ANNALES
POLONICI MATHEMATICI
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On the continuous dependence of solutions of some functional equations on given functions (I)

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In a series of our papers ([2], [3], [5], [6]) we have proved that under some assumptions concerning the given functions $f(x)$ and $G(x, y)$ the functional equation

$$\varphi[f(x)] = G\big(x, \varphi(x)\big) \tag{1}$$

possesses exactly one solution $\varphi(x)$ continuous in a one-sided neighbourhood of the point $x = b$ such that $f(b) = b$. The question arises in what manner this unique solution of equation (1) varies with the change of the given functions. The example of § 2 below shows that this dependence need not be continuous. In the sequel we shall prove, however, that under suitable assumptions the above-mentioned unique solution will vary in a continuous manner.

Equation $\varphi[f(x)] \pm \varphi(x) = F(x)$

§ 1. Now we shall consider the particular cases of equation (1)

$$\varphi[f(x)] + \varphi(x) = F(x) \tag{2}$$

and

$$\varphi[f(x)] - \varphi(x) = F(x)\,. \tag{3}$$

In the sequel we shall assume that

(i) The function $f(x)$ is defined, continuous and strictly increasing in an interval $\langle a, b\rangle$ and $f(x) > x$ for $x \in (a, b)$, $f(b) = b$.

(ii) The function $F(x)$ is defined and continuous in the interval $\langle a, b\rangle$ and $F(b) = 0$.

LEMMA I. *Under hypotheses* (i) *and* (ii):

1° *Equation* (2) *possesses at most one solution continuous in the interval* $\langle a, b\rangle$. *This solution, provided it exists, has the form* [1]

$$\varphi(x) = \sum_{\nu=0}^{\infty} (-1)^{\nu} F[f^{\nu}(x)]\,.$$

[1] Throughout the whole paper the symbol $f^{\nu}(x)$ denotes the ν-th iteration of the function $f(x)$.

2° *Equation* (3) *possesses at most one solution continuous in the interval* $(a, b\rangle$ *and fulfilling the condition* $\varphi(b) = 0$. *This solution, provided it exists, has the form*

$$\varphi(x) = -\sum_{\nu=0}^{\infty} F[f^{\nu}(x)] .$$

The proof of this lemma is to be found in [1], [5].

In the sequel by *the solution of equation* (2) *resp.* (3) we shall understand the solution fulfilling the conditions mentioned in the above lemma.

The next lemmas give some sufficient conditions of the existence of the solution of equation (2) resp. (3).

LEMMA II. *If hypotheses* (i) *and* (ii) *are fulfilled and the function* $F(x)$ *is monotonic in* (a, b), *then the solution of equation* (2) *necessarily exists.*

The proof of this lemma is to be found in [5].

LEMMA III. *If hypotheses* (i) *and* (ii) *are fulfilled and if there exists a bounded function* $G(x)$ *such that* $|F(x)| \leqslant G(x)$ *and*

$$\frac{G[f(x)]}{G(x)} < \vartheta < 1 \quad \text{for} \quad x \in (b-\eta, b) \tag{4}$$

where η *is a positive number, then the solution of equation* (2) *exists as well as the solution of equation* (3).

The proof of this lemma is to be found in [5] (this proof has been carried out there for equation (2), but it is also valid for equation (3)).

LEMMA IV. *If hypotheses* (i) *and* (ii) *are fulfilled and if, moreover, the functions* $f(x)$ *and* $F(x)$ *are of class* C^1 *in* $\langle a, b\rangle$ *and* $f'(b) \neq 1$, *then the solution of equation* (2) *exists as well as the solution of equation* (3).

Proof. Since $f(x) > x$, we have $f'(b) < 1$. Consequently there exist positive numbers η and ϑ, $\vartheta < 1$, such that $f'(x) < \vartheta$ in $(b-\eta, b\rangle$. From the mean-value theorem it follows that

$$|f(x)-f(b)| < \vartheta |x-b| \quad \text{for} \quad x \in (b-\eta, b) . \tag{5}$$

Since $F(x)$ is of class C^1, $|F(x)| \leqslant C|x-b|$ for $x \in \langle a, b\rangle$, where C is a constant. The function $G(x) \overset{\text{df}}{=} C|x-b|$ fulfils condition (4), because we have by (5)

$$\frac{G[f(x)]}{G(x)} = \frac{C|f(x)-b|}{C|x-b|} < \vartheta .$$

Consequently the existence of the solution of equation (2) resp. (3) follows immediately from lemma III.

