and hence, by (3),

$$\|f-k\|_G = I\Big(\int_H |f(x\xi)-k(x\xi)|\,d\xi\Big) \geqslant I(|\bar{f}|) = \tilde{I}(|f|) = \|f\|_G.$$

This proves that $\|f\|_G = \operatorname{dist}\{f, K\}$. Now, by (3), $\|f\|_G = \|g\|_{G/H}$ and hence $\|g\|_{G/H} = {}_K\|g\|_{G/H}$.

Proof of (b). By the triangle inequality it suffices to verify that

$${}_K\|g+h\|_{G/H} \geqslant {}_K\|g\|_{G/H} + {}_K\|h\|_{G/H}.$$

Let $r, t \in L^1(G)$ be such that $\bar{r} = g$, $\bar{t} = h$, where the supports S_r, S_t satisfy $S_rH \cap S_tH = \emptyset$. Then the inequality we wish to prove is equivalent to

$$\operatorname{dist}\{r+t, K\} \geqslant \operatorname{dist}\{r, K\} + \operatorname{dist}\{t, K\}. \tag{5}$$

It is easily seen that if $k \in K$, then the restricted functions $k^{(r)} = k\,|\,S_rH$ and $k^{(t)} = k\,|\,S_tH$ also belong to K, and this implies that

$$\|r+t-k\|_G = \tilde{I}(|r-k^{(r)}| + |t-k^{(t)}| + |k-k^{(r)}-k^{(t)}|)$$
$$\geqslant \|r-k^{(r)}\|_G + \|t-k^{(t)}\|_G \geqslant \operatorname{dist}\{r, k\} + \operatorname{dist}\{t, K\}.$$

Hence (5) follows and the proof of Theorem 4 is complete.

REFERENCES

[1] P. R. Halmos, *Measure theory*, New York 1951.

[2] L. H. Loomis, *An introduction to abstract harmonic analysis*, New York 1953.

[3] A. M. Macbeath and S. Świerczkowski, *Measures in homogeneous spaces*, Fundamenta Mathematicae 49 (1960), p. 15-24.

[4] H. Reiter, *Über L^1-Räume auf Gruppen I*, Monatshefte für Mathematik 58 (1954), p. 73-76.

[5] A. Weil, *L'intégration dans les groupes topologiques et ses applications*, Paris 1938.

MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 23. 3. 1960;
en version modifiée le 1. 4. 1960

COLLOQUIUM MATHEMATICUM

VOL. VIII 1961 FASC. 1

ON THE ALGEBRAS L_p OF LOCALLY COMPACT GROUPS

BY

W. ŻELAZKO (WARSAW)

Let G be locally compact group, and μ its left invariant Haar measure. Let L_p be the Banach space of complex functions defined on G, for which

$$\|f\|_p^p = \int |f(t)|^p\,d\mu(t) < \infty.$$

It is well known that L_1 is a Banach algebra if multiplication is defined as the convolution

$$f*g(t) = \int f(t\tau^{-1})g(\tau)\,d\mu(\tau).$$

It is also known that if the group G is compact, then the space L_2 is also a Banach algebra with the same multiplication (see [1], p. 156). Here I shall prove that this theorem and the converse theorem hold for all $p > 1$. More precisely I shall prove

THEOREM 1. *If the locally compact group G is compact, then for every p, $1 \leqslant p \leqslant \infty$, the space L_p is a Banach algebra under convolution.*

THEOREM 2. *If for a locally compact abelian group the space L_p is a Banach algebra under convolution, and $1 < p < \infty$, then the group G is compact.*

The following simple remark is useful in the proofs:

Let X be a Banach space with the norm $\|x\|$, and R a dense linear subspace, which is at the same time an algebra with the multiplication xy. Then

(A) *X is a Banach algebra with the same multiplication if and only if there exists such a number $C > 0$ that*

$$\|xy\| \leqslant C\|x\|\,\|y\| \quad \textit{for every} \quad x, y \in R.$$

Or, what is equivalent,

(A′) *The multiplication xy cannot be extended onto X in such a way that X is a Banach algebra if and only if for every $\varepsilon > 0$ there exist such $x, y \in R$ that*

