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*SELF-DEPENDENT ELEMENTS IN ABSTRACT ALGEBRAS*

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This paper is a contribution to the study of the general notion of independence introduced by E. Marczewski ([1], see also [2]).

Let us consider an arbitrary non-empty set $A$. Let $\boldsymbol{A}$ be a class of $A$-valued functions of finitely many variables running over $A$ such that 1° the functions defined by the formula $f(x_1, \ldots, x_n) = x_k$ $(n = 1, 2, \ldots; k = 1, 2, \ldots, n)$ belong to $\boldsymbol{A}$; 2° $\boldsymbol{A}$ is closed with respect to the superposition of functions. The system $(A, \boldsymbol{A})$ will be called an *algebra*.

By $\boldsymbol{A}^{(0)}$ we shall denote the set of the values of all constant functions belonging to $\boldsymbol{A}$. Further, by $\boldsymbol{A}^{(n)}$ $(n \geqslant 1)$ we shall denote the class of all functions of $n$ variables belonging to $\boldsymbol{A}$.

A set $I \subset A$ is called a set of *independent* elements if for every finite system of different elements $a_1, \ldots, a_n \in I$ and every pair of functions $f, g \in \boldsymbol{A}^{(n)}$ the equality

$$f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n)$$

implies $f = g$; otherwise $I$ is a set of dependent elements.

An element $a \in A$ is called *self-dependent* if the one-element set $\{a\}$ is a set of dependent elements, i. e. whenever there exist two functions $g, h \in \boldsymbol{A}^{(1)}$ such that $g(a) = h(a)$ and $g \neq h$.

If an algebra contains at least two elements, then all values of constant functions are self-dependent.

The following theorem is a direct consequence of the definition of $\boldsymbol{A}^{(0)}$:

THEOREM 1. *The set $\boldsymbol{A}^{(0)}$ is a subalgebra, i. e. for every $f \in \boldsymbol{A}^{(n)}$ we have the relation*

$$f(c_1, \ldots, c_n) \in \boldsymbol{A}^{(0)}$$

*whenever $c_1, \ldots, c_n \in \boldsymbol{A}^{(0)}$.*

We remark that the set of all self-dependent elements is not necessarily a subalgebra (see for example Theorem 4).

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THEOREM 2. *If the subalgebra generated by a self-dependent element is finite, then all its elements are self-dependent.*

Proof. Let $a_0$ be a self-dependent generator of the finite subalgebra $A_0 = \{a_0, a_1, \ldots, a_n\}$. We assume that $a_i \neq a_j$ whenever $i \neq j$ $(i, j = 0, 1, \ldots, n)$. Moreover, every element $a_k$ $(k = 0, 1, \ldots, n)$ can be represented as $a_k = f_k(a_0)$, where $f_k \in \boldsymbol{A}^{(1)}$ and $f_0$ is the identity function. By hypothesis there exist two functions $g, h \in \boldsymbol{A}^{(1)}$ such that $g \neq h$ and $g(a_0) = h(a_0)$. Since $g(a_0) \in A_0$, there exists an index $r$ $(0 \leqslant r \leqslant n)$ for which $g(a_0) = h(a_0) = a_r$. Hence all the functions

$$f_0, f_1, \ldots, f_{r-1}, g, h, f_{r+1}, \ldots, f_n$$

are different. Since subalgebra $A_0$ contains $n+1$ elements and for every element $a_k \in A_0$ all $n+2$ elements of the system

$$f_0(a_k), f_1(a_k), \ldots, f_{r-1}(a_k), g(a_k), h(a_k), f_{r+1}(a_k), \ldots, f_n(a_k)$$

belong also to $A_0$, we infer that at least two elements of this system are equal. Consequently, we have proved the existence of a pair of functions $g_1, h_1 \in \boldsymbol{A}^{(1)}$ $(g_1 \neq h_1)$ such that $g_1(a_k) = h_1(a_k)$. Thus all elements of $A_0$ are self-dependent.