§ **2.** Now let us consider the sequence of equations

$$\varphi[f_n(x)]+\varphi(x) = F_n(x) , \quad n = 1, 2, 3, \ldots \tag{6}$$

and

$$\varphi[f_n(x)]-\varphi(x) = F_n(x) , \quad n = 1, 2, 3, \ldots \tag{7}$$

We shall show by an example that the conditions

$$F_n(x) \underset{\langle a,b\rangle}{\Rightarrow} F(x) , \quad f_n(x) \underset{\langle a,b\rangle}{\Rightarrow} f(x) \tag{8}$$

are not sufficient to ensure the convergence of the sequence of the solutions $\varphi_n(x)$ of equations (7) to the solution $\varphi(x)$ of equation (3). An analogical example can be constructed for equations (6) and (2).

EXAMPLE. Let $f(x)$ be an arbitrary function fulfilling hypotheses (i). Let us take an arbitrary $x_0 \in (a, b)$ and let us write $x_\nu \overset{\text{df}}{=} f^{\nu}(x_0)$, $\nu = 1, 2, \ldots$ The sequence x_ν is (cf. [5]) increasing and converges to b. Further, let

$$F_n(x) \overset{\text{df}}{=} \begin{cases} \dfrac{1}{n} & \text{for} \quad x \in \langle a, x_{n-1}\rangle , \\ \dfrac{x_n - x}{n(x_n - x_{n-1})} & \text{for} \quad x \in (x_{n-1}, x_n) , \qquad n = 1, 2, \ldots \\ 0 & \text{for} \quad x \in \langle x_n, b\rangle , \end{cases}$$

$$f_n(x) \overset{\text{df}}{=} f(x) , \quad n = 1, 2, \ldots$$

and

$$F(x) \equiv 0 .$$

Relations (8) are evidently fulfilled. We have further

$$\varphi_n(x_0) = -\sum_{\nu=0}^{\infty} F_n[f^{\nu}(x_0)] = -\sum_{\nu=0}^{\infty} F_n(x_\nu) = -\sum_{\nu=0}^{n-1} \frac{1}{n} = -1 , \quad n = 1, 2, \ldots$$

and

$$\varphi(x_0) = 0 ,$$

where $\varphi_n(x)$ and $\varphi(x)$ are the solutions of equations (7) and (3) respectively. Consequently $\varphi_n(x_0)$ does not tend to $\varphi(x_0)$. By a slight modification of the definition of the functions $F_n(x)$ we can make them of class C^1. Similarly we can easily construct an example in which $|\varphi_n(x_0)| \to \infty$ while $\varphi(x_0) = 0$.

§ **3.** Now we shall prove the following

THEOREM I. *Let us assume that the functions* $f_n(x)$, $f(x)$ *and* $F_n(x)$, $F(x)$ *fulfil hypotheses* (i) *and* (ii) *and relations* (8). *Further, let the functions* $F_n(x)$ *and* $F(x)$ *be decreasing (increasing) in* (a, b). *Then*

$$\varphi_n(x) \underset{\langle c,b\rangle}{\Rightarrow} \varphi(x) \quad \text{for every} \quad c \in (a, b) \text{ (}^2\text{)} , \tag{9}$$

where $\varphi_n(x)$ *and* $\varphi(x)$ *are the solutions of equations* (6) *and* (2) *respectively.*

(²) In the case $f(a) \neq a$, c may also equal a.

Proof. Let us suppose that the functions $F_n(x)$ and $F(x)$ are decreasing (if they are increasing, the proof is quite analogical). The solutions $\varphi_n(x)$ and $\varphi(x)$ exist on account of lemma II. Further, by lemma I, we have

$$\varphi_n(x) = \sum_{\nu=0}^{\infty} (-1)^\nu F_n[f_n^\nu(x)] .$$

Since the function $F_n(x)$ is decreasing, we have the inequality [3]

$$(10) \quad \sum_{\nu=0}^{2p+1} (-1)^\nu F_n[f_n^\nu(x)] \leqslant \varphi_n(x) \leqslant \sum_{\nu=0}^{2p} (-1)^\nu F_n[f_n^\nu(x)] , \qquad p = 1, 2, \ldots$$

