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Investigation of some measures and sequences related to the extreme points

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I. Introduction. Let X be a Hausdorff space. We shall denote the points of this space by $a, x, y, \ldots$ Let Φ be a function defined on $X\times X$ and satisfying the conditions

1. $\Phi(x, y) = \Phi(y, x)$.
2. $\Phi(x, x) = +\infty$.
3. $\Phi(x, y)$ is continuous on $X\times X$.

We shall name this function Φ a *kernel*. The function $\omega(x, y) = \exp\big(-\Phi(x, y)\big)$ (where by $\exp(-\infty)$ we mean 0) will be named the *generating function*.

We fix in X a compact set E. We choose on E $n+1$ points $x_0, \ldots, x_n$ and we seek

$$\inf_{\{x_i\}\subset E} \sum_{\substack{i\neq j\\ 0\leqslant i,j\leqslant n}} \Phi(x_i, x_j)$$

or, which is the same,

$$\sup_{\{x_i\}\subset E} \prod_{\substack{i\neq j\\ 0\leqslant i<j\leqslant n}} \omega(x_i, x_j).$$

By the above conditions on Φ and E there exists at least one system of points $\{\eta_0^n, \ldots, \eta_n^n\}$ such that

$$\min_{\{x_i\}\subset E} \sum_{i\neq j} \Phi(x_i, x_j) = \sum_{i\neq j} \Phi(\eta_i^n, \eta_j^n).$$

This system $\{\eta_0^n, \ldots, \eta_n^n\}$ will be named the n-th *extreme system of E with respect to the kernel Φ*. The object of this paper is the investigation of some measures and sequences obtained by the extreme points. In the classical extreme points theory, formed by M. Fekete and developed by G. Polyà, G. Szegö and F. Leja, the extreme points have been used for the construction of some polynomials, X being the complex plane. Some sequences related to those polynomials converge to some functions which give the solution of the Dirichlet problem or are useful in the theory of double power series.

In the present paper we extend these investigations to the more general Hausdorff spaces. We base ourselves here on a different method of work, connected with the measure theory, which has been introduced in the potential theory by J. Radon and O. Frostman. Especially the last part contains the solution of the problem of convergence of a number sequence which is used in the classical extreme points theory and has not been solved by any other method.

II. Radon measures. Let X be the space described above. We denote by $\mathcal{C}$ the set of real functions which are continuous and have compact carriers.

We define a *measure* μ as a functional on $\mathcal{C}$ which satisfies the following conditions:

1° linearity condition:

$$\mu[af+bg] = a\mu[f]+b\mu[g]$$

for every pair of real numbers a and b and $f, g \in \mathcal{C}$;

2° non-negativity condition:

$$f \geqslant 0 \quad \text{implies} \quad \mu[f] \geqslant 0 .$$

A general, not necessarily Radon measure is defined as a real aggregate of two positive Radon measures.

In the following we shall deal chiefly with the non-negative measures. Therefore by the term measure we shall mean a non-negative one if the contrary is not required. Of course in place of μ we shall also use other symbols.

Let us fix the functional μ. We shall extend it to the class of lower semicontinuous functions. We shall denote this class by $\mathcal{D}$. We define for $f \in \mathcal{D}$

$$\mu[f] = \sup_g \mu[g] \quad \text{where} \quad g \in \mathcal{C}, \quad g \leqslant f .$$

Further, we denote the class of the upper semicontinuous functions by $\mathcal{G}$. For $f \in \mathcal{G}$ we put

$$\mu[f] = \inf_g \mu[g] \quad \text{where} \quad g \geqslant f \quad \text{and} \quad g \in \mathcal{D} .$$

For an arbitrary function f defined on X we put

$$f^+(x) = \max(0, f(x)) , \quad f^-(x) = f^+(x)-f(x) .$$

Of course $f^+(x) \geqslant 0, f^-(x) \leqslant 0$ and $f(x) = f^+(x)-f^-(x)$. For an arbitrary function $h \geqslant 0$ we define

$$\underline{\mu}[h] = \sup_f \mu[f] \quad \text{where} \quad f \leqslant h , \quad f \in \mathcal{G} ,$$

$$\overline{\mu}[h] = \inf_f \mu[f] \quad \text{where} \quad f \geqslant h , \quad f \in \mathcal{D} .$$

