analogous rôles (see (2.5)) for the Dirichlet and unitary products, respectively. The Legendre function $\varphi(x, n)$, which denotes the number of positive integers $\leqslant x$ prime to n, has the property ([1], (3.9)),

$$\varphi(x, n) = \frac{\varphi(n) x}{n} + O\big(\theta(n)\big), \tag{3.2}$$

where $\varphi(n) = \varphi(n, n)$ and $\theta(n)$ denotes the number of unitary divisors of n, the relation being uniform in x.

The generalized totient $\varphi_s(n)$, s real, is defined by

$$\varphi_s(n) \equiv \sum_{d\delta=n} \mu(d)\delta^s = n^s \prod_{p|n} (1-p^{-s}), \tag{3.3}$$

where the product ranges over the prime divisors p of n. In particular $\varphi_1(n) = \varphi(n)$. With $\zeta(s)$ denoting the Riemann zeta-function, we have also

$$\sum_{\substack{n=1\\(n,r)=1}}^{\infty} \frac{\mu(n)}{n^s} = \frac{r^s}{\zeta(s)\varphi_s(r)} \quad (s>1). \tag{3.4}$$

The proof of (3.4) is similar to that in the case $s=2$, sketched in [1], Lemma 5.1.

Finally, we recall the important estimate ([5], Theorem 8)

$$N_s(x) \equiv \sum_{n\leqslant x} \frac{1}{n^s} = \zeta(s) - \frac{1}{(s-1)x^{s-1}} + O\left(\frac{1}{x^s}\right), \quad s>0,\ s\neq 1. \tag{3.5}$$

This estimate may be generalized as follows.

LEMMA 3.1. *If $s>0$, $s\neq 1$, $x\geqslant 1$, then*

$$N_s(x,r) \equiv \sum_{\substack{n\leqslant x\\(n,r)=1}} \frac{1}{n^s} = \frac{\zeta(s)\varphi_s(r)}{r^s} - \frac{\varphi(r)}{r(s-1)x^{s-1}} + O\left(\frac{\theta(r)}{x^s}\right). \tag{3.6}$$

Proof. By (3.1)

$$N_s(x,r) = \sum_{n\leqslant x} \frac{\varepsilon\big((n,r)\big)}{n^s} = \sum_{n\leqslant x} \frac{1}{n^s} \sum_{\substack{d|(n,r)\\de=n}} \mu(d)$$

$$= \sum_{d|r} \frac{\mu(d)}{d^s} \sum_{e\leqslant x/d} \frac{1}{e^s} = \sum_{d|r} \frac{\mu(d)}{d^s} N_s\left(\frac{x}{d}\right).$$

Application of (3.5) leads to

$$N_s(x,r) = \zeta(s) \sum_{d|r} \frac{\mu(d)}{d^s} - \frac{1}{(s-1)x^{s-1}} \sum_{d|r} \frac{\mu(d)}{d} + O\left(\frac{\theta(r)}{x^s}\right),$$

and (3.6) results on the basis of (3.3).

It is convenient to write $c=a+b$, $\varrho=b/a$, $\sigma=a/b$, and to define

$$\Phi(s,r) = \frac{\varphi(r)\varphi_s(r)}{r^{s+1}}. \tag{3.7}$$

We are now ready to prove the following result concerning $\tau_{a,b}(n)$.

THEOREM 3.1. *If $b>a\geqslant 1$, then for $x\geqslant 1$*

$$T_{a,b}(x,r) = \alpha_r x^{1/a} + \beta_r x^{1/b} + O\big(x^{1/c}\theta(r)\big) + O\big(\theta^2(r)\big), \tag{3.8}$$

where $\alpha_r = \zeta(\varrho)\Phi(\varrho,r)$ and $\beta_r = \zeta(\sigma)\Phi(\sigma,r)$.