The finite sums occurring in inequality (10) converge as $n \to \infty$ uniformly in $\langle c, b\rangle$ to

$$(11) \qquad \sum_{\nu=0}^{2p+1} (-1)^\nu F[f^\nu(x)] \quad \text{and} \quad \sum_{\nu=1}^{2p} (-1)^\nu F[f^\nu(x)]$$

respectively. Both expressions (11) converge as $p \to \infty$ uniformly in $\langle c, b\rangle$ (cf. [5]) to

$$\varphi(x) = \sum_{\nu=0}^{\infty} (-1)^\nu F[f^\nu(x)] .$$

Consequently, for an arbitrary number $\varepsilon > 0$ we can find an index N such that for $n, p > N$ and $x \in \langle c, b\rangle$ both expressions in (10) differ from $\varphi(x)$ by less than $\varepsilon/2$. Hence it follows that for $n > N$ and $x \in \langle c, b\rangle$

$$|\varphi_n(x) - \varphi(x)| < \varepsilon ,$$

which was to be proved.

Theorem II. *Let us suppose that the functions $f(x)$ and $F_n(x)$, $F(x)$ fulfil hypotheses* (i) *and* (ii) *and*

$$f_n(x) = f(x) \quad \text{for} \quad x \in \langle a, b\rangle , \qquad n = 1, 2, \ldots$$
$$F_n(x) \underset{\langle a,b\rangle}{\Rightarrow} F(x) .$$

Moreover, for each $x \in (a, b)$, let the sequence $F_n(x)$ be decreasing (increasing). If the solutions $\varphi_n(x)$ and $\varphi(x)$ of equations (7) *resp.* (3) *exist, then relation* (9) *holds.*

Proof. Let us suppose that the sequence $F_n(x)$ is increasing for each $x \in (a, b)$ (if it is decreasing the proof is analogical) and that the

[3] From the monotonity of the functions $f_n(x)$ and $F_n(x)$ it follows that the sequence $F_n[f_n^\nu(x)]$ is — with fixed x and n — monotonic and consequently the series $\sum_{\nu=0}^{\infty} (-1)^\nu F_n[f_n^\nu(x)]$ is alternating.

solutions $\varphi_n(x)$ and $\varphi(x)$ exist. The hypotheses of the theorem imply that for each $x \in (a, b)$ the sequence $\varphi_n(x)$ is decreasing and

$$\varphi_n(x) \geqslant \varphi(x) , \qquad n = 1, 2, \ldots$$

Consequently the sequence $\varphi_n(x)$ converges. Passing to the limit as $n \to \infty$ in the relation

$$\varphi_n(x) - \varphi_n[f^p(x)] = -\sum_{\nu=0}^{p-1} F_n[f^\nu(x)]$$

we obtain

$$\lim_{n\to\infty} \varphi_n(x) - \lim_{n\to\infty} \varphi_n[f^p(x)] = -\sum_{\nu=0}^{p-1} F[f^\nu(x)] .$$

Now passing to the limit as $p \to \infty$ in the above relation we obtain on account of the inequality

$$\varphi_1[f^p(x)] \geqslant \varphi_n[f^p(x)] \geqslant \varphi[f^p(x)] , \qquad n = 1, 2, \ldots$$

the relation

$$\lim_{n\to\infty} \varphi_n(x) = \varphi(x) .$$

Since the sequence $\varphi_n(x)$ is decreasing and the functions $\varphi_n(x)$ and $\varphi(x)$ are continuous, the convergence is uniform in $\langle c, b\rangle$ for every $c \in (a, b)$. This completes the proof.

§ 4. The theorems of § 3 concerned only equations (6) and (2) or (7) and (3). Now we shall prove theorems valid for equations (6) and (2) as well as (7) and (3).

Theorem III. *Let us assume that the functions $f_n(x)$, $f(x)$ and $F_n(x)$, $F(x)$ fulfil hypotheses* (i) *and* (ii) *and relations* (8). *Let us assume, moreover, that there exists a bounded function $G(x)$ such that*

$$(12) \quad |F(x)| \leqslant G(x) , \quad |F_n(x)| \leqslant G(x) \quad \text{for} \quad x \in \langle a, b\rangle , \quad n = 1, 2, \ldots$$

and

$$(13) \quad \frac{G[f(x)]}{G(x)} < \vartheta , \quad \frac{G[f_n(x)]}{G(x)} < \vartheta , \quad \vartheta < 1 ,$$
$$\text{for} \quad x \in (b-\eta, b) , \quad n = 1, 2, \ldots \text{ [4]}$$

where η is a positive number. Then relation (9) *holds, where $\varphi_n(x)$ is the solution of equation* (6) *resp.* (7) *and $\varphi(x)$ is the solution of equation* (2) *resp.* (3).