If $\underline{\mu}[h] = \overline{\mu}[h]$ we say that h is *integrable* and we write

$$\mu[h] = \underline{\mu}[h] = \overline{\mu}[h] .$$

In the general case, where h is not necessarily non-negative, we decompose it into h^+ and h^- in the manner indicated above and we put $\mu[h] = \mu[h^+]-\mu[h^-]$, assuming of course that the right-hand members have their sense.

The value of a functional μ for a given function f will be named the integral. We shall use for it also the common notation $\int f(x)\,d\mu(x)$ or $\int f\,d\mu$.

We shall denote by ι_A the characteristic function of an arbitrary set A. If ι_A is integrable, we say that A is a measurable set and we write

$$\mu(A) = \int \iota_A d\mu = \mu[\iota_A] .$$

The number $\mu(A)$ we name the *μ-measure of the set A*.

Since the characteristic function of an open (resp. closed) set is lower (resp. upper) semicontinuous, the μ-measure of those sets is well defined. μ-measure considered as a function of a set is monotonic and completely additive. The proofs and details are to be found in [2], for instance.

A sequence of measures $\{\mu_n\}$ will be named *weakly convergent* or simply — *convergent* to a measure μ if for every function $f \in \mathcal{C}$ we have $\lim \mu_n[f] = \mu[f]$.

We shall now seek a norm of the functional μ, i. e.

$$\sup_f |\mu[f]|/\|f\|$$

where $\|f\|$ denotes the norm of f in $\mathcal{C}$: $\|f\| = \sup|f|$. The complementary of the sum of all open sets of μ-measure 0 will be named the *carrier* of μ. The μ-measure of this carrier will be named the *total mass of* μ.

LEMMA 1. *If μ is a measure of the compact carrier K, then*

3° *μ is monotonic, i.e. the inequality $f \geqslant g$ implies*

$$\mu[f] \geqslant \mu[g];$$

4° *the norm of the functional μ is equal to $\mu(K)$.*

Proof. The first proposition, 3°, follows directly from the linearity and non-negativity of μ. 3° yields directly

$$\left|\int f\,d\mu\right| \leqslant \int |f|\,d\mu \leqslant \int \|f\|\,d\mu = \|f\|\mu(K)$$

and hence

$$\|\mu\| \leqslant \mu(K). \tag{1}$$

By Uryson's lemma there exists in $\mathcal{C}$ a function f_0 which is equal to 1 on K and satisfies the condition $0 \leqslant f_0(x) \leqslant 1$. Of course

$$\int f_0 d\mu = \int 1\, d\mu = \|f_0\| \mu(K)$$

and in view of (1) we conclude that $\|\mu\| = \mu(K)$.

THEOREM 1. (The principle of choice). *If we have a sequence of measures $\{\mu_n\}$ their carries being contained in a common compact K and their norms being bounded by a number 1, then there exists a subsequence $\{\mu_{a_n}\}$ which converges weakly to some measure μ_0. The carrier of μ_0 is also contained in K and $\|\mu_0\| \leqslant 1$.*

Proof. This theorem follows directly from the general theorem which states that if S^* is an adjoint space over a linear space S then the unit sphere in S^* is a compact set. In our case S is $\mathcal{C}$ and $\{\mu \mid \|\mu\| \leqslant 1\}$ is a subset of a unit sphere in $\mathcal{C}^*$.

We shall prove some theorems on the convergence of the integrals. We consider a sequence of measures $\{\mu_n\}$ which converges to some measure μ. We assume that the carriers of all the measures μ_n are contained in a common compact K and $\mu_n(K) \leqslant 1$. Denote by $\mathcal{D}_K$ the class of the lower semicontinuous functions on K. Taking into consideration $f(x) - \min f(x)$ instead of $f(x)$ we may restrict ourselves to the investigation of the non-negative functions only.