Proof. We have

$$T_{a,b}(x,r) = \sum_{\substack{n\leqslant x\\(n,r)=1}} \tau_{a,b}(n) = \sum_{\substack{d^a\delta^b\leqslant x\\(d,r)=(\delta,r)=1}} 1,$$

and since, in the latter summation, d and δ cannot simultaneously assume values $> x^{1/c}$,

$$T_{a,b}(x,r) = \sum_{d\leqslant x^{1/c}} 1 + \sum_{\delta\leqslant x^{1/c}} 1 - \sum_{\substack{d\leqslant x^{1/c}\\\delta\leqslant x^{1/c}}} 1 \equiv \sum\nolimits_1 + \sum\nolimits_2 - \sum\nolimits_3 \tag{3.9}$$

where, in each summation, it is understood that $d^a\delta^b \leqslant x$, $(d,r)=(\delta,r)=1$. One obtains then

$$\sum\nolimits_1 = \sum_{\substack{d\leqslant x^{1/c}\\(d,r)=1}} \sum_{\substack{\delta\leqslant (x/d^a)^{1/b}\\(\delta,r)=1}} 1 = \sum_{\substack{d\leqslant x^{1/c}\\(d,r)=1}} \varphi\left(\frac{x^{1/b}}{d^{a/b}}, r\right),$$

so that, on applying a similar argument to $\sum_2$ and $\sum_3$,

$$\sum\nolimits_1 = L_{\sigma,x^{1/b}}(x^{1/c},r), \quad \sum\nolimits_2 = L_{\varrho,x^{1/a}}(x^{1/c},r), \quad \sum\nolimits_3 = \varphi^2(x^{1/c},r), \tag{3.10}$$

where

$$L_{s,z}(y,r) \equiv \sum_{\substack{n\leqslant y\\(n,r)=1}} \varphi\left(\frac{z}{n^s}, r\right), \quad s>0,\ s\neq 1,\ y\geqslant 1,\ z\geqslant 1. \tag{3.11}$$

By (3.2)

$$L_{s,z}(y,r) = \frac{z\varphi(r)}{r} N_s(y,r) + O\big(y\theta(r)\big),$$

and hence by Lemma 3.1,

$$L_{s,z}(y,r) = \zeta(s)z\Phi(s,r) - \frac{\varphi^2(r)z}{r^2(s-1)y^{s-1}} + O\left(\frac{z\theta(r)}{y^s}\right) + O\big(y\theta(r)\big). \tag{3.12}$$

From (3.10) and (3.12) it follows then that

$$\sum\nolimits_1 = \beta_r x^{1/b} - \frac{\varphi^2(r)x^{2/c}}{r^2(\sigma-1)} + O\big(x^{1/c}\theta(r)\big), \tag{3.13}$$

$$\sum\nolimits_2 = \alpha_r x^{1/a} - \frac{\varphi^2(r)x^{2/c}}{r^2(\varrho-1)} + O\big(x^{1/c}\theta(r)\big). \tag{3.14}$$

Again by (3.2) and (3.10)

$$\sum\nolimits_3 = \left(\frac{x^{1/c}\varphi(r)}{r} + O\big(\theta(r)\big)\right)^2,$$

which becomes (since $\varphi(r)\leqslant r$)

$$\sum\nolimits_3 = \frac{\varphi^2(r)x^{2/c}}{r^2} + O\big(x^{1/c}\theta(r)\big) + O\big(\theta^2(r)\big). \tag{3.15}$$

The theorem results on the basis of (3.9), (3.13), (3,14), and (3.15).

The case $r=1$ of the theorem yields

COROLLARY 3.1.1. (Franel-Landau [3], [2], p. 318). *If $b>a\geqslant 1$, then for $x\geqslant 1$,*

$$\sum_{n\leqslant x}\tau_{a,b}(n)=\zeta(\varrho)x^{1/a}+\zeta(\sigma)x^{1/b}+O(x^{1/c})\,. \tag{3.16}$$

The consideration of $T^*_{a,b}(x)$ is reduced to that of $T_{a,b}(x)$ by virtue of the following relation.

LEMMA 3.2. *If a,b are positive, then*

$$\tau^*_{a,b}(n)=\sum_{d^c\delta=n}\mu(d)\tau_{a,b}(\delta)\,. \tag{3.17}$$

Proof. By (3.1),

$$\tau^*_{a,b}(n)=\sum_{d^a\delta^b=n}\varepsilon\big((d,\delta)\big)=\sum_{d^a\delta^b=n}\sum_{\substack{D\Delta=d\\ DE=e}}\mu(D)=\sum_{D^c\Delta^aE^b=n}\mu(D)$$
$$=\sum_{D^c\delta=n}\mu(D)\sum_{\Delta^aE^b=\delta}1=\sum_{D^c\delta=n}\mu(D)\tau_{a,b}(\delta)\,.$$

The lemma is proved.

