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ON DEVELOPABLE SETS AND ALMOST-LIMIT POINTS

BY

E. MARCZEWSKI (WROCŁAW) AND T. TRACZYK (WARSAW)

Introduction. The notions and theorems contained in this paper have arisen from some problems of the approximation of sets by developable sets (i. e. sets which are F_σ and G_δ simultaneously) and the approximation of functions by functions of the first class.

Every measurable set is almost equal to an F_σ and to a G_δ; nevertheless it is easy to define a set H which is measurable (and even an F_σ or a G_δ) but which fails to be almost equal to any developable set. E. g. we may consider as H a set which is of positive measure on every interval, and whose complement has the same property. It turns out that this example is, roughly speaking, the most general one: A set E is almost equal to a developable set if and only if, for every closed F, there exists an interval I such that at least one of the sets EFI and $FI-E$ is of measure zero (see 3(i)).

This theorem, as well as all the others in this paper, is formulated not for the class of sets of measure zero but, more generally, for an arbitrary σ-ideal of sets. Thus, they are valid also for the class of all denumerable sets, the class of all sets of the first category, etc.

The reasoning used in sections 2 and 3 is analogous to that followed in the monograph of C. Kuratowski [2], especially p. 64-68.

In the following sections (4-7) we introduce the notions of almost continuity and almost-limit points and we apply them to the closure algebras in the sense of Sikorski [5].

Finally, in section 8, we formulate some simple connexions between those notions and the problem of approximation of functions by functions of the first class. The theorem converse to our proposition 8(iii) has recently been proved by Lederer [3]: If for a real function f and every closed set F, the partial function $f|F$ has at least one almost-limit point, then f is equal almost everywhere to a function of the first class. That is the answer to the problem raised by one of us. A generalization of this theorem for mappings in metric spaces is to be found in paper [6].

1. Generalized closure. We shall consider two separable metric spaces $\mathfrak{X}$ and $\mathfrak{Y}$. Moreover, we suppose that $\mathfrak{X}$ is complete; this hypothesis is essential only in sections 3 and 8. Let N be a σ-ideal of subsets of $\mathfrak{X}$, i. e. a hereditary and enumerable additive class of subsets of $\mathfrak{X}$. We denote the union, the intersection, the complement, the difference, and the symmetric difference of sets by $+$, $\cdot$, $'$, $-$, and $\dot{-}$ respectively. The following terminology will be used: if $A\subset\mathfrak{X}$ and $B\subset\mathfrak{X}$ then

(t_1) A and B are *almost equal* if $A\dot{-}B = AB'+A'\cdot B\epsilon N$,
(t_2) A and B are *almost disjoint* if $AB\epsilon N$,
(t_3) A is *almost contained* in B if $A-B\epsilon N$,
(t_4) two mappings f and g of $\mathfrak{X}$ into $\mathfrak{Y}$ are equal *almost everywhere* or, shortly, *almost-equal* if $f(x) = g(x)$ for $x\epsilon N'$, where $N\epsilon N$.

The symbol $P(E)$ will denote a *generalized closure* ([2], p. 34-35), i. e. a subset of $\mathfrak{X}$ such that $x\epsilon P(E)$ if and only if $EU\,\bar{\epsilon}\,N$ for every neighbourhood U of x.

If N contains only one element (the empty set) then $P(E)$ is identical with the closure $\bar{E}$ of E in the ordinary sense.

The following propositions are obvious:

(i) $P(0) = 0$,
(ii) $A\subset B \Rightarrow P(A)\subset P(B)$,
(iii) $P(A+B) = P(A)+P(B)$,
(iv) $PP(E) = P(E)$,
(v) $\overline{P(E)} = P(E)\subset \bar{E}$,
(vi) $E-P(E)\epsilon N$,
(vii) $A\dot{-}B\epsilon N \Rightarrow P(A) = P(B)$.

