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EQUICONTINUOUS AND RELATED FLOWS

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1. Introduction. In almost every aspect of topology the hypothesis of a compact space has led to interesting results and topological dynamics is no exception to this. The hypothesis of compactness is a natural one and is found in many of the classical theorems as a sufficiency condition. Equicontinuity of the dynamical system, a condition independent of compactness of the space, is also a natural hypothesis in topological dynamics. It is the purpose of this paper to secure consequences of equicontinuity with particular emphasis on results heretofore known only in the presence of compactness.

The notation of the paper follows that of [4] and is explained in Section 2. The intrinsic properties of an equicontinuous flow and the possibilities of a weaker concept have been found of sufficient interest to form the content of Section 3. In Section 4 the classical recurrence properties are taken up in a very general setting. In some instances these properties hold on a space as a result of holding on a dense subset, but this is more often the case in the presence of equicontinuity of the system. Limits of collections of orbits are considered in Section 5. As might be expected, the most interesting systems for study are those which employ the properties of real numbers, and some consequences of order are mentioned in Section 6.

2. Preliminary definitions and notation. Throughout the paper X, T, S, G, R and I will be used in the special sense explained in this section. X will denote a metric space with metric ϱ. T will denote a set of continuous transformations of X into itself. Usually T will be assumed closed under a binary operation which will always be assumed associative and commutative. When these conditions are imposed on T we shall refer to T as the semigroup S. It is often desirable to assume that S is a group and in this case we shall use G to denote this group. In this case, of course, each transformation of G is a homeomorphism of X onto itself. A topology, not necessarily metric, will be assumed on S and G in which the binary and inverse operations are continuous. In other words, S is a topological semigroup and G is a topological group. The notation $S=G$ means S is a group.

The special additive groups of the real numbers R and the integers I with their natural topology are often employed to yield results not always possible in the more general setting of a topological group. $S=R$ means S is isomorphic to R.

A continuous function of $X\times S$ onto X is a flow on X provided $x(s_1s_2)=(xs_1)s_2=xs_1s_2$ for all $x\in X$, $s_1, s_2\in S$. The algebraic rather than the functional notation will be used. The point set $xS\subset X$ will be called the *orbit* of x. This terminology will not be used in the case of simply a set of transformations T, but xt will always denote the image of x under the transformation t. In case $S=I$, the flow consists of the identity and the iterations of t and t^{-1}. This has often been called the *discrete flow*.

A semigroup $S'\subset S$ is said to be *replete* provided S' contains some translate of each compact set in S. A set $A\subset S$ is said to be *extensive* provided that A intersects every replete semigroup in S. A set $A\subset S$ is said to be *syndetic* provided that there exists a compact set $K\subset S$ such that $S=AK$ (this is the algebraic product, not the set intersection). It is remarked in [4] that when S is a group, a syndetic set is extensive.

In case $S=R$ or $S=I$ these definitions become simpler. A semigroup S' is replete provided S' contains a ray. A is extensive provided A contains a sequence

$$\{r_i\}:\ldots<r_{-2}<r_{-1}<r_0<r_1<r_2<\ldots$$

which is unbounded above and below. Such a sequence will be called an *extensive sequence*. A is syndetic provided that A contains an extensive sequence $\{r_i\}$ such that the sequence $\{r_i-r_{i-1}\}$ is bounded. Such a sequence will be called a *relatively dense sequence*.

A set $Y\subset X$ is said to be *invariant* provided $Yt\subset Y$ for each $t\in T$. If $T=I$, Y is seen to be invariant if and only if $Yt_1=Y$, where $t_1=1$.

The null set (of X) will be denoted by 0. $U_\varepsilon(x)$ denotes the set of all $y\in X$ such that $\varrho(x,y)<\varepsilon$.

3. Equicontinuity and related properties. T is said to be *equicontinuous at* $x\in X$ provided that to each real number $\varepsilon>0$ there corresponds a real number $\delta<0$ depending upon x and ε such that whenever $y\in X$, $\varrho(x,y)<\delta$, then $\varrho(xt,yt)<\varepsilon$ for each $t\in T$. T is said to be *equicontinuous on* $Y\subset X$ provided T is equicontinuous at each $y\in Y$. T is said to be *equiuniformly continuous on* $Y\subset X$ provided that to each real number $\varepsilon>0$ there corresponds a real number $\delta>0$, depending upon ε, such that whenever $y,y'\in Y$, $\varrho(y,y')<\delta$, then $\varrho(yt,y't)<\varepsilon$ for all $t\in T$.

It is not difficult to show that a necessary and sufficient condition that G be equicontinuous on the orbit xG is that G be equicontinuous at x.