LEMMA 3.3. *For all $s>0$ and all r,*

$$|\Phi(s,r)|\leqslant 1\,. \tag{3.18}$$

Proof. Actually, $0<\Phi(s,r)<1$, if $r>1$ because by (3.3),

$$\Phi(s,r)=\frac{\varphi(r)}{r}\prod_{p|r}\Big(1-\frac{1}{p^s}\Big)\,,\qquad r>1\,. \tag{3.19}$$

Analogous to $\Phi(s,r)$, we define

$$\Phi^*(s,r)=\frac{\varphi(r)\varphi_s(r)}{\varphi_{s+1}(r)}\qquad (s\neq -1)\,. \tag{3.20}$$

THEOREM 3.2. *If $b>a\geqslant 1$, then for $x\geqslant 2$,*

$$T^*_{a,b}(x,r)=a^*_rx^{1/a}+\beta^*_rx^{1/b}+O\big(\theta(r)x^{1/c}\log x\big)+O\big(\theta^2(r)x^{1/c}\big)\,, \tag{3.21}$$

where

$$a^*_r=\big(\zeta(\varrho)/\zeta(\varrho+1)\big)\Phi^*(\varrho,r)\quad \textit{and}\quad \beta^*_r=\big(\zeta(\sigma)/\zeta(\sigma+1)\big)\Phi^*(\sigma,r)\,.$$

Proof. By Lemma 3.2,

$$T^*_{a,b}(x,r)=\sum_{\substack{n\leqslant x\\ (n,r)=1}}\tau^*_{a,b}(n)=\sum_{\substack{d^c\delta\leqslant x\\ (d,r)=(\delta,r)=1}}\mu(d)\tau_{a,b}(\delta)\,,$$

from which

$$T^*_{a,b}(x,r)=\sum_{\substack{n\leqslant x^{1/c}\\ (n,r)=1}}\mu(n)\,T_{a,b}\Big(\frac{x}{n^c},r\Big)\,. \tag{3.22}$$

Application of Theorem 3.1 gives

$$T^*_{a,b}(x,r)=a_rx^{1/a}\sum_{\substack{n\leqslant x^{1/c}\\ (n,r)=1}}\frac{\mu(n)}{n^{\varrho+1}}+\beta_rx^{1/b}\sum_{\substack{n\leqslant x^{1/c}\\ (n,r)=1}}\frac{\mu(n)}{n^{\sigma+1}}$$
$$+O\Big(\theta(r)x^{1/c}\sum_{n\leqslant x^{1/c}}\frac{1}{n}\Big)+O\big(\theta^2(r)x^{1/c}\big)\,.$$

By Lemma 3.3, a_r and β_r are bounded as functions of r. It therefore follows that

$$T^*_{a,b}(x,r)=a_rx^{1/a}\sum_{\substack{n=1\\ (n,r)=1}}^{\infty}\frac{\mu(n)}{n^{\varrho+1}}+O\Big(x^{1/a}\sum_{n>x^{1/c}}\frac{1}{n^{\varrho+1}}\Big)+\beta_rx^{1/b}\sum_{\substack{n=1\\ (n,r)=1}}^{\infty}\frac{\mu(n)}{n^{\sigma+1}}$$
$$+O\Big(x^{1/b}\sum_{n>x^{1/c}}\frac{1}{n^{\sigma+1}}\Big)+O\big(\theta(r)x^{1/c}\log x\big)+O\big(\theta^2(r)x^{1/c}\big)\,.$$

The first two O-terms in the latter expression are $O(x^{1/c})$. The theorem results by virtue of (3.4).

COROLLARY 3.2.1 ($r=1$). *If $b>a\geqslant 1$, then for $x\geqslant 2$,*

$$\sum_{n\leqslant x}\tau^*_{a,b}(n)=\Big(\frac{\zeta(\varrho)}{\zeta(\varrho+1)}\Big)x^{1/a}+\Big(\frac{\zeta(\sigma)}{\zeta(\sigma+1)}\Big)x^{1/b}+O(x^{1/c}\log x)\,. \tag{3.23}$$

4. Estimates for $S_{a,b}(x)$ and $S^*_{a,b}(x)$. The notation of § 3 will be retained in this section. We first prive some lemmas.