2. Theorem on separation. A subset A of $\mathfrak{X}$ is said to be *developable* if there exists a transfinite decreasing sequence $\{F_\xi\}$ of closed sets such that

$$A = (F_0-F_1)+(F_2-F_3)+\ldots+(F_\xi-F_{\xi+1})+\ldots$$

Let A and B be two arbitrary subsets of $\mathfrak{X}$. We define through transfinite induction:

$$X_0 = \mathfrak{X},\qquad X_{\xi+1} = P(AX_\xi)\cdot P(BX_\xi)$$

and

$$X_\lambda = \prod_{\xi<\lambda} X_\xi,\qquad \text{if } \lambda \text{ is a limit number.}$$

Since by 1(ii) and 1(v)

$$(*)\qquad X_{\xi+1}\subset P(BX_\xi)\subset P(X_\xi)\subset \bar{X}_\xi = X_\xi,$$

$\{X_\xi\}$ is a decreasing sequence of closed sets. Consequently there exists an enumerable ordinal number α such that $X_\alpha = X_{\alpha+1}$, i. e.

$$(**)\qquad X_\alpha = P(AX_\alpha)\cdot P(BX_\alpha).$$

Let $Q_\xi = X_\xi-P(AX_\xi)$ and $R_\xi = X_\xi-P(BX_\xi)$. By $(*)$ $R = \sum_{\xi<\alpha} R_\xi$ is a developable set. On account of 1(vi)

$$AQ_\xi = A\big(X_\xi-P(AX_\xi)\big) = AX_\xi-P(AX_\xi)\epsilon N$$

and then A and $Q = \sum_{\xi<\alpha} Q_\xi$ are almost disjoint. In a similar way B and R are almost disjoint.

Since

$$\mathfrak{X} = Q+R+X_\alpha,$$

A is almost contained in $R+X_\alpha$.

From this fact and from $(**)$ follows the theorem on separation [1]:.

(i) *If the equality*

$$X = P(AX)\cdot P(BX)$$

implies $X\epsilon N$, then there exists a developable set R such that R and B are almost disjoint and A is almost contained in R.

3. Approximation by developable subsets.

(i) *The following conditions for a subset E of $\mathfrak{X}$ are equivalent:*

(d_1) *E is almost developable, i. e. there exists a developable set H almost equal to E;*

(d_2) *E is almost equal to a set which is F_σ and G_δ;*

(d_3) *for every closed non-empty set F the set $P(EF)\cdot P(E'\cdot F)$ is non-dense in F;*

(d_4) *for every closed non-empty set F*

$$P(EF)\cdot P(E'\cdot F) \neq F.$$

The implication $(d_1)\Rightarrow(d_2)$ follows from the fact that every developable set is an F_σ and a G_δ simultaneously [2].

[1] That is another form of a theorem of Sikorski [5], p. 173, 3.4.

[2] Compare a simple proof in [2], p. 269. Let us remark that, in the present proof, we do not use the converse implication (which follows from (i) in the case where N contains only one element, i. e. the empty set).

To prove the implication $(d_2) \Rightarrow (d_3)$ suppose that the set $K = P(EF)\cdot P(E'\cdot F)$ is not non-dense in F and that E and a developable set D are almost identical. On account of 1(vii) we have

$$(*) \qquad K = P(DF)\cdot P(D'\cdot F).$$

Since K is a closed set, there is a subset V of K which is open in F. By (*) and 1(v) DV and $D'\cdot V$ are F_σ and dense in V. This, however, contradicts the Baire theorem.

The implication $(d_3) \Rightarrow (d_4)$ is trivial and $(d_4) \Rightarrow (d_1)$ follows from the theorem on separation 2(i) by putting $A = E$ and $B = E'$. This completes the proof of the theorem.

4. Almost-limit points. A point $y_0 \epsilon \mathfrak{Y}$ is called an *almost-limit value* of a mapping f of $\mathfrak{X}$ into $\mathfrak{Y}$ at the point x_0, if there exists a mapping g of $\mathfrak{X}$ into $\mathfrak{Y}$, almost equal to f and having at x_0 the limit y_0. If there exists an almost-limit value of f at x_0 we say that x_0 is an *almost-limit point* of f.