THEOREM 2. *If $f \in \mathcal{D}_K$, then $\mu_n \to \mu$ implies*

$$\lim \int f d\mu_n \geqslant \int f d\mu .$$

Proof. By definition we have

$$\int f d\mu_n = \sup \int g\, d\mu, \qquad g \leqslant f, \qquad g \in \mathcal{C}.$$

Suppose that there exists in $\mathcal{D}_K$ a function f_0 such that for some subsequence $\{k_n\} \subset \{n\}$ we have

$$\lim \int f_0 d\mu_{k_n} < \int f_0 d\mu .$$

We take $g \in \mathcal{C}$, $g < f$ such that

$$\lim \int f_0 d\mu_{k_n} < \int g_0 d\mu \leqslant \int f_0 d\mu .$$

Since we have

$$\lim \int g_0 d\mu_n = \int g_0 d\mu ,$$

then for sufficiently large indices k_n we have the inequality

$$\int f_0 d\mu_{k_n} < \int g\, d\mu_{k_n},$$

which contradicts to the inequality $g \leqslant f$.

THEOREM 3. *If G is an open set, then $\mu_n \to \mu$ implies $\underline{\lim}\, \mu_n(G) \geqslant \mu(G)$. If H is a closed set, then $\overline{\lim}\, \mu_n(H) \leqslant \mu(H)$.*

Proof. The theorem follows by the fact that the characteristic function of an open (resp. closed) set is lower (resp. upper) semicontinuous. To these characteristic functions we apply theorem 2.

Let A be an arbitrary set in the space X. If ι_A is integrable, then we may define the *restriction μ^A of the measure μ to the set A* putting for integrable f

$$\int_A f d\mu = \mu^A[f] \overset{\text{df}}{=} \int \iota_A f d\mu .$$

The following theorem gives us a sufficient condition for the existence of the sequence $\mu_n(A)$ and the equality $\lim \mu_n(A) = \mu(A)$. This condition had been used by O. Frostmann in his definition of the convergence of measures. Denote the interior of A by A° and its closure by $\bar{A}$.

THEOREM 4. *If $\mu_n \to \mu$ and $\mu(\bar{A} - A^\circ) = 0$ then $\lim \mu_n(A)$ exists and equals to $\mu(A)$.*

Proof. Applying theorem 3 we have

$$\mu(A^\circ) \leqslant \underline{\lim}\, \mu_n(A^\circ) \leqslant \underline{\lim}\, \mu_n(A) \leqslant \overline{\lim}\, \mu_n(A) \leqslant \overline{\lim}\, \mu_n(\bar{A}) \leqslant \mu(\bar{A}). \tag{2}$$

Because

$$\mu(\bar{A}) - \mu(A^\circ) = \mu(\bar{A} - A^\circ) = 0$$

we conclude that all the members in (2) are equal.

THEOREM 5. *If $\{f_n\}$ is a sequence of functions of $\mathcal{C}$ which converges uniformly to the function f, then $\mu_n \to \mu$ implies*

$$\lim \int f_n d\mu_n = \int f d\mu .$$

Proof.

$$\begin{aligned}\left| \int f_n d\mu_n - \int f d\mu \right| &= \left| \int (f_n - f) d\mu_n + \int f d\mu_n - \int f d\mu \right| \\ &\leqslant \int |f_n - f|\, d\mu + \left| \int f d\mu_n - \int f d\mu \right| .\end{aligned}$$

The last member of this inequality may be made arbitrarily small if we take a sufficiently large n. Hence follows our proposition.

Let $\{f_n\}$ be a sequence of functions from $\mathcal{D}_K$. We assume that each f is continuous outside the set $\{x \mid f_n(x) = \infty\}$.