When T is isomorphic to R, the reals, one obtains the following characterization of equiuniform continuity:

THEOREM 1. *If $T=R$, a necessary and sufficient condition that R be equiuniformly continuous on X is that corresponding to each $\varepsilon>0$ there is a $\delta>0$ and an $N>0$ such that whenever $x,y\in X$, $\varrho(x,y)<\delta$, then $\varrho(xr,yr)<\varepsilon$ for all $r\in R$ for which $|r|>N$.*

Proof. The necessity is obvious. For the sufficiency let $\varepsilon>0$ be arbitrary and let δ and N be as guaranteed by the hypothesis. Now take δ', N' so that if $\varrho(x,y)<\delta'<\delta$ and $|r|>N'$, then $\varrho(xr,yr)<\delta$. If $|r|>N$, $\varrho(x,y)<\delta'<\delta$ and $\varrho(xr,yr)<\varepsilon$ by hypothesis. If $|r|\leqslant N$, let $M=3\max(N,N')$ so that $r\leqslant N\leqslant M/3<M/2$. Thus $M-r=(M/2-r)+ +M/2>N$ and $|r-M|>N$. If x, y are taken so that $\varrho(x,y)<\delta'$, then $\varrho(xM,yM)<\delta$ and $\varrho(xr,yr)=\varrho[(xM)(r-M),\,(yM)(r-M)]<\varepsilon$.

An example of Edrei [1] shows that the hypothesis "$|r|>N$" in Theorem 1 can not be weakened to "each r belonging to an extensive sequence".

S is said to be *weakly transitive on the closed set* $Y\subset X$ provided that there is an element $y\in Y$ for which $\overline{yS}=Y$ and *strongly transitive on the closed set* $Y\subset X$ provided that for each $y\in Y$, $\overline{yS}=Y$. Weak transitivity does not imply strong transitivity even where X is compact and $S=I$. The following theorems relate transitivity to equicontinuity in transformation groups.

THEOREM 2. *If there is an element $y\in\overline{xG}$ such that G is equicontinuous at y, then $\overline{xG}=\overline{yG}$.*

Proof. $\overline{yG}\subseteq\overline{xG}$ since $\overline{xG}$ is closed and invariant. To show that $\overline{xG}\subset\overline{yG}$ let $u\in xG=uG$, and let $\varepsilon>0$ be arbitrary. Since G is equicontinuous at y, there is a $\delta>0$ such that if $\varrho(y,v)<\delta$, then $\varrho(yg,vg)<\varepsilon$ for each $g\in G$. Now $y\in\overline{uG}$ so there is a $v\in uG$ such that $\varrho(y,v)<\delta$, and since G is a group there is a $g\in G$ such that $vg=u$. Since $\varrho(yg,u)=\varrho(yg,vg)<\varepsilon$, $u\in\overline{yG}$, and therefore $xG\subset\overline{yG}$ and $\overline{xG}\subset\overline{yG}$.

The next theorem follows at once from Theorem 2.

THEOREM 3. *If G is weakly transitive and equicontinuous on $Y\subset X$, then G is strongly transitive on Y. If G is equicontinuous on xG, then G is strongly transitive on $\overline{xG}$.*

Theorem 3 provides an opportunity to compare equicontinuity of G with compactness of X. A closed set $Y \subset X$ is said to be *minimal* provided that S (or G) is strongly transitive on Y. It is well-known that in case $G = R$ or $G = I$, if X is compact, each closed invariant set contains a minimal set [8]. Using theorem 3 and the fact that if Y is invariant then $\overline{yG} \subseteq Y$, we have the following analogous theorem for equicontinuity valid for any transformation group:

THEOREM 4. *If G is equicontinuous on X, a necessary and sufficient condition that a closed set $Y \subseteq X$ be a minimal set is that Y be an orbit closure (the closure of an orbit). Thus if G is equicontinuous on $Y \subset X$ and Y is closed and invariant, then Y contains a minimal set.*

Examples are easily constructed to show that X may be a compact orbit closure without being a minimal set.

If X is expressed as the union of disjoint minimal sets, these sets are said *to form a minimal set partition of* X. According to the definition of a minimal set, distinct minimal sets must be disjoint and so, from Theorem 4, we see that if G is equicontinuous on X, then the orbit closures form a minimal set partition of X.

The following property, weaker than equicontinuity, has been used by Wenjen [7] to secure necessary and sufficient conditions for the existence of a convergent sequence in a family of transformations.

$S \subset R$ is said to be *ε-related at* $x \in X$ provided that to each $\varepsilon > 0$ corresponds a $\delta > 0$ such that whenever $y \in X$, $\varrho(x, y) < \delta$, then there is an $N > 0$, depending on y, for which $\varrho(xs, ys) < \varepsilon$ for $s > N$. S is *ε-related on* $Y \subset X$ provided R is ε-related at each $y \in Y$. S is *uniformly ε-related on* $Y \subset X$ provided that to each $\varepsilon > 0$ there corresponds a $\delta > 0$ in such a fashion that if $x, y \in Y$, $\varrho(x, y) < \delta$, then there is an $N > 0$, depending upon x, y as well as ε, and $\varrho(xs, ys) < \varepsilon$ for $|s| > N$. It is obvious that if R (or I) is equicontinuous then R (or I) is ε-related. The following example demonstrates that the converse is not true by exhibiting a flow which is uniformly ε-related but not equicontinuous and none of the transformations is uniformly continuous.

Example 1. Let X be the following subset of E_2: all points $(i, 0)$ and $(i, 1/j)$ for $i = 0, \pm 1, \pm 2, \ldots$; $j = 1, 2, 3, \ldots$ Let $G = I$ be defined as follows:

when $x = (i, 1), i \leqslant 1$, define $x1 = (i, 1)$;
when $x = (i, 1), i > 2$, define $x1 = \big(i+1, 1/(i-1)\big)$;
when $x = (i, 1/i), i > 1$, define $x1 = (i+1, 1)$;
when $x = \big(i, 1/(i-1)\big), i > 1$, define $x1 = \big(i, 1/(i-1)\big)$;

for all other points $x = (i, p) \in X$, define $x1 = (i+1, p)$.