If $x_0 \epsilon P(\mathfrak{X})$ there is at most one almost-limit value of f at x_0. In fact, if in this case two mappings g_1 and g_2 are almost equal to f and have at x_0 the limits y_1 and y_2 respectively, then there exists in every neighbourhood of x_0 a point x' with

$$g_1(x') = g_2(x') = f(x'),$$

whence $y_1 = y_2$.

If $x_0 \notin P(\mathfrak{X})$, then the almost-limit value of f is not determined; more precisely, every point $y_0 \epsilon \mathfrak{Y}$ is the almost-limit value of f.

Let us denote by $Q_1, Q_2, \ldots$ and $R_1, R_2, \ldots$ the bases of neighbourhoods of $\mathfrak{X}$ and $\mathfrak{Y}$ respectively.

(i) *The following conditions for a point y_0 of $\mathfrak{Y}$ are equivalent:*

(l_1) *y_0 is the almost-limit value of f at x_0;*

(l_2) *for every subset A of $\mathfrak{X}$, if $x_0 \epsilon P(A)-A$ then $y_0 \epsilon \overline{f(A)}$;*

(l_3) *for every neighbourhood V of y_0 there exists a neighbourhood U of x_0 such that $U\cdot f^{-1}(V')-\{x_0\} \epsilon N$, in other words $x_0 \notin P\big(f^{-1}(V')-\{x_0\}\big)$;*

(l_4) *for every natural number n with $y_0 \epsilon R_n$ there exists a natural number m such that*

$$x_0 \epsilon Q_m \quad \textit{and} \quad Q_m f^{-1}(R'_n)-\{x_0\} \epsilon N.$$

The equivalence $(l_3) \Longleftrightarrow (l_4)$ being trivial, we will prove successively $(l_1) \Rightarrow (l_2) \Rightarrow (l_3) \Rightarrow (l_1)$.

If (l_1) is satisfied, then there exists a mapping g almost equal to f, having the limit y_0 at x_0. Let us suppose that $A \subset \mathfrak{X}$ and $x_0 \epsilon P(A)-A$, and put

$$A_0 = A\cdot\{x : f(x) = g(x)\}$$

whence $A - A_0 \epsilon N$ and, by 1(vii),

$$x_0 \epsilon P(A_0) - A_0 \subset \bar{A}_0 - A_0$$

and consequently

$$y_0 \epsilon \overline{g(A_0)} = \overline{f(A_0)} \subset \overline{f(A)}.$$

Condition (l_2) is thus satisfied.

In order to prove the implication $(l_2) \Rightarrow (l_3)$, let us suppose that (l_2) is satisfied and (l_3) is not. Therefore there exists a neighbourhood V of y_0, such that $x_0 \epsilon P(f^{-1}(V')-\{x_0\})$. Putting $A = f^{-1}(V')-\{x_0\}$ and applying (l_2) we obtain

$$y_0 \epsilon \overline{f(A)} \subset \overline{ff^{-1}(V')} \subset \overline{V'} = V',$$

which is not true.

Let us suppose, finally, that (l_3) is satisfied and let $V_1, V_2, \ldots$ be a sequence of neighbourhoods of y_0 such that

$$(*) \qquad y_0 = V_1\cdot V_2\cdot\ldots$$

By (l_3) there is, for every n, a neighbourhood U_n of x_0 such that $U_n f^{-1}(V'_n)-\{x_0\} \epsilon N$. Put

$$Z_n = U_n f^{-1}(V'_n)-\{x_0\}, \qquad Z = Z_1+Z_2+\ldots,$$

$$g(x) = \begin{cases} f(x) & \text{for} \quad x \epsilon \mathfrak{X}-Z, \\ y_0 & \text{for} \quad x \epsilon Z, \end{cases}$$

Consequently $U_n g^{-1}(V'_n)-\{x_0\} = 0$, whence $U_n-\{x_0\} \subset g^{-1}(V_n)$ and finally

$$g(U_n-\{x_0\}) \subset gg^{-1}(V_n) \subset V_n.$$

Hence, by (*), y_0 is the limit of g at x_0, and, since

$$f|\mathfrak{X}-Z = g|\mathfrak{X}-Z \quad \text{and} \quad Z \epsilon N,$$

condition (l_1) is fullfilled, q. e. d.