LEMMA 4.1. *For $s>0$,*

$$\Phi^*(s,r)=\prod_{p|r}\Big(1-\frac{p^s+p-2}{p^{s+1}-1}\Big)\,. \tag{4.1}$$

Proof. By (3,3) and (3.20).

LEMMA 4.2. *For all $s>0$ and all r,*

$$|\Phi^*(s,r)|\leqslant 1\,. \tag{4.2}$$

Proof. As a matter of fact, by (4.1), it is easily verified that $0<\Phi^*(s,r)<1$ if $r>1$.

Define now

$$R_s(q)=\sum_{n=1}^{\infty}\frac{\mu^*(n)\Phi^*(s,n)}{n^q}\,,\qquad s>0\,,\ q>1\,; \tag{4.3}$$

the series converges absolutely in view of (4.2) and the boundedness of $\mu^*(n)$. We now express $R_s(q)$ as an infinite product.

LEMMA 4.3. *If $s > 0$, $q > 1$, then $R_s(q) = \zeta(q)\zeta(s+1)J_s(q)$, where*

$$J_s(q) = \prod_p \left(1 - \frac{2}{p^q} + \frac{1}{p^{q+1}} + \frac{1}{p^{q+s}} - \frac{1}{p^{s+1}}\right), \tag{4.4}$$

the product ranging over all primes p.

Proof. By the multiplicativity of $\mu^*(n)$ and by (4.1), it follows that (cf. [4, § 17.4])

$$R_s(q) = \prod_p \left\{1 - \left(1 - \frac{p^s+p-2}{p^{s+1}-1}\right)\sum_{e=1}^{\infty}\frac{1}{p^{qe}}\right\}$$

$$= \prod_p \left\{2 - \frac{p^s+p-2}{p^{s+1}-1} - \left(1 - \frac{p^s+p-2}{p^{s+1}-1}\right)\sum_{e=0}^{\infty}\frac{1}{p^{qe}}\right\}$$

$$= \prod_p \left\{\frac{2-p^{-1}-p^{-s}}{1-p^{-s-1}} - \left(\frac{1-p^{-1}-p^{-s}+p^{-s-1}}{1-p^{-s-1}}\right)\left(\frac{1}{1-p^{-q}}\right)\right\},$$

and (4.4) results on factoring out $\prod(1-p^{-s-1})^{-1}(1-p^{-q})^{-1} = \zeta(s+1)\zeta(q)$.

LEMMA 4.4. *For all $\varepsilon > 0$, $\theta(n) = O(n^\varepsilon)$.*

Proof. It is well known ([4], Theorem 315) that $\tau(n) = O(n^\varepsilon)$ where $\tau(n) = \tau_{1,1}(n)$. Hence the lemma.

Place $k = ab$ and define

$$j_{a,b}(n) = \begin{cases} 1 & (n \in S_{a,b}), \\ 0 & (n \notin S_{a,b}). \end{cases} \tag{4.5}$$

Our estimation of $S_{a,b}(x)$ will be based on the following relation expressing $j_{a,b}(n)$ in terms of $\tau^*_{a,b}(n)$.

LEMMA 4.5. *If $b \geqslant a \geqslant 1$, $(a, b) = 1$, then*

$$j_{a,b}(n) = \sum_{\substack{d^k\delta = n \\ (d,\delta)=1}} \mu^*(d)\tau^*_{a,b}(\delta). \tag{4.6}$$

Proof. We evaluate the sum,

$$f_{a,b}(n) \equiv \sum_{\substack{d^k\delta = n \\ (d,\delta)=1}} j_{a,b}(\delta) = \sum_{\substack{d\delta = n \\ (d,\delta)=1}} \nu_k(d) j_{a,b}(\delta). \tag{4.7}$$

Since $\nu_k(n)$ and $j_{a,b}(n)$ are evidently multiplicative, it follows by Theorem 2.1, that $f_{a,b}(n)$ is also multiplicative. Therefore, it suffices to determine $f_{a,b}(n)$ in case $n = p^e$, p prime, $e > 0$. It is easily verified that $f_{a,b}(p^e) = 0, 1, 2$ according as e is divisible (i) by neither a nor b, (ii) by exactly one of the pair a, b, (iii) by both a and b. Thus $f_{a,b}(n) = \tau^*_{a,b}(n)$, and (4.6) results from 4.7 by Theorem 2.3.