It is well-known [8] that if X is compact and S is equicontinuous on X, then S is equiuniformly continuous on X. Similarly one can show that if X is compact and R is ε-related, then R is uniformly ε-related.

We now state a theorem needed in the next section. The proof is routine and will be omitted.

THEOREM 5. *If $f_n(x)$ is a mapping of a metric space into itself for $n = 1, 2, 3, \ldots$ and the sequence $\{f_n(x)\}$ is ε-related (uniformly ε-related) and converges pointwise to $F(x)$, then $F(x)$ is continuous (uniformly continuous) on X.*

In Theorem 5 the hypothesis that $\{f_n(x)\}$ is ε-related is stronger than is necessary for the conclusion. Let $\{f_n(x)\}$ have the property that corresponding to each $\varepsilon > 0$ and $x \in X$ there is a $\delta > 0$ such that whenever $y \in X$, $\varrho(x, y) < \delta$, then there is an increasing sequence $\{n_i\}$ of positive integers so that $\varrho\big(f_{n_i}(x), f_{n_i}(y)\big) < \varepsilon$. If $\{f_n(x)\}$ is a sequence of mappings of a metric space X into itself with this property and if $f_n(x)$ converges pointwise to $F(x)$, then one can show that $F(x)$ is continuous.

In Theorem 5 one can not prove that the convergence is uniform as this can not be inferred even when $\{f_n(x)\}$ is equiuniformly continuous.

4. Recurrence properties and their extensions to the closure of a set. In this section K denotes a dense subset of X, that is, $\overline{K} = X$. In most of the theorems of this section we assume that S has certain properties on K and show that this implies that S will have the same properties on X. The difficult question of extending the flow itself from K to X will not be considered. Thus, we will always assume that the flow is defined on all of X.

It is not difficult to show that if S is equiuniformly continuous on K, then S is equiuniformly continuous on X and that whenever R is uniformly ε-related on K, then R is uniformly ε-related on X. The word "uniformly" may not be omitted from the preceding observations, as is shown by the following examples wherein I is equicontinuous on K but not even ε-related on X, which is compact.

Example 2. Let K be the subset of the real numbers consisting of the points $\pm(1-1/2^n)$ for $n = 0, 1, 2, \ldots$ and let X consist of the points of K and the two points ± 1. Let $G = I$, defined as follows:

$$x1 = \begin{cases} 2x+1 & \text{for} \quad -1 \leqslant x < 0, \\ (x+1)/2 & \text{for} \quad 0 \leqslant x \leqslant 1. \end{cases}$$

I is not ε-related at $x = -1$.

S is said to be *periodic at* $x \in X$, or x is a *periodic point*, provided there exists a syndetic semigroup $A \subset S$ such that $xA = x$. An immediate consequence is that the orbit xS is compact. S is *pointwise periodic on* $Y\subset X$ if S is periodic at each $y\in Y$. S is *periodic on* $Y\subset X$ provided there exists a syndetic semigroup $A\subset S$ such that $yA=y$ for each $y\in Y$. It is easily shown that if $S=G$, A is a subgroup of G.

We now define almost periodicity and recurrence in S. S is said to be *almost periodic* (*recurrent*) *at* $x\in X$, or x is an *almost periodic* (*a recurrent*) *point*, provided that to each $\varepsilon>0$ corresponds a syndetic (an extensive) set $A\subseteq S$ such that $\varrho(x,xs)<\varepsilon$ for each $s\in A$. S is *pointwise almost periodic* (*pointwise recurrent*) *on* $Y\subset X$ if S is almost periodic (recurrent) at each $y\in Y$. S is *almost periodic* (*recurrent*) *on* $Y\subset X$ provided that to each $\varepsilon>0$ corresponds a syndetic (an extensive) set $A\subset S$ such that $\varrho(x,xs)<\varepsilon$ for each $s\in A$, $y\in Y$. It is clear that if S is periodic at x, then S is almost periodic at x and if S is almost periodic at x, then S is recurrent at x.

Before considering the extensions of these properties from K to X we recall [8] that if $S=I$ then S periodic on Y implies S is equicontinuous on Y and prove that this conclusion holds with the weaker hypothesis that I is almost periodic on X. If X is compact it is known [3] that equiuniform continuity of G and almost periodicity of G are equivalent.

THEOREM 6. *If I is almost periodic on X, then I is equicontinuous on X.*

Proof. Let $\varepsilon>0$ be arbitrary. From the almost periodicity there is a relatively dense sequence $\{i_n\}$ and a number B such that $0<i_n-i_{n-1}<B$ and $\varrho(x,xi_n)<\varepsilon/3$ for each integer n and each $x\in X$. To show the equicontinuity let $x\in X$ and $i\in I$ be arbitrary. Since for each n, i_n is continuous, there is a $\delta_n>0$ such that whenever $y\in X$ such that $\varrho(x,y)<\delta_n$, then $\varrho(xi_n,yi_n)<\varepsilon/3$. Let $\delta=\min\delta_n$ for $n=0,1,2,\ldots,B-1$. Let k be taken so that $i_{k-1}<-i\leqslant i_k$, and it follows that $0\leqslant i+i_k<B$. Now if y is taken such that $\varrho(x,y)<\delta\leqslant\delta_{i+i_k}$, then $\varrho(xii_k,yii_k)<\varepsilon/3$. From the almost periodicity $\varrho(xi,xii_k)=\varrho[(xi),(xi)i_k]<\varepsilon/3$, and similarly $\varrho(yi,yii_k)<\varepsilon/3$. It follows from the triangle inequality that $\varrho(xi,yi)\leqslant\varrho(xi,xii_k)+\varrho(xii_k,yii_k)+\varrho(yi,yii_k)<\varepsilon$, and thus I is equicontinuous on X.