THEOREM 6. *If the sequence f_n is convergent to a function f uniformly on the set $K-\{x|f(x)=\infty\}$, then $\mu_n\to\mu$ implies*

(3) $$\underline{\lim}\int f_n d\mu_n \geqslant \int f d\mu .$$

Proof. Let us put

$$f_n^M(x) = \min(M, f(x)) .$$

We take a number $\varepsilon>0$ and M so large that

$$\int f^M d\mu > \int f d\mu - \varepsilon .$$

We have

$$\underline{\lim}\int f_n d\mu_n \geqslant \lim \int f_n^M d\mu_n = \int f^M d\mu \geqslant \int f d\mu - \varepsilon .$$

Since ε is arbitrarily small, our proposition is valid. In a similar way we prove the theorem in the case where $\int f d\mu=\infty$.

In the following we shall make use of this theorem in the case $f_n(x) = \Phi(y_n, x)$.

Under the conditions of theorem 6 we shall prove

THEOREM 7. *A necessary and sufficient condition for the equality*

$$\lim \int f_n d\mu_n = \int f d\mu$$

is that for every $\varepsilon>0$ there exist an open set Z_ε such that f is finite on the set $K-Z_\varepsilon$ and that

$$\overline{\lim} \int_{Z_\varepsilon} f_n d\mu_n < \varepsilon .$$

Proof. Sufficiency. On $K-Z_\varepsilon$ almost all the f_n are continuous and the convergence is uniform there. Then by theorem 6 we have

$$\overline{\lim}\int f_n d\mu_n \leqslant \overline{\lim} \int_{K-Z_\varepsilon} f_n d\mu_n + \varepsilon \leqslant \int_{K-Z_\varepsilon} f d\mu + \varepsilon$$

and hence by $\varepsilon\to 0$ we obtain

$$\overline{\lim}\int f_n d\mu_n \leqslant \int f d\mu ,$$

which, compared with the opposite inequality (3), gives our proposition.

Necessity. If the condition of our theorem is not satisfied, then for some $\varepsilon>0$ and suitably chosen Z_ε we have

$$\overline{\lim}\int f_n d\mu_n - \int f d\mu$$
$$\geqslant \underline{\lim}\Big[\int_{K-Z_\varepsilon} f_n d\mu_n - \int_{K-Z_\varepsilon} f_n d\mu\Big] + \overline{\lim}\Big[\int_{Z_\varepsilon} f_n d\mu_n - \int_{Z_\varepsilon} f d\mu\Big] > 0 .$$

III. On some measures obtained by the extreme points.

Now we turn to the notations used in the introduction. E is a fixed compact in the space X, Φ is a kernel which satisfies the conditions 1, 2, 3, $\{\eta_0^n, \ldots, \eta_n^n\}$ denotes the n-th extreme system with respect to the kernel Φ. It has been proved in [1] that the sequence

$$\Big\{\exp\Big(2^{-1}n^{-1}(n+1)^{-1}\sum_{0\leqslant i<j\leqslant n}\Phi(\eta_i^n, \eta_j^n)\Big)\Big\} = \Big\{\Big(\prod_{0\leqslant i<j\leqslant n}\omega(\eta_i^n, \eta_j^n)\Big)^{2/n(n+1)}\Big\}$$

is a decreasing one. It converges to a nonnegative limit which is named the *span of E with respect to the generating function ω* and it is denoted usually by $v(E, \omega)$ [1].

In the following we shall assume that $v(E,\omega)>0$.

Transferring the above propositions into the formulae with Φ we see that the sequence

$$\Big\{(n+1)^{-2}\sum_{i\neq j}\Phi(\eta_i^n, \eta_j^n)\Big\}$$

converges to $\log\big(v(E,\omega)\big)^{-1}$. We denote this number by γ.

Let μ be a measure which is defined for the continuous functions by the formula

$$\mu_n[f] = \int f d\mu_n = \sum_{i=0}^{n} f(\eta_i^n) n^{-1}$$

we put

$$\Phi_0(x, y) = \begin{cases} 0 & \text{for} \quad x=y , \\ \Phi(x,y) & \text{for} \quad x\neq y . \end{cases}$$

The sequence $\{\mu_n\}$ contains a convergent subsequence $\{\mu_{a_n}\}$ and we denote its limit by μ_a. Our nearest purpose will be to prove the existence of

$$\lim \iint \Phi_0(x, y) d\mu_{a_n}(y) d\mu_{a_n}(x)$$

and to prove that μ_a realizes

(4) $$\inf_\mu \Big\{\iint \Phi(x, y) d\mu(y) d\mu(x)\Big\}$$

when μ varies over the set of positive Radon measures which have their carriers contained in E and their total masses are equal to 1. The class of these measures we denote by M.