Obviously

(ii) *If f and g are almost equal, then they have the same almost-limit points and the corresponding almost-limit values.*

5. Almost continuity. We say that a mapping f of $\mathfrak{X}$ into $\mathfrak{Y}$ is *almost continuous* at x_0 if there exists a mapping g of $\mathfrak{X}$ into $\mathfrak{Y}$ such that $f(x_0) = g(x_0)$, f and g are almost equal and g is continuous at x_0.

In other words, f is almost continuous at x_0 if and only if $f(x_0)$ is an almost-limit value of f at x_0.

Therefore, putting in 4(i) $y_0 = f(x_0)$ and simplifying for this case conditions (l_2), (l_3), (l_4), we find that,

(i) *The following conditions are equivalent*:

(c_1) *f is almost continuous at x_0,*

(c_2) *for every subset A of $\mathfrak{X}$, if $x_0 \in P(A)$, then $f(x_0) \in \overline{f(A)}$,*

(c_3) *for every neighbourhood V of $f(x_0)$ there exists a neighbourhood U of x_0 such that $Uf^{-1}(V') \in N$,*

(c_4) *for every natural number n with $f(x_0) \in R_n$ there exists a natural number m such that*

$$x_0 \in Q_m \quad \text{and} \quad Q_m f^{-1}(R_n') \in N.$$

Furthermore, let D_0 denote the set of points of $\mathfrak{X}$ at which the mapping f is not almost continuous, let $R_1, R_2, \ldots$ denote, as before, a basis of $\mathfrak{Y}$, and let us put $S_n = R_n'$. Condition (c_4) can now be formulated as follows: for every n either $f(x_0) \in S_n$ or $x_0 \notin P\big(f^{-1}(S_n)\big)$. Consequently

$$D_0' = \prod_{n=1}^{\infty} \{f^{-1}(S_n) + P\big(f^{-1}(S_n)\big)'\},$$

whence

$$\text{(ii)} \qquad D_0 = \sum_{n=1}^{\infty} \{P\big(f^{-1}(S_n)\big) - f^{-1}(S_n)\}.$$

6. The set of almost-limit points and the set of points of almost-continuity. Let D_1 denote the set of all points $x \in \mathfrak{X}$ which are not almost-limit points of a mapping f. Obviously

(i) $D_0 \supset D_1$.

In the subsequent part of this section we suppose that all one-point subsets of $\mathfrak{X}$ belong to the ideal N. Under this essential hypothesis, we obviously have

(ii) *x_0 is an almost-limit point of f if and only if there exists a mapping g continuous at x_0 and almost equal to f.*

(iii) *The set of points of almost continuity and the set of almost-limit points of a mapping f are almost equal; in other words $D_0 \dot{-} D_1 \in N$.*

In view of (i) it remains to prove that $D_0 - D_1 \in N$. If $x_0 \in D_0 - D_1$, then there exists a point $y_0 \in \mathfrak{Y}$ which is the almost-limit value of f at x_0, but which is different from $f(x_0)$. Thus, there exists a neighbourhood R_{n_0} of y_0 such that $f(x_0) \notin R_{n_0}$. In view of 4(i) (condition (l_4)) there exists a neighbourhood Q_{m_0} of x_0 such that

$$Q_{m_0} f^{-1}(R_{n_0}') - \{x_0\} \in N.$$

Since we supposed that $x_0 \in N$, we have $Q_{m_0} f^{-1}(R_{n_0}') \in N$ and, by the definition of R_{n_0}, $x_0 \in f^{-1}(R_{n_0}')$. Consequently

$$D_0 - D_1 \subset \sum Q_m f^{-1}(R_n'),$$

where the addition runs over all pairs (m, n) for which $Q_m f^{-1}(R_n') \in N$. Hence $D_0 \dot{-} D_1 \in N$, q. e. d.