We now prove

THEOREM 4.1. *If $b > a > 1$, $(a, b) = 1$, then for $x \geqslant 2$,*

$$S_{a,b}(x) = Ax^{1/a} + Bx^{1/b} + O(x^{1/c}\log x), \tag{4.8}$$

where $A = \zeta(\varrho)\zeta(b)J_\varrho(b)$, $B = \zeta(\sigma)\zeta(a)J_\sigma(a)$, and $J_s(q)$ is defined by (4.4).

Remark 4.1. The estimates in (3.21) and (4.8) are valid on the range $1 \leqslant x \leqslant 2$, if the logarithmic O-terms are all replaced by $O(1)$ for this range.

Proof. By (4.6) we have

$$S_{a,b}(x) = \sum_{n\leqslant x} j_{a,b}(n) = \sum_{\substack{d^k\delta\leqslant x \\ (d,\delta)=1}} \mu^*(d)\tau^*_{a,b}(\delta),$$

so that

$$S_{a,b}(x) = \sum_{n\leqslant x^{1/k}} \mu^*(n) T^*_{a,b}\left(\frac{x}{n^k}, n\right). \tag{4.9}$$

Applying Theorem 3.2, one obtains, since $|\mu^*(n)| \leqslant 1$, (cf. Remark 4.1)

$$S_{a,b}(x) = \frac{\zeta(\varrho)x^{1/a}}{\zeta(\varrho+1)}\sum_{n\leqslant x^{1/k}}\frac{\mu^*(n)\Phi^*(\varrho, n)}{n^b} + \frac{\zeta(\sigma)x^{1/b}}{\zeta(\sigma+1)}\sum_{n\leqslant x^{1/k}}\frac{\mu^*(n)\Phi^*(\sigma, n)}{n^a}$$
$$+ O\left(x^{1/c}\log x\sum_{n\leqslant(x/2)^{1/k}}\frac{\theta(n)}{n^{k/c}}\right) + O\left(x^{1/c}\sum_{n\leqslant x^{1/k}}\frac{\theta^2(n)}{n^{k/c}}\right) + O(x^{1/k}).$$

As $k > c$, it follows by Lemma 4.4 that the O-terms are $O(x^{1/c}\log x)$. By the boundedness of $\Phi^*(s, r)$ as a function of r (Lemma 4.2), we find then

$$S_{a,b}(x) = \frac{\zeta(\varrho)}{\zeta(\varrho+1)}R_\varrho(b)x^{1/a} + \frac{\zeta(\sigma)}{\zeta(\sigma+1)}R_\sigma(a)x^{1/b} + O\left(x^{1/a}\sum_{n>x^{1/k}}\frac{1}{n^b}\right)$$
$$+ O\left(x^{1/b}\sum_{n>x^{1/k}}\frac{1}{n^a}\right) + O(x^{1/c}\log x).$$

The first two O-terms are $O(x^{1/k})$, and therefore the theorem is a consequence of Lemma 4.3.

Corresponding to $j_{a,b}(n)$, we define $j^*_{a,b}(n)$ to be 1 or 0 according as n is or is not contained in $S^*_{a,b}$. We have the following analogue of Lemma 4.5.

LEMMA 4.6. *If $b \geqslant a \geqslant 1$, $(a, b) = 1$, then*

$$j^*_{a,b}(n) = \sum_{\substack{d^k\delta = n \\ (d,\delta)=1}} \mu^*(d) j_{a,b}(\delta). \tag{4.10}$$

Proof. As in the proof of Lemma 4.5, one may show that

$$\sum_{\substack{d^k\delta = n \\ (d,\delta)=1}} j^*_{a,b}(\delta) = j_{a,b}(n), \tag{4.11}$$

and (4.10) results by the inversion relation (2.6).