That pointwise periodicity may not be substituted for almost periodicity in Theorem 6 is illustrated by the following example where I is even pointwise periodic on a compact space X without being equicontinuous.

Example 3. Let X consist of the following subset of E_2 (expressed in polar coordinates (r,ϑ)): $p_{2n}=(1-1/2n,0)$, P is the set $(1,\vartheta)$ for $0\leqslant\vartheta<2\pi$, with the transformation I defined as

$$p_{2n}1=(1-1/2n,\vartheta+\pi/n)\quad\text{and}\quad p1=p\ \text{for}\ p\in P.$$

Let $X=(\bigcup_{n=1}^{\infty}p_{2n}I)\cup P$. I fails to be equicontinuous for any $p\in P$, $p\neq(1,0)$.

We turn now to the question of extending the properties of periodicity, almost periodicity and recurrence from K to X. From the continuity of the transformations it is obvious that if S is periodic on K, then S is periodic on X. We combine the almost periodic and recurrent cases into one theorem.

THEOREM 7. *If S is almost periodic* (*recurrent*) *on K, then S is almost periodic* (*recurrent*) *on X.*

Proof. Let $\varepsilon>0$, $x\in X$ be arbitrary. Then by hypothesis there is a syndetic (extensive) set $A\subseteq S$ such that $\varrho(k,ks)<\varepsilon/3$ for all $k\in K$, $s\in A$. Now let s be an arbitrary element of A. By the continuity of s at x, there is a $\delta>0$ such that $\delta<\varepsilon/3$ and $\varrho(xs,ys)<\varepsilon/3$ whenever $\varrho(x,y)<\delta$ and $y\in X$. Since $\overline{K}=X$, an element $k\in K$ may be selected so that $\varrho(x,k)<\delta<\varepsilon/3$. Then, by the triangle inequality, $\varrho(x,xs)\leqslant\varrho(x,k)+\varrho(k,ks)+\varrho(ks,xs)<\varepsilon$. Since s was an arbitrary element of the syndetic (extensive) set A, this establishes the almost periodicity (recurrence) of S on X.

In our next example we show that pointwise periodicity, pointwise almost periodicity and pointwise recurrence do not generally extend to X from K. In the example we give a flow I which has all of these properties on K but has none of these properties on the compact space X.

Example 4. Let X be the set of all functions from the integers onto the two symbols a,b, and let the metric be defined as follows, where functional notation is employed for clarity:

$$\varrho(x,y)=\begin{cases}0, & \text{when } x=y;\\ 1, & \text{when } x(0)\neq y(0);\\ 1/(j+1), & \text{when } x(n)=y(n) \text{ for } n=0,\pm1,\pm2,\ldots,\pm j \text{ but}\\ & x(j+1)\neq y(j+1) \text{ or } x(-j-1)\neq y(-j-1).\end{cases}$$

Let I be defined as follows: $x(n)1=x(n+1)$. This example is well-known and properties of the flow are found in [5]. If K is taken to be the set of all periodic points, it is easy to show that $\overline{K}=X$. I fails to be recurrent, even, at the point where $x(0)=a$, $x(n)=b$ for $n\neq0$.

Before considering pointwise recursive properties further we wish to recall the property of non-wandering, the weakest recursive property considered in the paper.

S is said to be *non-wandering at* $x\in X$, or x is a *non-wandering point*, provided that to each $\varepsilon>0$ corresponds an extensive set $A\subset S$ such that $U_\varepsilon(x)\cap U_\varepsilon(x)s\neq0$ for each $s\in A$. S is *pointwise non-wandering on*

$Y \subset X$ if S is non-wandering at each $y \in Y$. S is *non-wandering on* $Y \subset X$ provided that to each $\varepsilon > 0$ corresponds an extensive set $A \subseteq S$ such that $U_\varepsilon(y) \frown U_\varepsilon(y)s \neq 0$ for each $s \in A$, $y \in Y$. It is obvious that if S is recurrent at x, then S is non-wandering at x. Under certain hypotheses the converse is true.