Let us remark that (iii) (as well as (ii)) is not true if a certain one-point set $\{x_0\}$ does not belong to the ideal N. In fact, for a function f such that $f(x_0) = y_0$ and $f(x) = y_1 \neq y_0$ for $x \neq x_0$ we have $x_0 \in D_0 - D_1$, whence $D_0 - D_1 \notin N$.

7. Application to closure algebras. Let A be a Boolean σ-algebra. The Boolean operations on elements $A, B \in A$ will be denoted by $A+B$, AB and A'. The symbol $A-B$ will denote AB'. The symbol $\sum_{n=1}^{\infty} A_n$ will denote the union of elements $A_1, A_2, \ldots$ in A. The symbol $\prod_{n=1}^{\infty} A_n$ will denote $\big(\sum_{n=1}^{\infty} A_n'\big)'$.

A *closure algebra* is a Boolean σ-algebra A in which a closure operation is defined in such a way that the following axioms of Kuratowski are satisfied:

$$\text{I.} \quad \overline{A_1 + A_2} = \bar{A}_1 + \bar{A}_2, \qquad \text{II.} \quad A \subset \bar{A},$$
$$\text{III.} \quad \bar{O} = O, \qquad \text{IV.} \quad \overline{(\bar{A})} = \bar{A}.$$

The element $\bar{A}$ is called the *closure* of A, the element $\operatorname{Int}(A) = (\overline{A'})'$ is called the *interior* of A. An element A is said to be *closed* if $A = \bar{A}$, and *open* if $A = \operatorname{Int}(A)$.

We say that the closure algebra A has an *enumerable basis* if there exists an enumerable class $\{Q_n\}$ of open elements of A such that every open element of A is a union of elements of a subclass of Q_n.

If $\mathfrak{X}$ is a metric separable space, then the field $S(\mathfrak{X})$ of all subsets of $\mathfrak{X}$ is a closure algebra with an enumerable basis. If N is a σ-ideal of subsets of $\mathfrak{X}$, then the field $S(\mathfrak{X})$ modulo N will be denote by the symbol $S(\mathfrak{X})/N$. By $[X]$ we shall mean an element of $S(\mathfrak{X})/N$ determined by X, i. e. a class of sets, $X_1 \subset \mathfrak{X}$ such that $X \dot{-} X_1 \in N$.

If $Q_1, Q_2, \ldots$ is a basis in $S(\mathfrak{X})$, then the class $[Q_1], [Q_2], \ldots$ induces in the Boolean algebra $S(\mathfrak{X})/N$ a closure operation. Namely, let us denote by $\operatorname{Int}([X])$ the union of all $[Q_m]$ which are contained in $[X]$ and let

$[\overline{X}] = (\mathrm{Int}([X']))'$, by definition. The closure operation defined as above satisfies axioms I-IV and $[Q_1], [Q_2], \ldots$ is a basis in $\boldsymbol{S}(\mathfrak{X})/\boldsymbol{N}$ (see [5], p. 172).

Let $\boldsymbol{B}(\mathfrak{Y})$ be the field of Borel subsets of the metric separable space $\mathfrak{Y}$. $R_1, R_2, \ldots$ will denote a basis of neighbourhoods of $\mathfrak{Y}$, and let us put $S_n = R_n'$. By h we denote a σ-homomorphism of $\boldsymbol{B}(\mathfrak{Y})$ in $\boldsymbol{S}(\mathfrak{X})/\boldsymbol{N}$, i. e. a mapping of $\boldsymbol{B}(\mathfrak{Y})$ into $\boldsymbol{S}(\mathfrak{X})/\boldsymbol{N}$ such that

$$h\Big(\sum_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{n} h(A_n) \quad \text{and} \quad h(A') = h(A)'$$

for every $A, A_n \epsilon \boldsymbol{B}(\mathfrak{Y})$. Let φ denote a mapping of $\mathfrak{X}$ into $\mathfrak{Y}$ that induces the homomorphism h (i. e. $[\varphi^{-1}(Y)] = h(Y)$ for every $Y \epsilon \boldsymbol{B}(\mathfrak{Y})$). In this paper we suppose that the mapping φ exists. (It always exists if the space $\mathfrak{Y}$ is homeomorphic to a Borel subset of the Hilbert cube; see [4], p. 19).