LEMMA 2. *If $\mu\in M$, then for every extreme system we have*

$$n^{-1}(n+1)^{-1}\sum_{i\neq j}\Phi(\eta_i^n, \eta_j^n) \leqslant \iint \Phi(x,y) d\mu(y) d\mu(x) .$$

Proof. Let $x_0, \ldots, x_n$ be an arbitrary system of points of the set E. By the definition of the extreme system we have

$$\sum_{i<j} \Phi(\eta_i^n, \eta_j^n) \leqslant \sum_{i<j} \Phi(x_i, x_j).$$

Let μ be a measure of M. We apply to the both members the operator $\int \ldots \int d\mu(x_1) \ldots d\mu(x_n)$. This yields

$$\begin{aligned}\sum_{i<j} \Phi(\eta_i^n, \eta_j^n) &= \int \ldots \int \sum_{i<j} \Phi(\eta_i^n, \eta_j^n) d\mu(x_0) \ldots d\mu(x_n) \\ &\leqslant \int \ldots \int \sum_{i<j} \Phi(x_i, x_j) d\mu(x_0) \ldots d\mu(x_n) \\ &= \sum_{i<j} \iint \Phi(x_i, x_j) d\mu(x_i) d\mu(x_j) \\ &= \tfrac{1}{2}(n+1)n \iint \Phi(x, y) d\mu(y) d\mu(x),\end{aligned}$$

which gives directly our lemma.

COROLLARY 1. *The constant γ is not larger than the lower bound* (4).

LEMMA 3. *The measure μ_a of every one-point set is equal to* 0.

Proof. By the monotony of measure treated as a function of the set it suffices to prove that every point has a neighbourhood of the arbitrarily small μ_a measure. Let a be a point of E and let U be its open neighbourhood. We put

$$\lim_{E\times E} \Phi(x, y) = \delta.$$

We have

$$\begin{aligned}\gamma - \delta &= \lim \iint \big(\Phi_0(x, y) - \delta\big) d\mu_{a_n}(y) d\mu_{a_n}(x) \\ &\geqslant \lim \int_U \int_U (\Phi_0 - \delta) d\mu_{a_n} d\mu_{a_n} \\ &\geqslant \lim \big(\mu_{a_n}(U)\big)^2 \big(\inf_{U\times U} \Phi(x, y) - \delta\big) \\ &\geqslant \big(\mu_a(U)\big)^2 \big(\inf_{U\times U} \Phi(x, y) - \delta\big),\end{aligned}$$

which yields

$$\mu_a(U) \leqslant \Big((\gamma - \delta)\big(\inf_{U\times U} \Phi(x, y) - \delta\big)^{-1}\Big)^{1/2}. \tag{5}$$

In view of properties 2 and 3 of the kernel Φ the quantity $\inf \Phi(x, y)$ is as large as we like if we choose the neighbourhood U suitably. Then by (5) follows our lemma.

COROLLARY 2. *Since, for every $x \in E$, Φ_0 differs from Φ outside a set of the points y with the μ_a-measure* 0, *we have*

$$\int \Phi_0(x, y) d\mu_a(y) = \int \Phi(x, y) d\mu_a(y),$$
$$\iint \Phi_0(x, y) d\mu_a(y) d\mu_a(x) = \iint \Phi(x, y) d\mu(y) d\mu(x).$$

THEOREM 8. *The measure μ_a realises the lower bound* (4) *and this lower bound is equal to $\gamma = \log\big(v(E, \omega)\big)^{-1}$.*

Proof. By theorem 2 and corollaries 1 and 2 we have

$$\iint \Phi d\mu_a d\mu_a = \underline{\lim} \iint \Phi_0 d\mu_a d\mu_a = \gamma \leqslant \inf_{\mu\in M} \iint \Phi d\mu d\mu \leqslant \iint \Phi d\mu_a d\mu_a.$$

This immediately yields our theorem.