THEOREM 8. *If S is equicontinuous and non-wandering at $x \in X$, then S is recurrent at x. If S is equiuniformly continuous and non-wandering on $Y \subseteq X$, then S is recurrent on Y.*

Proof. Let $\varepsilon > 0$ be arbitrary. Then by the equicontinuity at x, there is a $\delta > 0$ such that $\delta < \varepsilon/2$ and $\varrho(xs, ys) < \varepsilon/2$ for each $s \in S$ whenever $\varrho(x, y) < \delta$. Now since S is non-wandering at x, there is an extensive set $A \subset S$ such that $U_\delta(x) \frown U_\delta(x)s \neq 0$ for each $s \in A$. Let $s \in A$ be arbitrary, and corresponding to this s let $v \in U_\delta(x) \frown U_\delta(x)s$. Then there is a $y \in U_\delta(x)$ such that $v = ys$, and so $\varrho(xs, v) < \varepsilon/2$. Since $\varrho(x, v) < \varepsilon/2$, it follows that $\varrho(x, xs) \leqslant \varrho(x, v) + \varrho(v, xs) < \varepsilon$, and since s was any element of the extensive set A, S is recurrent at x.

The proof of the second half of the theorem is similar.

We are now ready to give the theorem for non-wandering S analogous to Theorem 7. Non-wandering differs from the other recursive properties in that pointwise non-wandering on K implies this property on X.

THEOREM 9. *If S is non-wandering on K, then S is non-wandering on X. If S is pointwise non-wandering on K, then S is pointwise non-wandering on X.*

Proof. Let $\varepsilon > 0$ be arbitrary. By hypothesis there is an extensive set $A \subset S$ such that $U_{\varepsilon/2}(k) \frown U_{\varepsilon/2}(k)s \neq 0$ for each $s \in A$, $k \in K$. Let $x \in X$ be arbitrary. Then there exists a $k \in K$ for which $\varrho(x, k) < \varepsilon/2$. Then $U_\varepsilon(x) \supset U_{\varepsilon/2}(k)$ and $U_\varepsilon(x)s \supset U_{\varepsilon/2}(k)s$ for each $s \in A$. Therefore $U_\varepsilon(x) \frown U_\varepsilon(x)s \supset U_{\varepsilon/2}(k) \frown U_{\varepsilon/2}(k)s \neq 0$, so that S is non-wandering on X.

The proof for the second part is similar. For the case $S = G$, it also follows from a remark in [4].

In the presence of equicontinuity of S on X or equiuniform continuity on K, either pointwise recurrence or pointwise almost periodicity on K implies the respective property on X.

THEOREM 10. *If S is equicontinuous on X or equiuniformly continuous on K and if S is pointwise almost periodic (pointwise recurrent) on K, then S is pointwise almost periodic (pointwise recurrent) on X.*

Proof. Assume S is equicontinuous on X. Let $\varepsilon > 0$, $x \in X$ be arbitrary. Then there is a $\delta > 0$ such that $\delta < \varepsilon/3$ and $\varrho(xs, ys) < \varepsilon/3$ whenever $y \in X$, $\varrho(x, y) < \delta < \varepsilon/3$. Since $\bar{K} = X$, there is an element $k \in K$ for which $\varrho(x, k) < \delta < \varepsilon/3$. Now, using the hypothesis of pointwise almost periodicity (pointwise recurrence) there is a syndetic (extensive) set $A \subseteq S$ such that $\varrho(k, ks) < \varepsilon/3$ for each $s \in A$. Applying the triangle inequality, $\varrho(x, xs) \leqslant \varrho(x, k) + \varrho(k, ks) + \varrho(ks, xs) < \varepsilon$, the almost periodicity (recurrence) of the arbitrary point x is established.

In case S is equiuniformly continuous on K then, as has been observed, S is equiuniformly continuous on X and the theorem is proved.

If $S = R$ or $S = I$, the hypotheses of Theorem 10 may be weakened.

THEOREM 11. *If R is ε-related on X or uniformly ε-related on K and if R is pointwise almost periodic (pointwise recurrent) on K, then R is pointwise almost periodic (pointwise recurrent) on X.*

The proof of Theorem 11 is similar to that of Theorem 10 and will be omitted.

The convergence of ε-related sequences of transformations was considered in Theorem 5. Using that theorem we are able to secure the following result:

THEOREM 12. *If X is complete and $f_n(X) \subset X$ for $n = 1, 2, \ldots$ and if the sequence $\{f_n(X)\}$ is ε-related on X (uniformly ε-related on X) and converges to $F(x)$ on K, then $f_n(x)$ converges on X, and the limit function is continuous (uniformly continuous) on X.*

Proof. Let $\varepsilon > 0$, $x \in X$ be arbitrary. Then there is a $\delta > 0$ such that if $\varrho(x, y) < \delta$, $y \in X$, then there is an $N = N(y) > 0$ for which $\varrho(f_n(x), f_n(y)) < \varepsilon/3$, whenever $n > N$. Since $\bar{K} = X$, there is an element $k \in K$ for which $\varrho(x, k) < \delta$. Also, since $f_n(k)$ converges to $F(k)$, there is a number $N' > 0$ such that $\varrho(f_m(k), f_n(k)) < \varepsilon/3$ for $m, n > N'$. Let $m, n > \max(N, N')$. Then by the triangle inequality

$$\varrho(f_m(x), f_n(x)) \leqslant \varrho(f_m(x), f_m(k)) + \varrho(f_m(k), f_n(k)) + \varrho(f_n(k), f_n(x)) < \varepsilon,$$

so that $\{f_n(x)\}$ is a Cauchy sequence and consequently convergent since X is complete. As a consequence of Theorem 5, $F(x)$ is continuous on X (uniformly continuous on X).