We shall say that the homomorphism h is *continuous* if $\overline{h(Y)} \subset h(\overline{Y})$ for every $Y \epsilon \boldsymbol{B}(Y)$.

Let $D(h) = \sum_{n=1}^{\infty}\{h(R_n) - \mathrm{Int}(h(R_n))\}$.

(i) *h is continuous if and only if $D(h) = 0$* (see [5], p. 177).

(ii) $D(h) = \sum_{n=1}^{\infty}\{\overline{h(S_n)} - h(S_n)\} = \sum_{n=1}^{\infty}\{[P(\varphi^{-1}(S_n))] - [\varphi^{-1}(S_n)]\}$.

Since

$$h(R_n) - \mathrm{Int}(h(R_n)) = h(S_n') - (\overline{h(S_n)})' = \overline{h(S_n)} - h(S_n),$$

and since h is induced by the mapping φ, we have

$$h(S_n) = [\varphi^{-1}(S_n)] \quad \text{and} \quad \overline{h(S_n)} = \overline{[\varphi^{-1}(S_n)]} = [P(\varphi^{-1}(S_n))]$$

(see [5], p. 180). This completes the proof.

From (ii) and 5(ii) immediately follows

(iii) $D(h) = [D_0]$,

where D_0 has been defined in sections 5 and 6.

From (i) and (iii) we infer that

(iv) *The homomorphism h is continuous if and only if the mapping φ is almost everywhere almost continuous* (3).

(3) Professor R. Sikorski has remarked that this theorem and theorem 21.1 of his paper [5] imply the following interesting statement: φ is almost continuous almost everywhere if and only if there exists a set $N \epsilon \boldsymbol{N}$ such that $\varphi|\mathfrak{X} - N$ is continuous. It seems desirable to find a direct proof of this equivalence (i. e. without any use of closure algebras).

In the case where $\boldsymbol{N}$ contains all one-point sets, we may, on account of 6(iii), modify (iii) and (iv) by replacing D_0 by D_1 and the almost-continuity points by limit points.

8. Approximation by a mapping of the first class. A mapping f of a metric space $\mathfrak{Z}$ into a metric space $\mathfrak{Y}$ is called a *mapping of the first class* if the set $f^{-1}(G)$ is an F_σ in $\mathfrak{Z}$ for every open subset G of $\mathfrak{Y}$.

Let $\boldsymbol{P}$ denote the class of all mappings of $\mathfrak{X}$ into $\mathfrak{Y}$ which are almost equal to mappings of the first class. Let $\boldsymbol{M}$ denote the class of mappings which are of the first class if one disregards the sets belonging to $\boldsymbol{N}$; in other words, $f \epsilon \boldsymbol{M}$ if there exists a set $Z \epsilon \boldsymbol{N}$ such that the mapping $f|\mathfrak{X} - Z$ is of the first class on $\mathfrak{X} - Z$. Obviously

(i) $\boldsymbol{P} \subset \boldsymbol{M}$.

We will prove the following equivalence:

(ii) *$f \epsilon \boldsymbol{M}$ if and only if the set $f^{-1}(G)$ is almost equal to an F_σ for every open subset G of $\mathfrak{Y}$.*

The first implication being trivial, it remains to prove the second one. Let $\{R_n\}$ be an open basis of $\mathfrak{Y}$ and $K_1, K_2, \ldots$ a sequence of F_σ-sets such that $K_n \dot{-} f^{-1}(R_n) \epsilon \boldsymbol{N}$. Let us put

$$Z = \sum_{n=1}^{\infty}\big(K_n \dot{-} f^{-1}(R_n)\big).$$

It is easily seen that the mapping $f|\mathfrak{X} - Z$ is of the first class, q. e. d.

