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ON THE APPROXIMATIONS OF MAPPINGS BY BAIRE MAPPINGS

BY

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Let f be a mapping of a metric separable space $\mathfrak{X}$ into a metric separable and complete space $\mathfrak{Y}$. Let $\boldsymbol{I}$ be an arbitrary σ-ideal of subsets of $\mathfrak{X}$, i. e. a hereditary and enumerable additive class of subsets of $\mathfrak{X}$, and let $|u-v|$ denote the distance between points u and v in $\mathfrak{Y}$. Denote by B_α the set of mappings of $\mathfrak{X}$ into $\mathfrak{Y}$ of the Baire class α in the sense of Kuratowski (see [1], p. 280).

Definition (cf. [2], p. 85). *We say that f has property D_α at $x_0 \in \mathfrak{X}$ if for every $\varepsilon > 0$ there exist a neighbourhood G of x_0 and a mapping $g \in B_\alpha$ such that*

$$|f(x)-g(x)| < \varepsilon \text{ a. e. in } G,$$

i. e. for $x \in A'$, where $A \in \boldsymbol{I}$.

Note 1. If $\boldsymbol{I}$ is the ideal of subsets of measure zero and $\alpha > 2$, then the mapping having property D_α at x_0 has property D_2 at x_0.

Note 2. If for every $\varepsilon > 0$ there exists a neighbourhood G of x_0 such that

$$|f(x)-f(x_0)| < \varepsilon \text{ a. e. in } G,$$

(i. e. if the mapping f is almost continuous (see [3], section 5) at x_0), then it has property D_0.

The purpose of this paper is to prove the following theorem, which is a generalization for mappings in metric spaces of an analogous theorem concerning real functions and due to Lederer [2].

THEOREM. *If $\alpha > 0$ and for any non-empty closed set $F \subset \mathfrak{X}$ the mapping $f|F$ has property D_α with respect to F at least at one point, then there exists a mapping $g \in B_\alpha$ such that $f(x) = g(x)$ a. e.*

First we prove the following lemma:

Under the conditions of the theorem, for every $\varepsilon > 0$ there exists a mapping $\Phi \epsilon B_a$ such that

$$|f(x) - \Phi(x)| < \varepsilon \text{ a. e.}$$

Let $\{R_n\}$ be a basis of neighbourhoods in $\mathfrak{X}$. Let us denote by N_0 the set of all n for which there exists a mapping $\Phi_n \epsilon B_a$ such that

$$|f(x) - \Phi_n(x)| < \varepsilon \text{ a. e. in } R_n.$$

The set N_0 is obviously infinite:

$$N_0 = \{n_k\}, \quad k = 1, 2, \ldots$$

Now we define the mapping Φ as follows:

$$\Phi(x) = \begin{cases} \Phi_{n_1}(x) & \text{for} \quad x \epsilon R_{n_1}, \\ \Phi_{n_k}(x) & \text{for} \quad k > 1 \text{ and } x \epsilon R_{n_k} - \sum\limits_{i<k} R_{n_i}. \end{cases}$$

Since $\Phi_{n_k} \epsilon B_a$ and $a > 0$, we easily prove that $\Phi \epsilon B_a$.

It remains to prove that $\mathfrak{X} = \sum\limits_{k=1}^{\infty} R_{n_k}$. Suppose it is not true. Since the set

$$F = \mathfrak{X} - \sum_{k=1}^{\infty} R_{n_k}$$

is non-empty and closed, there exist, by hypothesis, a point $x_0 \epsilon F$, a neighbourhood R_m of x_0, and a mapping Φ_m of F into $\mathfrak{Y}$ which is of Baire class a such that

$$|f(x) - \Phi_m(x)| < \varepsilon \quad \text{a. e. in } FR_m.$$

Now let

$$\Psi(x) = \begin{cases} \Phi_m(x) & \text{for} \quad x \epsilon F, \\ \Phi(x) & \text{for} \quad x \epsilon \mathfrak{X} - F. \end{cases}$$

It is evident that $\Psi \epsilon B_a$ and

$$|f(x) - \Psi(x)| < \varepsilon \quad \text{a. e. in } R_m.$$

Hence $m \epsilon N_0$. This, however, contradicts the definition of F and x_0.

Proof of the theorem. It is well known (cf. [1], p. 294, th. 3) that for each $\varphi \epsilon B_a$ and $a > 0$ there exists a sequence $\{\varphi_n\}$ of mappings such that: $\varphi_n \to \varphi$ uniformly, the set $\varphi_n(\mathfrak{X})$ is isolated in $\mathfrak{Y}$, and $\varphi_n \epsilon B_a$ for every n.

There exists therefore, by the lemma, a sequence $\{f_n\}$ of mappings, such that for each n, $f_n \epsilon B_a$, the sets $f_n(\mathfrak{X})$ are isolated, and

$$|f(x) - f_n(x)| < 2^{-n} \text{ a. e.}$$

Hence the set

$$H_{n,1} = \{x: |f_n(x) - f_{n-1}(x)| < 3 \cdot 2^{-n}\}$$

is a Borel set of the additive class a. Using the fact that $f_n(\mathfrak{X})$ and $f_{n-1}(\mathfrak{X})$ are isolated, it is not difficult to prove that $H'_{n,1}$ is of the same class.