[4] In [4] we have proved that the function $G(x)$ can always be chosen decreasing. Consequently if the sequence $f_n(x)$ is monotonic, then it is sufficient to assume the relation

$$\frac{G[f_1(x)]}{G(x)} < \vartheta \quad \text{or} \quad \frac{G[f(x)]}{G(x)} < \vartheta$$

according to whether the sequence $f_n(x)$ is decreasing or increasing.

Proof. The solutions $\varphi_n(x)$ and $\varphi(x)$ exist on account of lemma III. Evidently $\lim_{x\to b} G(x) = 0$. Consequently for an arbitrary $\varepsilon > 0$ we can choose $\eta_1 > 0$, $\eta_1 < \eta$, such that

$$G(x) < \varepsilon(1-\vartheta)/4 \quad \text{for} \quad x \in (b-\eta_1, b). \tag{14}$$

We can also find for an arbitrary $c \in (a, b)$ an index N_1 such that

$$f^\nu(x) \in (b-\eta_1, b) \quad \text{for} \quad x \in \langle c, b), \quad \nu \geqslant N_1. \tag{15}$$

We can find at last an index $N_2 > N_1$ such that

$$f_n^\nu(x) \in (b-\eta_1, b) \quad \text{for} \quad x \in \langle c, b), \quad n, \nu \geqslant N_2. \tag{16}$$

We have

$$|\varphi_n(x)-\varphi(x)| \leqslant \sum_{\nu=0}^{\infty} |F_n[f_n^\nu(x)]-F[f^\nu(x)]|,$$

whence

$$|\varphi_n(x)-\varphi(x)| \leqslant \sum_{\nu=0}^{N_2-1} |F_n[f_n^\nu(x)]-F[f^\nu(x)]| + \sum_{\nu=N_2}^{\infty} |F_n[f_n^\nu(x)]| + \sum_{\nu=N_2}^{\infty} |F[f^\nu(x)]|. \tag{17}$$

We have further by (13), (15) and (16) for $x \in \langle c, b)$ and $n, \nu \geqslant N_2$

$$|F_n[f_n^\nu(x)]| \leqslant \vartheta^{\nu-N_2} G[f_n^{N_2}(x)]$$

and

$$|F[f^\nu(x)]| \leqslant \vartheta^{\nu-N_2} G[f^{N_2}(x)],$$

whence, by (14), (15), (16) and (17)

$$|\varphi_n(x)-\varphi(x)| \leqslant \sum_{\nu=0}^{N_2-1} |F_n[f_n^\nu(x)]-F[f^\nu(x)]| + \frac{\varepsilon}{2}. \tag{18}$$

On account of relations (8) we can find $N > N_2$ such that

$$|F_n[f_n^\nu(x)]-F[f^\nu(x)]| < \varepsilon/2N_2 \quad \text{for} \quad x \in \langle c, b\rangle, \quad n > N, \quad \nu = 0, \ldots, N_2-1.$$

Hence we have by (18)

$$|\varphi_n(x)-\varphi(x)| < \varepsilon \quad \text{for} \quad x \in \langle c, b\rangle, \quad n > N,$$

which was to be proved.

THEOREM IV. *Let us assume that the functions $f_n(x)$, $f(x)$ and $F_n(x)$, $F(x)$ fulfil hypotheses* (i) *and* (ii) *and relations* (8). *Let us assume, moreover, that these functions are of class C^1 in $\langle a, b\rangle$, $f'(b) \neq 1$ and*

$$f_n'(x) \underset{\langle a,b\rangle}{\Rightarrow} f'(x), \quad F_n'(x) \underset{\langle a,b\rangle}{\Rightarrow} F'(x). \tag{19}$$

Then relation (9) *holds, where $\varphi_n(x)$ is the solution of equation* (6) *resp.* (7) *and $\varphi(x)$ is the solution of equation* (2) *resp.* (3).