It follows from (ii) that, in the case where $\boldsymbol{N}$ is the class of sets of Lebesgue measure zero, the class $\boldsymbol{M}$ is that of all measurable functions. The class $\boldsymbol{P}$ is, in this case, smaller, e. g. the characteristic function of the set H quoted in the Introduction belongs to $\boldsymbol{M} - \boldsymbol{P}$.

It follows directly from 4(ii) and from the Baire theorem on functions of the first class (see e. g. [2], p. 300) that

(iii) *If $f \epsilon \boldsymbol{P}$, then for every closed subset F of $\mathfrak{X}$, the mapping $f|F$ has at least one almost-limit point.*

The converse theorem is also true (cf. Introduction, [3] and [6]). Here we will prove only a weaker relation (4):

(iv) *If, for a mapping f of $\mathfrak{X}$ into $\mathfrak{Y}$ and for every non-void closed set $F \subset \mathfrak{X}$, the mapping $f|F$ possesses at least one almost-limit point, then $f \epsilon \boldsymbol{M}$.*

Let F be a closed subset of $\mathfrak{Y}$ and let

$$\mathfrak{Y} - F = F_1 + F_2 + \ldots$$

(4) It is a generalization of one part of the theorem of Baire quoted above. Our proof is a modification of that of Kuratowski [2], p. 301-302.

where the sets F_j are closed in $\mathfrak{Y}$. We will prove that, for every n, if

$$(*) \qquad P(Zf^{-1}(F))\cdot P(Zf^{-1}(F_n)) = Z,$$

then $Z = 0$. Let us suppose (*) and $Z \neq 0$. The set Z being closed, there exists an almost-limit point x_0 of the mapping $g = f|Z$. In view of 4(i) there is a point $y_0 \epsilon \mathfrak{Y}$ such that if $x_0 \epsilon P(A)$ then $y_0 \epsilon \overline{f(A)}$. Hence

$$x_0 \epsilon Z \subset P(Zf^{-1}(F)) = P(g^{-1}(F)),$$

and consequently $y_0 \epsilon \overline{gg^{-1}(F)} \subset \overline{F} = F$.

Analogously we obtain $y_0 \epsilon F_n$, which is impossible, the sets F and F_n being disjoint. We obtain therefore $Z = 0$.

Now we apply the theorem on separation 2(i) and obtain a developable set H_n almost containing $f^{-1}(F)$ and almost disjoint with $f^{-1}(F_n)$. Thus the set

$$H = H_1\cdot H_2\cdot\ldots$$

is a G_δ almost containing $f^{-1}(F)$ and almost disjoint with $f^{-1}(F_1+F_2+\ldots) = f^{-1}(F')$. The set $f^{-1}(F')$ is then almost equal to an F_σ. Applying (ii) we obtain theorem (iv).

Note that the analogue of (iii) for almost continuity points is not generally true. More precisely:

(v) *Let N be the class of all sets of Lebesgue measure zero in the unit interval I. There exists a real function f on I which is equal almost everywhere to a function of the first class, and which does not possess any almost continuity point.*

Let $K = K_1+K_2+\ldots$ be the sum of a sequence of disjoint, closed, non-dense subsets K_n of I, having the following property:

(a) for every interval $J \subset I$ and every n we have either $JK_n = 0$ or $|JK_n| > 0$,

(b) $|K| = 1$,

where $|\ |$ denotes the Lebesgue measure.

Let us put

$$f(x) = g(x) = \frac{1}{n} \quad \text{for} \quad x \epsilon K_n,$$

$$f(x) = 2 \text{ and } g(x) = 0 \quad \text{for} \quad x \epsilon I-K.$$

It is easy to verify that g is a function of the first class. Consequently, in view of (b), f is almost equal to a function of the first class.

We shall prove that, on the other hand, f has no almost-continuity point.

If $x_0 \epsilon I-K$ then, by (b), $x_0 \epsilon P(K)$ and simultaneously $f(x_0) = 2 \notin \overline{f(K)}$, since if $x \epsilon K$ then $f(x) \leqslant 1$. Hence, by condition (c_2) of 5(i), x_0 is not an almost-continuity point of f.