The uniqueness of the measure realising (4) obtained by the above method as a limit distribution of the extreme points has not been proved. In order to study this problem see e.g. [4].

We denote the carrier of the measure μ_a by E_a. Evidently $E_a \subset E$.

We take into consideration an arbitrary compact F in X. If there exists no measure ν of the positive total mass and no carrier contained in F such that

$$\int_F \int_F \Phi d\nu d\nu < \infty$$

then we shall name F a *polar set*. A set of type F_σ will be named polar if every compact contained in it is polar.

THEOREM 9. *The μ_a-measure of every polar set is equal to* 0.

Proof. Suppose that there exists a polar set H such that $\mu_a(H) > 0$. Putting $\inf_{E\times E} \Phi(x, y) = \delta$ we have

$$\infty > \iint (\Phi - \delta) d\mu_a d\mu_a \geqslant \int_H \int_H \Phi d\mu_a d\mu_a - \delta\big(\mu_a(H)\big)^2,$$

which contradicts the definition of the polar set.

THEOREM 10. *The function $u(x) = \int \Phi(x, y) d\mu_a(y)$ is lower semi-continuous on X.*

Proof. We take a point $x \in X$ and any sequence of points $x_n \to x$. By theorem 6 we have

$$\underline{\lim} \int \Phi(x_n, y) d\mu_a(y) \geqslant \int \Phi(x, y) d\mu_a(y).$$

THEOREM 11. *On E_a (the carrier of μ_a) we have $u(x) \leqslant \gamma$ and strong inequality holds at most on a polar set.*

Proof. Suppose that there exists a non-polar set $H \subset E$ on which we have $u(x) < \gamma$. By the semicontinuity of u the sets $\{x|u(x) \leqslant \gamma - \varepsilon\}$ are F_σ sets for an arbitrary positive ε. Of course

$$H = \bigcup_{\varepsilon > 0} \{x|u(x) \leqslant \gamma - \varepsilon\}.$$

Then there exists a non-polar compact K and $\bar{\varepsilon} > 0$ such that on K we have the inequality

$$\int \Phi(x, y)\, d\mu_a(y) \leqslant \gamma - 2\bar{\varepsilon}.$$

Let ν be a positive measure of the carrier contained in K and of the total mass 1 such that $\iint \Phi\, d\nu d\nu < \infty$. By the equality $\iint \Phi\, d\mu_a d\mu_a = \gamma$ it follows that there exists on E_a such a point a that $u(a) > \gamma - \bar{\varepsilon}$ and by the semicontinuity of u this inequality remains valid in some neighbourhood U of a. We write $\mu_a(U) = m$ and we define a new measure σ putting for $f \in \mathcal{C}$

$$\sigma[f] = m\nu[f] - \mu_a[\iota_U f].$$

Of course σ is a signed measure (it is not non-negative!) and its total mass is equal to 0. Let t be a positive small parameter. Then $\mu_a + t\sigma$ is a positive measure and has the total mass 1. Consider the difference

$$\Delta(t) = \iint \Phi\, d\mu_a d\mu_a - \iint \Phi\, d(\mu_a + t\sigma) d(\mu_a + t\sigma),$$

which is non-positive by theorem 8. By a simple computation we find

$$\begin{aligned}\Delta(t) &= -2t \int u(x)\, d\sigma(x) - t^2 \iint \Phi\, d\sigma d\sigma \\ &\geqslant t\Big(2m(\gamma - \bar{\varepsilon}) - 2m(\gamma - 2\bar{\varepsilon}) - t\iint \Phi\, d\sigma d\sigma\Big) \\ &= t\Big(2m\bar{\varepsilon} - t\iint \Phi\, d\sigma d\sigma\Big).\end{aligned}$$