(1) $$\|x\| < \varepsilon, \quad \|y\| < \varepsilon \quad \text{and} \quad \|xy\| \geqslant C > 0.$$

Now we pass to the easy

Proof of theorem 1. If the group G is compact, then we may assume that $\mu(G) = 1$. Hence (see [1], p. 156)

(2) $$\|f\|_1 \leqslant \|f\|_p, \quad 1 \leqslant p,$$

for every complex function defined on G, and

(3) $$R = L_\infty \cap L_1 \subset L_p,$$

We also have (see [1], p. 121-122)

(4) $$\|f*g\|_p \leqslant \|f\|_1 \|g\|_p \quad \text{for every} \quad f \in L_1,\ g \in R_p.$$

And by (2), (3), and (4) we get

$$\|f*g\|_p \leqslant \|f\|_p \|g\|_p \quad \text{for} \quad f, g \in R.$$

Hence, by the remark (A), L_p is a Banach algebra, q. e. d.

Proof of theorem 2. At the beginning we assume that $\mu(t) = 1$ for every $t \in G$ ($\equiv G$ is discrete). We shall prove the following remark

(B) *If $1 < p < \infty$, and L_p is a Banach algebra, then the group G is finite.*

Indeed, in this case the algebra L_p is a commutative Banach algebra with the unit element (the unity of algebra L_p is the characteristic function of the unit element of the group G), and by the Gelfand theory there exists a non-zero multiplicative linear functional F. The functional F may be written in the form $F(f) = \int f(t) g(t) dt$, where the function g is a member of L_q, $1/p + 1/q = 1$. On the other hand, for the discrete group we have $L_1 \subset L_p$. The functional F restricted to L_1 may be written in the form $F(f) = \int f(t) \chi(t) dt$ for $f \in L_1$, where $\chi(t)$ is the character of the group G [3]. It follows that $\chi = g$ and $\chi \in L_q$. But if $p > 1$, then $q < \infty$, and therefore the group G must be finite, because $\|\chi\|_q^q = \int \chi(t)^q dt = \mu(G) < \infty$.

By (B) and (A′) we get the following remark:

(B′) *If the discrete abelian group G is infinite, then for a given p, $1 < p < \infty$, and for every $\varepsilon > 0$, there exist two functions x, y with a compact ($\equiv$ finite) support and a $C > 0$, such that for the norm $\|x\|_p$ inequalities (1) hold.*

Now let G be an arbitrary locally compact abelian group. Let V be any compact neighbourhood of the unit e of G. Let G_0 be the sub-

group of G generated by V. The structure of G_0 is well known by a certain theorem of Pontriagin (see [2], p. 274). It follows from this theorem that G_0 is either compact or it contains such an element a that the subgroup generated by a is discrete and infinite. We shall discuss these two cases.

1° Let G_0 be a compact subgroup of G. Then $\mu(G_0) < \infty$. On the other hand, $\mu(G_0) \geqslant \mu(V) > 0$, and we may assume that $\mu(G_0) = 1$. Let L_p be the Banach algebra of the group G. We shall consider the subalgebra $L_p' \subset L_p$ of all $f \in L_p$ which are constant on the cosets tG_0, $t \in G$. The subalgebra L_p' is isometric with the algebra l_p of the discrete group G/G_0, and by the remark (B) the number of cosets must be finite, whence the group G must be compact.

2° Let $a \in G$, and the subgroup generated by a be discrete. It is to be shown that L_p is not a Banach algebra if $1 < p < \infty$. It may easily be proved that in G there exists a symmetric neighbourhood U of the unit e such that for integer m, n

(5) $$a^n U^2 \cap a^m U^2 = \emptyset \quad \text{if} \quad m \neq n,$$

where $\emptyset$ denotes the void set.

We may assume that $\mu(U) \leqslant 1$.