If $x_0 \epsilon K$, then $x_0 \epsilon K_{n_0}$ and $f(x_0) = 1/n_0$. The sets K_n being non-dense, it easily follows from (a) and (b) that $x_0 \epsilon P(I-K_{n_0})$. The set $f(I-K_{n_0})$ contains only the numbers 2 and $1/n$ for $n \neq n_0$, whence

$$f(x_0) = \frac{1}{n_0} \notin \overline{f(I-K_{n_0})}.$$

Thus, as in the preceding case, x_0 is not an almost-continuity point of f.

The proof of (v) is now complete.

The set of almost-limit points of the function f defined in the proof of (v) is obviously a residual set. Therefore proposition (v) proves that the relations between the notion of continuity and that of limit points contained in [1] are not true for the almost-continuity and almost-limit points (cf. especially p. 166 and 167). It is worth noticing that, nevertheless,

(vi) *If the ideal N is a class of sets of the first category in $\mathfrak{X}$ and if $f \epsilon \boldsymbol{P}$, then f is almost continuous at one point at least.*

Let g be a mapping of the first class almost equal to f. The set

$$E = \{x : f(x) \neq g(x)\}$$

and the set D of discontinuity points of f are of the first category (see e. g. [2], p. 301) and f is almost continuous at every point of $\mathfrak{X}-(E+D)$. The space $\mathfrak{X}$ being complete, theorem (vi) follows from the theorem of Baire.

It follows directly from (vi) that, in some particular cases, the analogue of (iii) is true:

(vii) *If the ideal N is the class of denumerable sets* (or, more generally, a class of sets always of the first category; see e. g. [2], p. 423 and 424) *then for every $f \epsilon \boldsymbol{P}$ and every closed subset F of $\mathfrak{X}$, the mapping $f|F$ is almost continuous at one point at least.*

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ON THE APPROXIMATIONS OF MAPPINGS BY BAIRE MAPPINGS

BY

T. TRACZYK (WARSAW)

Let f be a mapping of a metric separable space $\mathfrak{X}$ into a metric separable and complete space $\mathfrak{Y}$. Let $\boldsymbol{I}$ be an arbitrary σ-ideal of subsets of $\mathfrak{X}$, i. e. a hereditary and enumerable additive class of subsets of $\mathfrak{X}$, and let $|u-v|$ denote the distance between points u and v in $\mathfrak{Y}$. Denote by B_α the set of mappings of $\mathfrak{X}$ into $\mathfrak{Y}$ of the Baire class α in the sense of Kuratowski (see [1], p. 280).

Definition (cf. [2], p. 85). *We say that f has property D_α at $x_0 \epsilon \mathfrak{X}$ if for every $\varepsilon > 0$ there exist a neighbourhood G of x_0 and a mapping $g \epsilon B_\alpha$ such that*

$$|f(x)-g(x)| < \varepsilon \text{ a. e. in } G,$$

i. e. for $x \epsilon A'$, where $A \epsilon \boldsymbol{I}$.

Note 1. If $\boldsymbol{I}$ is the ideal of subsets of measure zero and $\alpha > 2$, then the mapping having property D_α at x_0 has property D_2 at x_0.

Note 2. If for every $\varepsilon > 0$ there exists a neighbourhood G of x_0 such that

$$|f(x)-f(x_0)| < \varepsilon \text{ a. e. in } G,$$

(i. e. if the mapping f is almost continuous (see [3], section 5) at x_0), then it has property D_0.

The purpose of this paper is to prove the following theorem, which is a generalization for mappings in metric spaces of an analogous theorem concerning real functions and due to Lederer [2].

THEOREM. *If $\alpha > 0$ and for any non-empty closed set $F \subset \mathfrak{X}$ the mapping $f|F$ has property D_α with respect to F at least at one point, then there exists a mapping $g \epsilon B_\alpha$ such that $f(x) = g(x)$ a. e.*

First we prove the following lemma: