Les ANNALES POLONICI MATHEMATICI constituent une continuation des ANNALES DE LA SOCIÉTÉ POLONAISE DE MATHÉMATIQUE (vol. I-XXV) fondées en 1921 par Stanisław Zaremba.

Les ANNALES POLONICI MATHEMATICI publient, en langues des congrès internationaux, des travaux consacrés à l'Analyse Mathématique, la Géométrie et la Théorie des Nombres. Chaque volume paraît en 3 fascicules.

Les manuscrits dactylographiés sont à expédier à l'adresse:
Rédaction des ANNALES POLONICI MATHEMATICI
KRAKÓW (Pologne), ul. Solskiego 30.

Toute la correspondance concernant l'échange et l'administration est à expédier à l'adresse:
ANNALES POLONICI MATHEMATICI
WARSZAWA 10 (Pologne), ul. Śniadeckich 8.

Le prix de ce fascicule est 2.50 \$.
Les ANNALES sont à obtenir par l'intermédiaire de
ARS POLONA
WARSZAWA 5 (Pologne), Krakowskie Przedmieście 7.

PRINTED IN POLAND

DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

ANNALES
POLONICI MATHEMATICI
XI (1961)

On some functional equations containing iterations of the unknown function

by M. Kuczma (Kraków)

In the present paper I give a construction of the general solution, and general continuous solution of some functional equations containing iterations of the unknown function. The unknown function will be denoted in this paper by $\varphi(x)$, $\varphi^n(x)$ will denote the n-th iteration of the function $\varphi(x)$.

§ 1. Let X be a space of arbitrary elements and let $g(x)$ be a function defined in a subset F of the space X and assuming values from X. Further, let E be an arbitrary subset of the space X.

Definition I. We say that a function $\varphi(x)$ (defined in E and assuming values from X) *satisfies the functional equation*

(1) $$\varphi^2(x) = g[\varphi(x)]$$

in the set E if for each $x \in E$ the expressions $\varphi^2(x)$ and $g[\varphi(x)]$ have a meaning and both these expressions are equal. Every function $\varphi(x)$ satisfying equation (1) in the set E will be called *a solution of equation* (1) *in* E.

In the simple case $g(x) = x$ equation (1) has been treated by S. Gołąb [2] and G. M. Ewing and W. R. Utz [1] (cf. § 4 below).

Definition II. We denote by $\boldsymbol{V}$ the class of all sets V such that

(2) $$g(V) \subset V \subset E \cap F .$$

Lemma I. *If a function $\varphi(x)$ satisfies equation* (1) *in E, then the set $\varphi(E)$ belongs to the class $\boldsymbol{V}$.*

Proof. Let $y_0 \in \varphi(E)$. It means that there exists an $x_0 \in E$ such that $y_0 = \varphi(x_0)$. Putting $x = x_0$ in equation (1) we obtain

(3) $$\varphi(y_0) = g(y_0) .$$

Consequently $y_0 \in E$ and $y_0 \in F$, i.e. $y_0 \in E \cap F$. Hence

$$\varphi(E) \subset E \cap F .$$

Moreover it follows from (3) that $g(y_0) \in \varphi(E)$, whence

$$g[\varphi(E)] \subset \varphi(E)\,,$$

which completes the proof.

The following corollary is an immediate consequence of the above lemma:

COROLLARY. *A necessary condition for equation* (1) *to possess a solution in E is that the class $\boldsymbol{V}$ be non-empty.*

We shall show that the above condition is also sufficient.

DEFINITION III. For an arbitrary set $V \in \boldsymbol{V}$ we denote by $\boldsymbol{H}_V$ the class of all functions $h(x)$ that are defined in $E-V$ and fulfil the condition

$$h(E-V) \subset V\,. \tag{4}$$

We shall prove the following

THEOREM I. *The general solution of equation* (1) *in E is given by the formula*

$$\varphi(x) = \begin{cases} g(x) & \text{for} \quad x \in V, \\ h(x) & \text{for} \quad x \in E-V, \end{cases} \tag{5}$$

where V is an arbitrary set from the class $\boldsymbol{V}$ and $h(x)$ an arbitrary function from the class $\boldsymbol{H}_V$.

Proof. Inclusions (2) and (4) imply that the function $\varphi(x)$, defined by formula (5), satisfies equation (1) in E. Moreover, we must show that every solution of equation (1) in E is contained in the family of functions determined by formula (5).