Let (a_n), (β_n), $n = 0, \pm 1, \pm 2, \ldots$, be two finite sequences of complex numbers such that

$$\Big(\sum |a_n|^p\Big)^{1/p} < \varepsilon, \quad \Big(\sum |\beta_n|^p\Big)^{1/p} < \varepsilon, \quad \Big(\sum |\gamma_n|^p\Big)^{1/p} = 1,$$

where

$$\gamma_n = \sum_k a_{n-k} \beta_k,$$

their existence, for every $\varepsilon > 0$, is proved by the remark (B′).

We put

$$f(t) = \sum_n a_n \chi_{a^{-n}U}(t), \quad g(t) = \sum_n \beta_{-n} \chi_{a^n U}(t),$$

where

$$\chi_A(t) = \begin{cases} 1 & \text{for} \quad t \in A, \\ 0 & \text{for} \quad t \notin A. \end{cases}$$

We have using (5)

(6) $$\|f\|_p = \Big(\int \Big|\sum_n a_n \chi_{a^{-n}U}(t)\Big|^p dt\Big)^{1/p} = \Big(\sum_n |a_n|^p \mu(U)^p\Big)^{1/p}$$
$$\leqslant \Big(\sum_n |a_n|^p\Big)^{1/p} < \varepsilon \quad \text{and} \quad \|g\|_p < \varepsilon.$$

On the other hand

(7) $$h(t) = f*g = \sum_{nk} a_n\beta_{-k}\int \chi_{a^{-n}U}(t\tau^{-1})\chi_{a^kU}(\tau)\,d\tau$$

$$= \sum_{nk} a^n\beta_{-k}\mu(a^nUt\frown a^kU)$$

$$= \sum_{nk} a_n\beta_{-k}\,\mu(a^{n-k}Ut\frown U)$$

$$= \sum_{ns} a_n\beta_{s-n}\mu(a^sUt\frown U)$$

$$= \sum_{s}\mu(a^sUt\frown U)\sum_{n} a_n\beta_{s-n}$$

$$= \sum_{s}\gamma_s\mu(a^sUt\frown U).$$

But, as is well known, the function $\varphi(t) = \mu(Ut\frown U)$ is continuous, and $\varphi(e) = \mu(U)$. Therefore there exists such a neighbourhood V of e that

$$\varphi(t) \geqslant \frac{\mu(U)}{2} \quad \text{for} \quad t\in V.$$

Hence

(8) $$\mu(a^sUt\frown U) \geqslant \frac{\mu(U)}{2} \quad \text{for} \quad t\in a^{-s}V.$$

By (5) the functions $\varphi_s(t) = \varphi(a^st)$ have disjoint supports and by (6), (7), and (8) we have

$$\|h\|_p = \Big\|\sum_s \gamma_s\varphi_s(t)\Big\|_p = \Big(\int\Big|\sum_s \gamma_s\varphi_s(t)\Big|^p dt\Big)^{1/p}$$

$$= \Big(\sum_s |\gamma_s|^p\int|\varphi_s(t)|^p dt\Big)^{1/p}$$

$$\geqslant \Big(\sum_s |\gamma_s|^p\int_{a^{-s}V}\varphi_s^p(t)\,dt\Big)^{1/p}$$

$$\geqslant \Big(\sum |\gamma_s|^p\mu(V)\frac{\mu(U)^p}{2^p}\Big)^{1/p}$$

$$= \Big(\sum_s|\gamma_s|^p\Big)^{1/p}\frac{\mu(U)}{2}\mu(V)^{1/p} = \frac{\mu(U)}{2}\mu(V)^{1/p} > 0.$$

Hence, by (A′), L_p is not a Banach algebra, q. e. d.

In case $p = 2$ the proof of theorem 2 may be obtained in another way, where the form of the group G need not be discussed. To this aim let us assume that the space L_2 of locally compact abelian group G is a Banach algebra with convolution. By Raikov's inequality (see [3], p. 45, formula (6.1)) we get

$$\sqrt[n]{\|f^n\|_2} \geqslant \sqrt[n]{\frac{\|f\|_2^2}{\|f^2\|_2}}\cdot\frac{\|f^2\|_2}{\|f\|_2}, \quad \text{where} \quad f^n = \underbrace{f*f*\ldots*f}_{n\ \text{times}},$$