THEOREM 3. *There exist a finite algebra generated by a non-self-dependent element $a_0$ such that all other its elements are self-dependent.*

Proof. Let $A$ be the three-elements set $\{0, 1, 2\}$. We define the functions $f_0, f_1, f_2$ as follows: $f_0(x) = x$ for any $x \in A$, $f_1(0) = f_1(1) = 1$, $f_1(2) = 2$, $f_2(0) = f_2(1) = 2$, $f_2(2) = 1$. Let $\boldsymbol{A}$ be the class of all functions $f$ of the form

$$f(x_1, x_2, \ldots, x_n) = f_j(x_k) \quad (1 \leqslant k \leqslant n;\ j = 0, 1, 2).$$

Since the class $\boldsymbol{A}$ is closed with respect to the superposition, the system $(A, \boldsymbol{A})$ is an algebra.

The element 0 is a generator of our algebra. In fact, we have the equalities $0 = f_0(0)$, $1 = f_1(0)$ and $2 = f_2(0)$. Since $f_0$, $f_1$ and $f_2$ are the only functions belonging to $\boldsymbol{A}^{(1)}$, the last equalities imply that the element 0 is not self-dependent. Finally, the equalities $f_0(1) = f_1(1)$ and $f_0(2) = f_1(2)$ imply that the elements 1 and 2 are self-dependent. The theorem is thus proved.

As is shown by the following theorem, the assumption of finiteness of an algebra in Theorem 2 is essential:

THEOREM 4. *There exists an algebra generated by a self-dependent element $a_0$ such that all its elements different from $a_0$ are non-self-dependent.*

Proof. Let $A$ be the set of all zero-one systems $\langle i_1, i_2, \ldots, i_n \rangle$ satisfying the condition $i_1 = 1$ if $n > 1$. For any finite system of indices $j_1, j_2, \ldots, j_k$ $(j_s = 0, 1;\ s = 1, 2, \ldots, k)$ we define the function $f_{j_1, j_2, \ldots, j_k}$ by the formula



$$(*) \qquad f_{j_1, j_2, \ldots, j_k}(\langle i_1, i_2, \ldots, i_n \rangle) = \begin{cases} \langle 1, j_2, \ldots, j_k \rangle & \text{if } \langle i_1, \ldots, i_n \rangle = \langle 0 \rangle, \\ \langle i_1, \ldots, i_n, j_1, \ldots, j_k \rangle & \text{in other cases.} \end{cases}$$

Let $\boldsymbol{A}$ be the class of all functions of the form

$$f(x_1, \ldots, x_n) = f_{j_1, j_2, \ldots, j_k}(x_r) \quad (1 \leqslant r \leqslant n)$$

or

$$f(x_1, \ldots, x_n) = x_r \quad (1 \leqslant r \leqslant n).$$

It is easy to see that the class $\boldsymbol{A}$ is closed with respect to the superposition of functions. Thus the system $(A, \boldsymbol{A})$ is an algebra.

First of all we shall prove that the element $\langle 0 \rangle$ is a self-dependent generator of our algebra. Denoting by $e$ the identity function which of course belongs to $\boldsymbol{A}$ we have the equality $\langle 0 \rangle = e(\langle 0 \rangle)$. Further, for every element $\langle 1, j_1, \ldots, j_n \rangle \in A$ we have the equality

$$f_{1, j_1, \ldots, j_n}(\langle 0 \rangle) = \langle 1, j_1, \ldots, j_n \rangle.$$

Consequently, $\langle 0 \rangle$ is a generator of our algebra. Since $f_0 \neq f_1$ and $f_0(\langle 0 \rangle) = \langle 1 \rangle = f_1(\langle 0 \rangle)$, the element $\langle 0 \rangle$ is self-dependent.

Now we shall prove that all other elements are non-self-dependent. By formula $(*)$ every function belonging to $\boldsymbol{A}$ is uniquely determined by its value on an arbitrary element $\langle i_1, \ldots, i_n \rangle \neq \langle 0 \rangle$. Hence it follows that all these elements are non-self-dependent, which completes the proof of our Theorem.

Remark. The last algebra contains an infinite set of independent elements. For instance, all the elements $\langle 1, 0, 1 \rangle$, $\langle 1, 0, 0, 1 \rangle$, $\langle 1, 0, 0, 0, 1 \rangle, \ldots$ are independent.

*REFERENCES*

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