We now summarize those results of this section that pertain to extending properties of S from K to $\bar{K} = X$. If, on K, S is equiuniformly continuous, uniformly ε-related, periodic, almost periodic, recurrent, non-wandering, or pointwise non-wandering, then S has the respective property on X. If S is pointwise almost periodic or pointwise recurrent on K and if, also, S is either equiuniformly continuous on K or equicontinuous on X, then S has the respective recursive property on X. We have not shown sufficient conditions for S pointwise periodic on K to imply S pointwise periodic on X. According to Example 4, compactness is not enough to insure this inheritance. We now give an example to show that S may be pointwise periodic on K, equiuniformly continuous on the compact space X yet S is not pointwise periodic on X. This example was suggested by E. E. Floyd [2].

Example 5. Let X be all $(x_1, x_2, \ldots)$ such that

(1) x_i is an integer $1, 2, 3, \ldots, 2^i$ and $x_{i+1} = x_i (\operatorname{mod} 2^i)$

or

(2) x_i is an integer $1, 2, 3, \ldots, 2^i$ and $x_{i+1} = x_i (\operatorname{mod} 2^i)$, $i = 1, 2, \ldots, m-1$ while $x_i = 0$, $i = m, m+1, \ldots$

For example, X contains the following three points: $(1, 1, 5, \ldots)$, $(1, 3, 7, \ldots)$, $(1, 3, 3, 0, 0, \ldots)$.

If $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ are points of X,

$$\varrho(x, y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{4^i}$$

provides a metric in X under which X is compact.

For $x = (x_1, x_2, \ldots) \in X$ define $S = I$ as $x1 = y = (y_1, y_2, \ldots)$ wherein $y_i = x_i - 1$ reduced $\operatorname{mod} 2^i$ in case $x_i \neq 0$. If $x_i = 0$ let $y_i = 0$. Under S the points of (2) are periodic and it is easily seen that they are dense in X. Since S is equiuniformly continuous on the points of (2), S is equiuniformly continuous on X. However, no point of (1) is periodic.

5. Limits of collections of orbits. Let H be any infinite collection of subsets of X not necessarily distinct. The set of all $x \in X$ such that for every $\varepsilon > 0$, $U_\varepsilon(x)$ contains at least one point of infinitely many sets of H is called the *limit superior* of H, written $\limsup H$. The set of all $x \in X$ such that for every $\varepsilon > 0$, $U_\varepsilon(x)$ contains at least one point of all but a finite number of sets of H is called the *limit inferior* of H, written $\liminf H$. In case $\liminf H = \limsup H$, then H is said *to converge*, and the set is called the *limit* of H, written $\lim H$.

It is noted in [8] that $\liminf H \subset \limsup H$, and that each set is closed. It is easily seen that in case H is a collection of orbits, then $\limsup H$ and $\liminf H$ are invariant sets.

THEOREM 13. *If H is a collection of orbits of points of X such that $\liminf H = L$, and if G is equicontinuous at some point $x \in L$, then $L = \overline{xG}$ so that G is weakly transitive on L. Thus, if G is equicontinuous on L, then L is a minimal set.*

Proof. Since L is closed and invariant, it follows that $\overline{xG} \subset L$. To complete the proof, let $\varepsilon > 0$, $y \in L$ be arbitrary. Then by the equicontinuity there is a $\delta > 0$ such that if $\varrho(x, u) < \delta$, then $\varrho(xg, ug) < \varepsilon/2$ for each $g \in G$. Since $x, y \in L$, each of the sets $U_\delta(x)$ and $U_{\varepsilon/2}(y)$ intersects all but a finite number of the orbits in H so that there is an orbit $zG \in H$ intersecting both $U_\delta(x)$ and $U_{\varepsilon/2}(y)$. Let $u \in zG \cap U_\delta(x)$ and $ug \in zG \cap U_{\varepsilon/2}(y)$. Then by the equicontinuity $\varrho(xg, ug) < \varepsilon/2$, and, using the triangle inequality, $\varrho(y, xg) \leqslant \varrho(y, ug) + \varrho(ug, xg) < \varepsilon$, $L \subset \overline{xG}$, and therefore $L = \overline{xG}$, from which it follows that G is weakly transitive on L.

Since a periodic orbit is compact and therefore closed, the next theorem follows from Theorem 13.

THEOREM 14. *If H is a collection of orbits of points of X such that $\liminf H = L$ and if G is equicontinuous and periodic at some point $x \in L$, then L is an orbit.*

In Theorem 14 almost periodicity may not be substituted for periodicity at a point even with additional hypotheses, as is shown in the following example, where X is compact, $G = I$ equiuniformly continuous and almost periodic on X, and $\lim H$ exists but is not an orbit.

Example 6. Let X be the subset of E_2 for which (in polar coordinates (r, ϑ)) $r = 1$ and let I be defined as the rotation through 1 radian, $(1, \vartheta)1 = (1, \vartheta + 1)$. It is easily seen that X is compact and the I is equiuniformly continuous, and it follows from a theorem in [3] that I is almost periodic on X. If H is taken to be the set of all orbits of points of X, it is obvious that $\lim H = X$ is not an orbit.

Thus far the question of the convergence of a collection of orbits has not been considered. In the presence of equicontinuity a sufficient condition for convergence is now shown.