An estimate for $S^*_{a,b}(x)$ can now be easily deduced.

THEOREM 4.2. *If $b > a > 1$, $(a, b) = 1$, then for $x \geqslant 2$,*

$$S^*_{a,b}(x) = A^* x^{1/a} + B^* x^{1/b} + O(x^{1/c} \log x), \tag{4.12}$$

where $A^* = A\zeta(b)U(b)$, $B^* = B\zeta(a)U(a)$, and

$$U(s) = \prod_p \left(1 - \frac{2}{p^s}\right), \quad s > 1. \tag{4.13}$$

Proof. By Lemma 4.6, it follows that (cf. (4.6))

$$S^*_{a,b}(x) = \sum_{n \leqslant x} j^*_{a,b}(n) = \sum_{n \leqslant x^{1/k}} \mu^*(n) S_{a,b}\left(\frac{x}{n^k}\right), \tag{4.14}$$

and hence by Theorem 4.1 and the boundedness of $\mu^*(n)$ (cf. Remark 4.1),

$$S^*_{a,b}(x) = Ax^{1/a} \sum_{n \leqslant x^{1/k}} \frac{\mu^*(n)}{n^b} + Bx^{1/b} \sum_{n \leqslant x^{1/k}} \frac{\mu^*(n)}{n^a} + O\left(x^{1/c} \log x \sum_{n \leqslant (x/2)^{1/k}} \frac{1}{n^{k/c}}\right) + O(x^{1/k}).$$

By an argument similar to that of Lemma 4.3, it is seen that

$$\sum_{n=1}^{\infty} \frac{\mu^*(n)}{n^s} = \zeta(s) \prod_p \left(1 - \frac{2}{p^s}\right), \quad s > 1. \tag{4.15}$$

The proof now proceeds like that of Theorem 4.1.

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Zeta functions of quadratic forms

by

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Dedicated to the memory of Dr. R. Vaidyanathaswamy

§ 1. Introduction. The Riemann zeta function has been generalized in two directions; one generalization concerns the zeta functions of algebraic number fields and algebras and the other concerns the zeta functions of Lerch-Epstein associated with definite quadratic forms and of Siegel associated with indefinite quadratic forms. Our object in this paper is to study the zeta functions associated with quadratic forms over involutorial algebras. We deal here with commutative algebras only reserving the non-commutative case for the second part.

Let K be an algebraic number field and σ an automorphism of K whose square is the identity. Let k be the fixed field of σ. Then $(K: k) = 1$ or 2 according as σ is or is not the identity automorphism. For any matrix A of m rows and columns with elements in K let A^σ denote the matrix, (a_{kl}^σ) where $A = (a_{kl})$. We say that A is symmetric (hermitian) if σ is (or not) the identity and $A' = A^\sigma$. If α is a m-rowed vector with elements in K we call $\alpha' A \alpha^\sigma$ the quadratic (hermitian) form associated with A. Let first $\sigma = 1$ the identity automorphism. Let S be symmetric, m-rowed and non-singular over K. Let K have r_1 real and r_2 complex infinite prime spots and let S be definite at $r_1 - l$ of the real infinite prime spots of K, $0 \leqslant l \leqslant r_1$. For every $g \neq 0$ in K which can be represented by S we associate a vector $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_l)$, $\varepsilon_i = \pm 1$ where $\varepsilon_k = g^{(k)}/|g^{(k)}| = \operatorname{sgn} g^{(k)}$. We call ε the signature of g. With each ε we associate the zeta function

$$\zeta_\varepsilon(S, \mathfrak{a}, s) = N\mathfrak{a}^{2s} \sum_g \frac{M(S, \mathfrak{a}, g)}{(N|g|)^s}$$

where $\mathfrak{a} \neq 0$ is an ideal of K, $M(S, \mathfrak{a}, g)$ is the measure of representation of g by S (see § 4) and the summation runs over all g with signature ε which are representable by S such that for no two g_1, g_2 in the summation $g_1 = \varepsilon^2 \bar{\varepsilon} g_2$ holds, ε being a unit in K. There are clearly 2^l such Dirichlet series. It is shown (§ 3) that they converge for $\sigma > m/2$ and define in this half plane regular analytic functions of s. By generalizing suitably