Let $\varphi(x)$ be an arbitrary solution of equation (1) in E and let us put

$$V \overset{\text{df}}{=} \varphi(E)\,. \tag{6}$$

Lemma I implies that $V \in \boldsymbol{V}$. It follows from relation (3) that for $x \in V$

$$\varphi(x) = g(x)\,.$$

Further, we have by (6)

$$\varphi(E-V) \subset V\,,$$

which proves that function $\varphi(x)$ restricted to the set $E-V$ belongs to the class $\boldsymbol{H}_V$. This completes the proof.

§ 2. Now let X be the space of real numbers and let the set E be an interval (¹). Further, let the function $g(x)$ be defined and continuous in an interval F. Since we are going to give the general continuous solution of equation (1) in E, we have to modify a little the definitions of the classes $\boldsymbol{V}$ and $\boldsymbol{H}_V$.

(¹) By an interval we shall also understand a half-axis, the whole space X, or a point.

DEFINITION IV. We denote by $\mathcal{V}$ the class of all intervals V fulfilling condition (2). For an arbitrary interval $V \in \mathcal{V}$ and an arbitrary function $f(x)$, defined and continuous in the interval V, we denote by $\mathcal{H}_{V,f}$ the class of all functions $h(x)$ that are defined and continuous in the set $E-\operatorname{int}V$ and fulfil relation (4) and the condition

$$h(u) = \lim_{x\to u,\, x\in V} f(x) \tag{7}$$

for each $u \in \overline{V} \cap E-\operatorname{int}V$ ($\overline{V}$ denotes the closure of the set V). If for some u of this kind limit (7) does not exist, we consider the class $\mathcal{H}_{V,f}$ to be empty.

Since the image of an interval by a continuous function is an interval, $\varphi(E) \in \mathcal{V}$ for every continuous solution $\varphi(x)$ of equation (1) in E.

Similarly to theorem I, we can prove

THEOREM II. *The general continuous solution of equation* (1) *in E is given by formula* (5), *where V is an arbitrary interval from the class $\mathcal{V}$ and $h(x)$ an arbitrary function from the class $\mathcal{H}_{V,g|V}$ ($g|V$ denotes the restriction of the function $g(x)$ to the set V).*

§ 3. Equation (1) is a particular case of the equation

$$\varphi^n(x) = g[\varphi(x)]\,. \tag{8}$$

By an argument similar to that of §§ 1-2 we obtain the following

THEOREM III. *The general solution of equation* (8) *in E is given by the formula*

$$\varphi(x) = \begin{cases} \psi(x) & \text{for} \quad x \in V, \\ h(x) & \text{for} \quad x \in E-V, \end{cases} \tag{9}$$

where V is an arbitrary set from the class $\boldsymbol{V}$, $\psi(x)$ an arbitrary solution of the equation

$$\psi^{n-1}(x) = g(x) \tag{10}$$

in V and $h(x)$ an arbitrary function from the class $\boldsymbol{H}_V$.

In the particular case where X is the space of real numbers, the set E is an interval and the function $g(x)$ is continuous in an interval F, the general continuous solution of equation (8) *is given by formula* (9), *where V is an arbitrary interval from the class $\mathcal{V}$, $\psi(x)$ an arbitrary continuous solution of equation* (10) *in V and $h(x)$ an arbitrary function from the class $\mathcal{H}_{V,\psi}$.*

The above theorem enables us to reduce the solving of equation (8) to the solving of equation (10). The general solution of equation (10) in a set V such that $g(V) = V$ has been given by S. Łojasiewicz [3]. As far as it is known to me, the general solution of equation (10) in a set V such that $g(V) \subset V$ as well as the general continuous solution of equation (10) have not been given till now. They will be the subject of one of my next papers. (In the simple case where $g(x) = x$ the general continuous solution of equation (10) can be found in paper [1]).

§ 4. Equation (8) is a generalization of the equation

$$\varphi^n(x) = \varphi(x)\,, \tag{11}$$

the general continuous solution of which is given by G. M. Ewing and W. R. Utz [1]. Ewing and Utz also show by an example that the equation

$$\varphi^n(x) = \varphi^m(x) \tag{12}$$

can have a solution which is not a solution of equation (11) for any iterative exponent n. We may ask how to construct the general solution (or general continuous solution) for equation (12) or for a more general equation

$$\varphi^n(x) = g[\varphi^m(x)]\,, \qquad m < n\,. \tag{13}$$

At first we shall show

LEMMA II. *If $\varphi(E) \subset E$, then for every positive integer k*

$$\varphi^{k+1}(E) \subset \varphi^k(E)\,. \tag{14}$$

Proof. Let $y_0 \in \varphi^{k+1}(E)$. It means that there exists an $x_0 \in E$ such that $y_0 = \varphi^{k+1}(x_0)$. Let us put $z_0 \overset{\mathrm{df}}{=} \varphi(x_0)$. Evidently $z_0 \in E$ and $y_0 = \varphi^k(z_0)$, whence $y_0 \in \varphi^k(E)$. Consequently relation (14) holds, which was to be proved.