THEOREM 15. *Let* (a) *$Y = \{y_n\}$, where $y_n \in X$ and the sequence y_n converges to $x \in X$,* (b) *H be the collection of orbits yG for all $y \in Y$,* (c) *G equicontinuous at x, then $\lim H = \overline{xG}$.*

Proof. It is obvious that $x \in \liminf H$, which is a closed invariant set, so that $\overline{xG} \subset \liminf H \subset \limsup H$. Now let $\varepsilon > 0$, $p \in \limsup H$ be arbitrary. By (c) there is a $\delta > 0$ such that whenever $\varrho(x, u) < \delta$, then $\varrho(xg, ug) < \varepsilon/2$ for each $g \in G$. Select a sequence $\{y_n\}$ such that each $y_n \in Y$ and $y_n G \cap U_{\varepsilon/2}(p) \neq 0$. Then by (a) for some n, $\varrho(x, y_n) < \delta$. Let $y_n g \in U_{\varepsilon/2}(p)$ so that $\varrho(p, y_n g) < \varepsilon/2$. By the triangle inequality $\varrho(p, xg) \leqslant \varrho(p, y_n g) + \varrho(y_n g, xg) < \varepsilon$, and $\limsup H \subset \overline{xG} \subset \liminf H \subseteq \limsup H$, from which it follows that $\lim H = \overline{xG}$.

The strength of the hypothesis of equicontinuity is illustrated by the following example where X is compact, I is pointwise periodic but the limit does not exist.

Example 7. To the space of Example 3 add the fixed points $p_{2n-1} = \big(1 - 1/(2n-1), 0\big)$ for $n = 1, 2, 3, \ldots$ The sequence of points $\{p_n\}$ converges to the point $(1, 0)$, but $\liminf p_n G$ is the point $(1, 0)$ whereas $\limsup p_n G$ is the set P of all points $(1, \vartheta)$ for $0 \leqslant \vartheta < 2\pi$.

In case C is a component of $Y \subset X$ and $Y = \bigcup_{c \epsilon C} cS$, then Y is called a *component orbit*. It is obvious that the orbit of a point is a component orbit and that a component orbit is an invariant set. The following theorem is an immediate consequence of the continuity of the transformations and gives a characterization of component orbits:

THEOREM 16. *A necessary and sufficient condition that an invariant set $Y \subset X$ be a component orbit is that $yG \cap C \neq 0$ for each component C of Y and each $y \epsilon Y$.*

In the case where X is compact, $S = I$, and I is pointwise periodic on X, Schweigert [6] has shown that the limit of a sequence of component orbits is itself a component orbit. It is easy to see that the same result is valid for the limit of any collection of component orbits. It is interesting to note that this is one situation where equicontinuity is no substitute for compactness. This is shown in the following example where I is periodic and equiuniformly continuous on the complete space X.

Example 8. Let X be E_2 and let H consist of the sequence of ellipses $x^2/(1-1/n)^2+y^2/n^2 = 1$ for $n = 2, 3, 4, \ldots$

I is defined so that $(x, y)1 = (x, y)$. Thus every point of X is fixed. Each ellipse is obviously a component orbit, but $\lim H$ is the pair of parallel lines $x^2-1 = 0$, which is not a component orbit.

The case where G is connected (including the case $G = R$) is easily treated. In this case pointwise periodicity plays no role.

THEOREM 17. *If X is compact, G is connected and H is a convergent sequence of component orbits, then $\lim H$ is a component orbit.*

Proof. Since G is connected, each component orbit is connected. It is known [8] that under these conditions $\lim H$ is connected. By a previous remark H is invariant and consequently a component orbit.

Using the fact that if G is connected an orbit closure is also connected and Theorem 13 we secure the following theorem:

THEOREM 18. *If H is a collection of orbits of points of X such that $\liminf H = L$ and if G is connected and equicontinuous at some point $x \epsilon L$, then L is a component orbit.*

That the last theorem is not valid for a collection of component orbits is shown by Example 8, defining G exactly as I so that each transformation of G is the identity transformation.

6. Special properties of ordered transformation groups. Whenever S is simply ordered, recurrence properties may be defined in terms of the order relationship. This has already been done in case $S = R$ or $S = I$. Some additional properties of these special cases will be considered in this section.

R is *positively recurrent at $x \epsilon X$* provided that to each $\varepsilon > 0$ corresponds an unbounded sequence $\{r_n\}$ such that $0 < r_n - r_{n-1} < r_n$, $r_n \epsilon R$, and $\varrho(x, xr_n) < \varepsilon$ for $n = 1, 2, 3, \ldots$ If, in addition, $\{r_n - r_{n-1}\}$ is bounded, R is said to be *positively almost periodic at $x \epsilon X$*. Negative recurrence and other related properties are defined in an obvious manner. If follows immediately from the group property that if R is positively almost periodic at $x \epsilon X$, then R is positively almost periodic at xr for each $r \epsilon R$, and similarly for the other related properties. In the presence of equicontinuity a stronger property may be shown.