Proof. The solutions $\varphi_n(x)$ and $\varphi(x)$ exist on account of lemma IV. According to (19) there exists a constant C such that

$$|F_n'(x)| \leqslant C \quad \text{and} \quad |F'(x)| \leqslant C \quad \text{for} \quad x \in \langle a, b\rangle, \quad n = 1, 2, \ldots$$

Similarly there exist numbers η and ϑ, $\vartheta < 1$, such that

$$|f_n(x)-b| < \vartheta|x-b| \quad \text{and} \quad |f(x)-b| < \vartheta|x-b|$$
$$\text{for} \quad x \in (b-\eta, b) \quad \text{and } n \text{ sufficiently large}.$$

Hence it follows, as in the proof of lemma IV, that the function $G(x) \overset{\text{df}}{=} C|x-b|$ fulfils relations (12) and (13). Thus, on account of theorem III, relation (9) holds, which was to be proved.

Remark. It is apparent that if in the example of § 2 the functions $F_n(x)$ and $f(x)$ are of class C^1, $f'(b) \neq 1$, then all the hypotheses of the above theorem will be fulfilled with the only exception that the convergence $F_n'(x) \to F'(x)$ will not be uniform!

§ 5. Up to now in our considerations we assumed that

$$F(b) = 0 \quad \text{and} \quad F_n(b) = 0, \quad n = 1, 2, \ldots \tag{20}$$

For equations (3) and (7) this assumption is necessary. But one may ask whether for equations (2) and (6) the theorems proved above remain valid when conditions (20) are not fulfilled.

Let $\varphi(x)$ and $\varphi_n(x)$ be the solutions of equations (2) and (6) respectively. Then the functions $\psi(x) \overset{\text{df}}{=} \varphi(x) - \frac{1}{2}F(b)$ and $\psi_n(x) \overset{\text{df}}{=} \varphi_n(x) - \frac{1}{2}F_n(b)$ are the solutions of the equations

$$\psi[f(x)] + \psi(x) = F(x) - F(b)$$

and

$$\psi[f_n(x)] + \psi(x) = F_n(x) - F_n(b),$$

respectively. Since under the condition $F_n(b) \to F(b)$ the convergence $\varphi_n(x) \Rightarrow \varphi(x)$ is equivalent to the convergence $\psi_n(x) \Rightarrow \psi(x)$, we see that the case of arbitrary $F_n(b), F(b)$ may always be reduced to the case in which relations (20) are fulfilled.

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Sur les périodes des solutions de l'équation différentielle $x''+g(x) = 0$

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1. Considérons l'équation différentielle non linéaire du second ordre

$$x''+g(x) = 0 \tag{1}$$

où — comme d'habitude — $x'' = d^2x/dt^2$. Supposons que la fonction $g(x)$ soit définie et continue dans tout l'intervalle $(-\infty, +\infty)$ et telle que l'on ait

$$xg(x) > 0, \tag{2}$$

quel que soit $x \neq 0$. Il en résulte, en particulier, que $g(0) = 0$. Désignons par $G(x)$ la fonction primitive de $g(x)$ qui s'annule pour $x = 0$. On a donc

$$G(x) = \int_0^x g(s)\,ds ,$$

et, en vertu de l'hypothèse (2), $G(x) > 0$ pour tout $x \neq 0$. Supposons de plus que l'on ait

$$\lim_{|x|\to\infty} G(x) = +\infty . \tag{3}$$

Dans ces hypothèses l'équation (1), équivalente au système de deux équations différentielles du premier ordre

$$x' = y , \quad y' = -g(x) \tag{4}$$

n'a qu'un seul point singulier, à savoir l'origine des coordonnées. Les solutions du système (4) sont les lignes de niveau de la fonction $y^2+2G(x)$, dont chacune est déterminée par l'équation

$$\tfrac{1}{2}y^2+G(x) = C \tag{5}$$

où C est une constante non négative. Si $C > 0$, toutes ces courbes sont fermées (en vertu de (3)), symétriques par rapport à l'axe des x et contiennent l'origine des coordonnées dans leurs intérieurs. Pour tout $s > 0$, la courbe (5) qui correspond à $C = G(s)$ détermine une solution $x(t; s)$ de l'équation (1), telle que

$$x(0; s) = s , \quad x'(0; s) = 0 . \tag{6}$$