which holds for every function from $L_1\frown L_2$ satisfying $f(t^{-1}) = f(t)$, $t\in G$, e. g. for a characteristic function of a compact symmetric neighbourhood of e. Hence algebra L_2 is not radical, and there exists a non-zero multiplicative linear functional F. The functional F may be written in the form $F(f) = \int f(t)\varphi(t)\,dt$, where φ is a member of L_2. By the multiplicativity of F and the Fubini theorem we have

$$F(f*g) = \int\varphi(t)\int f(t\tau^{-1})g(\tau)\,d\tau = \int\int\varphi(tp)f(t)g(p)\,dt\,dp$$

$$= \int f(t)\varphi(t)\,dt\cdot\int g(p)\varphi(p)\,dp$$

for every $f, g\in L_2$.

It follows that

(9) $$\varphi(pt) = \varphi(p)\varphi(t) \quad \text{a. e. } (\mu\times\mu).$$

Since φ is not a. e. equal to zero, there is a set $C\subset G$ such that

$$\int_C\varphi(t)\,dt \neq 0 \quad \text{and} \quad \int_C\varphi*\varphi\,dt < \infty.$$

Let $Q\subset G$ be any set of σ-finite measure μ such that the (σ-finite) support of φ is contained in Q. Then, by Fubini's theorem and by (9),

$$\infty > \int_C\varphi*\varphi\,dt = \int_C\int_Q\varphi(t\tau^{-1})\varphi(\tau)\,d\tau\,dt$$

$$= \iint_{C\times Q}\varphi(t\tau^{-1})\varphi(\tau)\,d\tau\,dt$$

$$= \iint_{C\times Q}\varphi(t)\,d\tau\,dt = \mu(Q)\int_C\varphi(t)\,dt.$$

So we get $\mu(Q) = \int_C\varphi*\varphi\,dt/\int_C\varphi\,dt$. Hence $\mu(G) < \infty$, and the group G is compact.

REFERENCES

[1] L. H. Loomis, *An introduction to abstract harmonic analysis*, Toronto-New York-London 1953.

[2] Л. С. Понтрягин, *Непрерывные группы*, Москва 1954.

[3] Д. А. Райков, *Гармонический анализ на коммутативных группах с мерой Хаара и теория характеров*, Труды Математического Института им. Стеклова XIV, Москва-Ленинград 1945.

[4] R. Sikorski, *Funkcje rzeczywiste*, tom II, Warszawa 1958.

Reçu par la Rédaction le 20. 10. 1959

COLLOQUIUM MATHEMATICUM

VOL. VIII 1961 FASC. 1

A PROOF OF A THEOREM OF ŻELAZKO ON L^p-ALGEBRAS

BY

K. URBANIK (WROCŁAW)

Let G be a locally compact Abelian topological group. For each $p \geqslant 1$, we define the space $L^p(G)$ as the space of all measurable complex-valued functions f on G such that $|f|^p$ is integrable with respect to the Haar measure m on G. Obviously, $L^p(G)$ is a Banach space under the norm

$$\|f\|_p = \Big(\int_G |f(x)|^p m(dx)\Big)^{1/p}.$$

In the sequel we shall denote by fg the convolution of functions f and g, i. e.

$$(fg)(x) = \int_G f(y) g(xy^{-1}) m(dx) \qquad (x \in G).$$

In this note we shall give a simple proof of the following theorem, proved by W. Żelazko in paper [3]:

If, for a number $p > 1$, $L^p(G)$ is a topological ring under the convolution multiplication, then G is a compact group.

Proof. Let R be the extension of $L_p(G)$ to a topological ring with a unit element (see [1], p. 158). The norm in R, which is an extension of the norm in $L^p(G)$, will henceforth be denoted by $\| \|_p$. It is well known that the norm

$$\|f\| = \sup_{g \in R, \|g\|_p = 1} \|fg\|_p \tag{1}$$

makes R a normed ring (see [1], p. 168).

First we shall prove that the ring $L^p(G)$ admits a non-trivial continous homomorphism into the complex field. To prove this it is sufficient to show that $L^p(G)$ contains an element which does not belong to the radical of R.