THEOREM 19. *If R is equicontinuous at $x \epsilon X$ and either positively or negatively almost periodic (positively or negatively recurrent) at x, then R is almost periodic (recurrent) at x. Thus, if R is equicontinuous on $Y \subset X$ and is either positively or negatively almost periodic (positively or negatively recurrent) on Y, then R is pointwise almost periodic (pointwise recurrent) on Y.*

Proof. Let R be positively almost periodic at x and let $\varepsilon > 0$ be arbitrary. By the equicontinuity there is a $\delta > 0$ such that whenever $\varrho(x, y) < \delta$, then $\varrho(xr, yr) < \varepsilon$ for each $r \epsilon R$. Then by hypothesis there is an unbounded sequence $\{r_n\}$ such that $0 < r_n - r_{n-1} < r_n$, $\{r_n - r_{n-1}\}$ is bounded and $\varrho(x, xr_n) < \delta$ for $n = 1, 2, 3, \ldots$, and it follows that $\varrho(xr, xr_n r) < \varepsilon$ for each $r \epsilon R$, and in particular $\varrho[x(-r_n), xr_n(-r_n)] = \varrho[x(-r_n), x] < \varepsilon$, so that R is negatively almost periodic at x and consequently almost periodic at x. The proof in the other case is identical.

The following theorem is easily established by a proof similar to that for Theorem 19:

THEOREM 20. *If R is equiuniformly continuous on $Y \subset X$ and either positively or negatively almost periodic (positively or negatively recurrent) on Y, then R is almost periodic (recurrent) on Y.*

A point $y \epsilon X$ is called an *ω-limit point* (*α-limit point*) *of the orbit xR* provided that there exists an unbounded sequence $\{r_n\}$ such that

$$0 < r_n - r_{n-1} < r_n \quad (0 > r_n - r_{n-1} > r_n), \qquad r_n \epsilon R \quad \text{for} \quad n = 1, 2, \ldots$$

and $\lim xr_n = y$. The set of all ω-limit points (α-limit points) of the orbit xR will be denoted by xR_ω (xR_α). It is easily seen that each of the sets xR_ω and xR_α is closed and invariant and that $\overline{xR} = xR \cup xR_\omega \cup xR_\alpha$. It is remarked in [4] under more restrictive conditions that if R is positively (negatively) recurrent at x, then $\overline{xR} = xR_\omega$ (xR_α) and that if R is recurrent at x, then $\overline{xR} = xR_\omega = xR_\alpha$. The same result is valid here. It is now shown that in the presence of equicontinuity, it holds without recurrence.

THEOREM 21. *If R is equicontinuous on $\overline{xR}$ and $xR_\omega \cap xR_a \neq 0$, then $\overline{xR} = xR_\omega = xR_a$ and R is recurrent at x.*

Proof. Assume $xR_\omega \neq 0$ and let $y \in xR_\omega$, $\varepsilon > 0$ be arbitrary. By hypothesis there is a $\delta > 0$ such that whenever $\varrho(y, u) < \delta$, then $\varrho(yr, ur) < \varepsilon$ for each $r \in R$. Since $y \in xR_\omega$, there is an unbounded increasing positive sequence $\{r_n\}$ such that $\varrho(y, xr_n) < \delta$ from which it follows that

$$\varrho[y(-r_n), xr_n(-r_n)] = \varrho[y(-r_n), x] < \varepsilon,$$

so that it is seen that $x \in yR_a \subset \overline{yR}$. Now since $y \in xR_\omega$, a closed and invariant set, then $\overline{yR} \subset xR_\omega$, and therefore $x \in xR_\omega$ which is also closed and invariant so that $\overline{xR} \subset xR_\omega$, and it follows that $\overline{xR} = xR_\omega$.

Since $x \in xR_\omega$, then $x \in xR_a$ from Theorem 19, and similarly $\overline{xR} = xR_a$. The proof is identical in case $xR_a \neq 0$.

All of the results in this section are established in the same manner when G is any simply ordered group, with appropriate modifications of the definitions.

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PROBLEMS ON SEMIGROUPS

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P 326. Is it possible to construct a continuous associative multiplication on the closed n-cell ($n \geqslant 2$) such that the boundary consists of exactly those elements satisfying $x^2 = x$?

P 327. Is it possible to construct a continuous associative multiplication on an n-sphere in such a way that (i) every element is the product of two elements, (ii) there is a zero-element.

For $n = 1$ the answer is negative, see [3].

P 328. If G is a compact totally disconnected metrizable group does there exist a compact connected-acyclic one-dimensional metrizable space T and on T a continuous associative multiplication with a two-sided unit such that the maximal subgroup of T which contains the unit coincides with G and such that G is the set of endpoints of T?

If G is the Cantor group the answer is affirmative (unpublished). A related question has been considered and solved by Koch and McAuley (also unpublished).

P 329. Suppose that Euclidean n-space R^n is supplied with a continuous associative multiplication with unit and that there exists a compact connected subset G of R^n which contains the unit and which is a subgroup of R^n under the given multiplication. Is it possible that G can be "self-linked" in any reasonable way? (Cf. [1] for $n = 3$).

P 330. If S is a compact connected locally connected metrizable one-dimensional semigroup with unit, then it is known that S is either a dendrite or contains exactly one simple closed curve which coincides with the minimal ideal of S. (The details of the proof are unpublished but see [6]). Is there an analogous proposition for higher dimensions?

P 331. If S is a compact connected commutative semigroup with unit, all of whose elements satisfy $x^2 = x$, does S have the fixed point property?

P 332. If S is a compact semigroup then the minimal ideal K of S is a retract of S in the sense of Borsuk (see [9]). Examples will show