It t is sufficiently near 0, then the last term of this inequality is positive, which contradicts theorem 8. Then the strong inequality $u(x) < \gamma$ holds at most on a polar set. Suppose now that there exists a point $b \in E_a$ such that $u(b) > \gamma + \varepsilon > \gamma$. By the semicontinuity of u this last inequality remains valid in some neighbourhood V of the point b and $\mu_a(V) > 0$. We compute

$$\begin{aligned}\gamma = \int u(x)\, d\mu_a(x) &= \int_{E-V} u\, d\mu_a + \int_V u\, d\mu_a \\ &\geqslant \mu(E-V) + (\gamma + \varepsilon)\mu(V) = \gamma + \varepsilon\mu(V),\end{aligned}$$

which is absurd. Thus our theorem is proved.

Let C be a compact set such that $u(x)$ is continuous on C.

THEOREM 12. *For an arbitrary number $\varepsilon > 0$ there exists an index N_ε such that for $n > N_\varepsilon$ and $x \in C$ we have* $\int \Phi(x, y)\, d\mu_{a_n}(y) > u(x) - \varepsilon$.

Proof. Suppose the contrary. Then there exist a sequence $\{x_n\} \in C$ and a sequence of indices k_n such that we have

$$\int \Phi(x_{k_n}, y)\, d\mu_{k_n}(y) \leqslant u(x_{k_n}) - \varepsilon\,. \tag{6}$$

Assuming that x_n converges to some point x_0 we obtain by theorem 6

$$u(x_0) \leqslant \varliminf \int \Phi(x_{k_n}, y)\, d\mu_{k_n}(y),$$

which compared with (6) is absurd.

THEOREM 13. *The sequence $\{\int u(x)\, d\mu_{a_n}(x)\}$ converges to γ.*

Proof. By theorem 2 we have

$$\varliminf \int u\, d\mu_{a_n} \geqslant \int u\, d\mu_a = \gamma\,. \tag{7}$$

Suppose now that we have $\varlimsup \int u\, d\mu_{a_n}$. In order to avoid the new notation we assume that $\lim \int u\, d\mu_{a_n} > \gamma$. We write

$$E_a^{\hat{}} = \{x|u(x) = \gamma\} \cap E.$$

It follows easily by theorem 10 that $E_a^{\hat{}}$ is an F_σ set and

$$\int_{E_a^{\hat{}}} u\, d\mu_a = \gamma\,.$$

Let ε be a positive number. We choose a compact $C \subset E_a^{\hat{}}$ such that $\mu_a(E - C) < \varepsilon$. Then by theorem 12 we have for sufficiently large n and $x \in C$ the inequality

$$\int \Phi_0(x, y)\, d\mu_{a_n}(y) > \gamma - \varepsilon\,.$$

Hence we obtain by integration

$$\iint \Phi_0\, d\mu_{a_n} d\mu_{a_n} \geqslant \int_C d\mu_{a_n}(y) \int_E \Phi_0(x, y)\, d\mu_{a_n}(x) \geqslant \int_C u\, d\mu_{a_n} - \varepsilon,$$

which yields in the limit

$$\gamma \geqslant \lim \int_C u\, d\mu_{a_n} - \varepsilon\,.$$

Since ε is arbitrary and $\int_C$ approaches $\int_E$ as near as we like, we conclude that

$$\varlimsup \int u\, d\mu_{a_n} - \varepsilon\,.$$

This inequality together with the opposite one (7) gives our proposition.

IV. On some number sequences. Let $\widetilde{\mu}_n$ denote the measure defined for $f \in \mathcal{C}$ by the formula

$$\int f\, d\widetilde{\mu}_n = \sum_{i=1}^{n} f(\eta_i^n)/n\,.$$

It is easy to see that $\widetilde{\mu}_{a_n}$ converges to μ^a and that we have

$$\lim \int u\, d\widetilde{\mu}_{a_n} = \gamma \tag{8}$$

(theorem 12).