Now we define a double sequence of mappings

$$\begin{array}{lll} g_{1,1} & & \\ g_{2,1} & g_{2,2} & \\ g_{3,1} & g_{3,2} & g_{3,3} \\ \cdots & \cdots & \cdots \end{array}$$

as follows:

$$g_{1,1} = f_1$$

and for $n > 1$

$$g_{n,1}(x) = \begin{cases} f_n(x) & \text{for} \quad x \epsilon H_{n,1}, \\ f_{n-1}(x) & \text{for} \quad x \epsilon H_{n,1}. \end{cases}$$

Suppose that the mappings $g_{n,1}, g_{n,2}, \ldots, g_{n,m-1}$ for $n = 1, 2, \ldots$ are already defined. Let us write for $m > 2$

$$H_{n,m-1} = \{x: |g_{n,m-1}(x) - g_{n-1,m-1}(x)| < 3 \cdot 2^{-n}\}.$$

We put

$$g_{n,m}(x) = \begin{cases} g_{n,m-1}(x) & \text{for} \quad x \epsilon H_{n,m-1}, \\ g_{n-1,m-1}(x) & \text{for} \quad x \epsilon H'_{n,m-1}. \end{cases}$$

Since, by the above,

$$|g_{n,m}(x) - g_{n-1,m-1}(x)| < 3 \cdot 2^{-n}$$

for each $x \epsilon \mathfrak{X}$ and $n, m = 2, 3, \ldots$ and the space $\mathfrak{Y}$ is complete, $\{g_{n,n}\}$ converges uniformly over $\mathfrak{X}$.

Since $f_n \epsilon B_a$ and the set $f_n(\mathfrak{X})$ $(n = 1, 2, \ldots)$ is isolated, by induction the sets $g_{n,m}(\mathfrak{X})$ are isolated in $\mathfrak{Y}$, the sets $H_{n,m}$ and their complements are Borel sets of the additive class a, and all the mappings $g_{n,m}$ belong to B_a.

Thus the mapping $g = \lim\limits_{n\to\infty} g_{n,n}$ also belongs to B_a.

Let

$$X = \sum_{n=1}^{\infty} \{x: |f(x) - f_n(x)| \geqslant 2^{-n}\}.$$

Evidently $X \in \boldsymbol{I}$. If $x \in \mathfrak{X} - X$ then $|f_n(x) - f_{n-1}(x)| < 3 \cdot 2^{-n}$, whence $x \in H_{n,1}$ and $g_{n,1}(x) = f_n(x)$.

By induction with respect to m, $g_{n,m}(x) = f_n(x)$ for $x \in \mathfrak{X} - X$ and for each n and $m = 1, 2, \ldots, n$. It follows for $x \in X'$ that

$$g(x) = \lim_{n\to\infty} g_{n,n}(x) = \lim_{n\to\infty} f_n(x) = f(x).$$

In consequence

$$g(x) = f(x) \quad \text{a. e.}$$

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CONCERNING DISTANCES OF SETS AND DISTANCES OF FUNCTIONS

BY

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Let $(X, \boldsymbol{M}, \mu)$ be a σ-finite σ-measure space. H. Steinhaus has introduced the following distance of measurable sets of finite measure:

$$\sigma_\mu(A, B) = \begin{cases} \dfrac{\mu(A \dot{-} B)}{\mu(A+B)} & \text{if} \quad \mu(A+B) > 0, \\ 0 & \text{if} \quad \mu(A) = \mu(B) = 0, \end{cases} \tag{1}$$

where $A \dot{-} B$ denotes the symmetric difference of A and B, i. e. the set $(A+B) - AB$. The distance (1) is discussed in [1] and [2]; it has been applied by biologists.

In connection with a question raised recently by J. B. Faliński from the Botanical Institute of the Polish Academy of Sciences, J. Perkal observes that there exist finite sequences of sets arbitrarily near each other (in the sense of the distance σ_μ) and yet having an empty intersection. Namely, it suffices to consider the sequence $S_1, \ldots, S_n$ of all $(n-1)$-element subsets of a fixed n-element set X and to adopt the number of elements of $A \subset X$ as the measure $\mu_0(A)$. Then

$$\sigma_{\mu_0}(S_i, S_j) = 2/n \quad \text{for} \quad i \neq j \text{ and } S_1 \cdot \ldots \cdot S_n = 0.$$

We provide here the answer to a problem of J. Perkal by proving that, *for every σ-measure μ, if $\sigma_\mu(A_i, A_j) < 2/n$, then $A_1 \cdot \ldots \cdot A_n \neq 0$* (see 1.2(ii)). The preceding example of Perkal proves that our inequalities in section 1.2 may be considered as the strongest ones.

The second part of this paper contains analogous considerations concerning the distance of functions.