COLLOQUIUM MATHEMATICUM

VOL. VIII 1961 FASC. 2

# A THEOREM ON THE DISCRETE GROUPS AND ALGEBRAS $L_p$

BY

W. ŻELAZKO (WARSAW)

It has been proved in [2] that for $p > 1$ the space $L_p$ of all complex valued functions defined on a locally compact Abelian group $G$ and integrable with $p$-th power with respect to Haar measure $\mu$ is a normed algebra under convolution

$$f*g(t) = \int f(\tau)g(t\tau^{-1})d\mu(\tau) \tag{1}$$

if and only if the group $G$ is compact. Here is a result for $0 < p < 1$. This result is a consequence of two propositions formulated for measures more general than the Haar measure.

In the sequel we assume $p$ be a real number satisfying $0 < p < 1$, and $L_p(\mu, G)$ is the space of all complex functions defined ón $G$ such that

$$\|f\| = \int |f(t)|^p d\mu(t) < \infty, \tag{2}$$

where $G$ is an arbitrary group and $\mu$ — an arbitrary measure defined on any field of subsets of $G$. The space $L_p(\mu, G)$ is an $F$-space with the norm (2) (see [1]).

PROPOSITION 1. *If $G$ is a group with the measure $\mu$ such that $\mu(t) = 1$ for every $t \in G$, then $L_p(\mu, G)$ is a topological algebra under the convolution* (1).

Proof. We have

$$\|f\| = \sum_{\tau \in G} |f(\tau)|^p$$

and

$$\|f*g\| = \sum_t \Big|\sum_\tau f(\tau)g(t\tau^{-1})\Big|^p \leqslant \sum_t \sum_\tau |f(\tau)|^p |g(t\tau^{-1})|^p$$
$$= \sum_\tau |f(\tau)|^p \sum_t |g(t\tau^{-1})|^p = \|f\|\cdot\|g\|.$$

Hence $L_p(\mu, G)$ is a topological algebra [1], q. e. d.

[1] The theory of topological algebras of this type is developed in the papers [3] and [4].

Proposition 2. *Let $G$ be a group with a measure $\mu$ such that there exist a symmetric subset $V$ containing the unit element and a family of $\mu$-measurable sets $V_n$ with $\mu(V_n) > 0$ $(n = 1, 2, \ldots)$ satisfying*

(a) $0 < \mu(V^2) < \infty$,

(b) $V_n \subset V$ *for* $n = 1, 2, \ldots$,

(c) $V_n \cap V_k = \emptyset$ *for* $n \neq k$;

*then $L_p(\mu, G)$ is not an algebra under the convolution* (1).

Proof. We shall find two functions $f, g \in L_p(\mu, G)$ such that $f*g$ is not in $L_p(\mu, G)$. By our assumptions we have $\sum_{n=1}^{\infty} \mu(V_n) < \infty$. Hence, by passing if necessary to the subsequence, we may assume that $\mu(V_n) \leqslant 2^{-n}$ for $n = 1, 2, \ldots$ We put now

$$f(t) = \sum_{n=1}^{\infty} [\mu(V_n)\cdot n^2]^{-1/p} \chi_{V_n}(t),$$

where $\chi_A$ designates the characteristic function of the set $A$, and

$$g(t) = \chi_{V^2}(t).$$

We have

$$\|f\| = \sum_{n=1}^{\infty} [\mu(V_n) n^2]^{-1} \int \chi_{V_n}(t)\, d\mu(t) = \sum_{n=1}^{\infty} n^{-2} < \infty,$$

and

$$\|g\| = \mu(V^2) < \infty.$$

Hence $f, g \in L_p(\mu, G)$. On the other hand,

$$f*g(t) = \int f(\tau) g(t\tau^{-1})\, d\mu(\tau) = \int_{V^2 t} f(\tau)\, d\mu(\tau)$$

and for every $t \in V$ we have

$$f*g(t) = \sum_{n=1}^{\infty} [\mu(V_n) n^2]^{-1/p} \cdot \mu(V_n) \geqslant \sum_{n=1}^{\infty} \frac{2^{n(1/p-1)}}{n^2} = \infty.$$

Hence $f*g$ cannot be in $L_p(\mu, G)$, because $V$ contains a measurable subset of positive measure, q. e. d.

Applying the propositions to the locally compact group with the left invariant Haar measure $\mu$, we get the following

Theorem. *Let $G$ be a locally compact group with the left invariant Haar measure $\mu$. Then the space $L_p(\mu, G)$ is a topological algebra under convolution if and only if the group $G$ is discrete.*



*REFERENCES*

[1] S. Rolewicz, *On a certain class of linear metric spaces*, Bulletin de l'Académie Polonaise des Sciences, Cl. III, 5 (1957), p. 479-484.

[2] W. Żelazko, *On the algebras $L_p$ of locally compact groups*, Colloquium Mathematicum 8 (1960), p. 115-120.

[3] — *On the locally bounded and m-convex topological algebras*, Studia Mathematica 19 (1960), p. 333-356.

[4] — *On the radicals of p-normed algebras*, ibidem (in print).

MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

*Reçu par la Rédaction le 28. 4. 1960*