Now we shall define a class of functions $\boldsymbol{H}_V^m$. For a fixed set $V \in \boldsymbol{V}$ we assign to the class $\boldsymbol{H}_V^m$ all the functions $h(x)$ (defined in $E-V$) formed in the following manner (²):

Let $U_0, \ldots, U_i$, $i \leqslant m$, be sequence of sets such that

$$U_0 = E\,, \quad U_i = V\,, \quad U_{k+1} \subset U_k\,, \quad U_{k+1} \neq U_k \quad \text{for} \quad k = 0, \ldots, i-1.$$

We impose the function $h(x)$ arbitrarily on each of the sets $U_k - U_{k+1}$, $k = 0, \ldots, i-1$, but in such a manner that

$$h(U_k - U_{k+1}) \subset U_{k+1} \quad \text{for} \quad k = 0, \ldots, i-1\,.$$

Now we can prove

THEOREM IV. *The general solution of equation* (13) *is given by formula* (9), *where V is an arbitrary set from the class $\boldsymbol{V}$, $\psi(x)$ is an arbitrary solution of the equation*

$$\psi^{n-m}(x) = g(x) \tag{15}$$

in V and $h(x)$ is an arbitrary function from the class $\boldsymbol{H}_V^m$.

(²) If $V = E$ the class $\boldsymbol{H}_V^m$ is of course understood to be empty.

Proof. It can easily be verified that every function $\varphi(x)$ defined by formula (9) satisfies equation (13) in E. On the other hand, let $\varphi(x)$ be a solution of equation (13) in E. We put $V \overset{\mathrm{df}}{=} \varphi^m(E)$. One can easily verify that $V \in \boldsymbol{V}$. Similarly it is evident that for $x \in V$ $\varphi(x) = \psi(x)$, where $\psi(x)$ satisfies equation (15) in V. It remains only to prove that in the case $V \neq E$ the function $\varphi(x)$ restricted to the set $E-V$ belongs to the class $\boldsymbol{H}_V^m$.

Let us put

$$U_0 \overset{\mathrm{df}}{=} E\,, \qquad U_k \overset{\mathrm{df}}{=} \varphi^k(E)\,, \qquad k = 1, 2, \ldots$$

Since $\varphi(x)$ satisfies equation (13) in E, $\varphi(E) \subset E$. Thus we have by lemma II $U_{k+1} \subset U_k$. Moreover, if $U_{k_0} = U_{k_0+1}$, then $U_k = U_{k_0}$ for all $k > k_0$. Consequently there exists an index i, $1 \leqslant i \leqslant m$, such that

$$U_{k+1} \neq U_k \quad \text{for} \quad k = 0, \ldots, i-1$$

and

$$U_i = V\,.$$

Evidently

$$\varphi(U_k - U_{k+1}) \subset U_{k+1} \quad \text{for} \quad k = 0, \ldots, i-1\,.$$

This completes the proof.

§ 5. Now let X be the space of real numbers, E an interval and let $g(x)$ be a function defined and continuous in an interval F. For an arbitrary interval $V \in \mathcal{V}$ and an arbitrary function $f(x)$, defined and continuous in the interval V, we denote by $\mathcal{H}_{V,f}^m$ the class of all functions $h(x)$ (defined and continuous in $E - \operatorname{int} V$), formed in an analogical manner to the functions from the class $\boldsymbol{H}_V^m$, but with the following additional conditions:

1° The sets U_k, $k = 0, \ldots, i$, are intervals.

2° For each $u \in \overline{V} \cap E - \operatorname{int} V$ condition (7) is fulfilled.

Then one can prove

THEOREM V. *The general continuous solution of equation* (13) *in E is given by formula* (9), *where V is an arbitrary interval from the class $\mathcal{V}$, $\psi(x)$ an arbitrary continuous solution of equation* (15) *in V and $h(x)$ an arbitrary function from the class $\mathcal{H}_{V,\psi}^m$.*

References

[1] G. M. Ewing and W. R. Utz, *Continuous solutions of the functional equation $f^n(x) = f(x)$*, Canadian J. Math. 5 (1953), p. 101-103.

[2] S. Gołąb, *Über eine Funktionalgleichung der Theorie der geometrischen Objekte*, Wiadomości Matematyczne 45 (1938), p. 97-137.

[3] S. Łojasiewicz, *Solution générale de l'équation fonctionnelle $f^n(x) = g(x)$*, Ann. Soc. Pol. Math. 24 (1951), p. 88-91.

Reçu par la Rédaction le 3. 6. 1960