We introduce the notation

$$D_j^n = \sum_{0 \leqslant \nu \leqslant n} \Phi(\eta_\nu^n, \eta_j^n)/n\,.$$

We numerate the points η_i^n in such a way that

$$D_0^n = \max_{0 \leqslant j \leqslant n} D_j^n\,. \tag{9}$$

Evidently

$$D_0^n = \int \Phi(\eta_0^n, y)\, d\widetilde{\mu}_{a_n}(y)\,.$$

LEMMA 4.

$$D_0^n = \min_{x \in E} \sum_{\nu=1}^{n} \Phi(\eta_0^n, \eta_\nu^n)/n\,.$$

Proof. The proposition follows by the definition of the extreme points and by the representation

$$\frac{1}{2}\sum_{i \neq j} \Phi(\eta_i^n, \eta_j^n) = \sum_{0 \leqslant i < j \leqslant n} \Phi(\eta_i^n, \eta_j^n) = nD_0^n + \sum_{1 \leqslant i < j \leqslant n} \Phi(\eta_i^n, \eta_j^n)\,.$$

THEOREM 14. *The sequence D_0^n converges to the number γ.*

Proof. Suppose the contrary. We choose a sequence of indices $\{\beta_n\}$ such that μ_{β_n} converges to some measure μ_β and $\lim D_0^{\beta_n} \neq \gamma$. In view of lemma 4 we have by $x \in E$

$$D_0^{\beta_n} \leqslant \int \Phi(x, y)\, d\widetilde{\mu}_{\beta_n}(y)\,.$$

The integration of this inequality by $d\mu_{\beta_n}(x)$ yields

$$D_0^{\beta_n} \leqslant \int\int \Phi_0\, d\widetilde{\mu}_{\beta_n} d\mu_{\beta_n}$$

and in view of (8) we have

$$\overline{\lim}\, D_0^{\beta_n} \leqslant \gamma\,. \tag{10}$$

Now we choose on E_a a point d such that $u(d) = \gamma$. Let $\eta_{j_n}^{\beta_n}$ be a sequence of extreme points which converges to d as $n \to \infty$. Using theorem 6 we obtain

$$\gamma = u(a) \leqslant \underline{\lim} \int \Phi_0(\eta_{j_n}^{\beta_n}, y)\, d\mu_{\beta_n}(y) = \underline{\lim}\, D_{j_n}^{\beta_n}\,.$$

In view of (9) we have $D_0^{\beta_n} \geqslant D_{j_n}^{\beta_n}$ and in consequence

$$\underline{\lim}\, D_0^{\beta_n} \geqslant \gamma\,,$$

which in view of (10) yields $\lim D_0^{\beta_n} = \gamma$, which contradicts the hypothesis.

Using the notation which has commonly been used in the theory of the extreme points (see e. g. [1]) we write $\Delta_n^0 = \exp(-nD_0^n)$. Theorem 14 translated into these terms states that

$$\lim \sqrt[n]{\Delta_n^0} = v(E, \omega) = e^{-\gamma}\,.$$

In this form the theorem has been used in the papers of Leja and Górski concerning the Dirichlet problem and related.

THEOREM 15. *If the sequence of the extreme points $\eta_{j_n}^{a_n}$ converges to a point b such that $u(b) = \gamma$, then $\lim D_{j_n}^{a_n} = \gamma$.*

Proof. By theorem 14 we have

$$\lim D_{j_n}^{a_n} \leqslant \lim D_0^{a_n} = \gamma\,,$$

and by theorem 6 $\underline{\lim}\, D_{j_n}^{a_n} \geqslant u(b) = \gamma$.

THEOREM 16. *If c is such a point of E that $u(c) > \gamma$, then c is not an accumulation point of the extreme points.*

Proof. The supposition that there exists a sequence $\eta_{j_n}^{a_n}$ which converges as $n \to \infty$ to the point c in question leads to a contradiction because

$$\gamma = \lim D_0^{a_n} \geqslant \lim D_{j_n}^{a_n} \geqslant u(c) > \gamma\,.$$

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