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ON INTERPOLATION BY ALMOST PERIODIC FUNCTIONS

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In connection with the preceding paper of Mycielski [2] I will point out a class of "bad" sequences of positive integers d_n, if "bad" means that one can assign to each n a value 0 or 1 so as to make it impossible to find not only a periodic continuous function but even a Bohr almost periodic function assuming these values on d_n. There is no problem of giving an example of a bad sequence: for this purpose we can put $d_n = n$, since for every Bohr function f the sequence $f(n)$ is almost periodic, thus having special properties. However, it appears that also sequences increasing more rapidly can be bad, according to the following

THEOREM 1. *There is a sequence a_n of zeros and unities such that for every integer $k > 0$ and every Bohr function f one has $f(n^k) \neq a_n$ for some n.*

This theorem asserts that the sequences $d_n = n^k$ $(n = 1, 2, \ldots)$ are "uniformly bad", and the sequence a_n will be shown to satisfy all requirements if the lower density of zeros and that of unities in it is 0. Assume this, fix an integer $k > 0$ and let f be a Bohr function with Fourier series $\sum_n c_n e^{2\pi i \mu_n t}$. Take numbers λ_j so that $1, \lambda_1, \lambda_2, \ldots$ be arithmetically independent and that for every i there exist rational numbers $r_j^{(i)}$ $(j = 0, 1, \ldots, s_i)$ with

$$\mu_i = r_0^{(i)} + r_1^{(i)}\lambda_1 + \ldots + r_{s_i}^{(i)}\lambda_{s_i}. \tag{1}$$

Denote in general by $[a]$ the integer part and by $\{a\}$ the fractional part of the number a. There are an integer $N > 0$ and a $\delta > 0$ $(\delta < 1)$ such that

$$\{\mu_i \tau\} < \delta \qquad (i = 1, 2, \ldots, N) \tag{2}$$

implies $|f(t+\tau) - f(t)| < 1$ for every t. Denote by Q a common denominator of the numbers $r_j^{(i)}$ $(j = 0, 1, \ldots, s_i;\ i = 1, 2, \ldots, N)$. We thus have

$$r_j^{(i)} = \frac{p_j^{(i)}}{Q} \qquad (p_j^{(i)} \text{ integers}). \tag{3}$$

There must be integers $u_0 \geqslant 0$, $u_0 < Q$ and $\nu \geqslant 0$ such that

$$b_n = (\nu + nQ)^k \equiv u_0 \,(\mathrm{mod}\, Q) \qquad (n = 1, 2, \ldots). \tag{4}$$

As is well known, for every system $\lambda_1, \ldots, \lambda_m$ such that $1, \lambda_1, \ldots, \lambda_m$ are arithmetically independent, the sequence of points

$$p_n = (\{\lambda_1 b_n\}, \ldots, \{\lambda_m b_n\})$$

is uniformly distributed in the m-dimensional unit cube ([3], Satz 14). Hence putting

$$m = \max_{1 \leqslant i \leqslant N} s_i, \qquad M = \max |r_j^{(i)}| \qquad (j = 1, 2, \ldots, s_i;\ 1 \leqslant i \leqslant N), \tag{5}$$

we can prove the existence of a sequence n_ν of positive density such that

$$\{b_{n_\nu} \lambda_j\} < \frac{\delta}{2Mm}, \qquad [b_{n_\nu} \lambda_j] \equiv 0 \ (\mathrm{mod}\, Q) \qquad (j = 1, 2, \ldots, m). \tag{6}$$

In fact, let $\varphi(t_1, \ldots, t_m)$ be the characteristic function of the set

$$\bigcup_{n_1, \ldots, n_j = 0}^{\infty} \bigcap_{j=1}^{m} \left\{ n_j Q \leqslant t_j < n_j Q + \frac{\delta}{2Mm} \right\} \subset E^m.$$

Then (6) is equivalent to

$$\varphi(b_{n_\nu} \lambda_1, \ldots, b_{n_\nu} \lambda_m) = 1. \tag{6'}$$

But $\varphi(t_1 Q, \ldots, t_m Q)$ is of period 1 in each variable and thus, since $\lambda_1/Q, \ldots, \lambda_m/Q, 1$ are arithmetically independent, we have

$$\lim_{l \to \infty} \frac{1}{l} \sum_{n=1}^{l} \varphi(b_n \lambda_1, \ldots, b_n \lambda_m) = \left(\frac{\delta}{2QMm} \right)^m$$

according to Weyl's theorem. Then (6') must be satisfied by a sequence of positive density.

If we had $f(n^k) = a_n$ for every n, then there would be a value $\nu = \nu_1$ such that $f(b_{n_{\nu_1}}) = 0$ and another $\nu = \nu_2$ for which $f(b_{n_{\nu_2}}) = 1$. If $\tau = b_{n_{\nu_2}} - b_{n_{\nu_1}}$, we find by (4) and (3)

$$r_0^{(i)} \tau \equiv 0 \,(\mathrm{mod}\, 1)$$

and by (3), (5) and (6)

$$\sum_{j=1}^{s_i} \{\tau r_j^{(i)} \lambda_j\} < \delta \qquad (1 \leqslant i \leqslant N).$$

Thus, in virtue of (1) we get (2), and finally

$$|f(b_{n_{\nu_1}} + \tau) - f(b_{n_{\nu_1}})| = |f(b_{n_{\nu_2}}) - f(b_{n_{\nu_1}})| < 1,$$

which is a contradiction.

It is quite easy to see that if for a sequence $a_n \uparrow \infty$ of real numbers and a fixed number λ the multiplies λa_n are uniformly distributed mod 1, then it is impossible to prescribe a non-constant 0-1 sequence a_n in such a way as to find a continuous function with period $1/\lambda$ and $f(a_n) = a_n$. In fact, by reducing a_n mod $1/\lambda$ we get a dense set in $(0, 1/\lambda)$.

On account of [3] (Satz 21) λa_n are uniformly distributed for almost all λ. Thus, for a "good sequence" and an arbitrarily prescribed non-constant sequence of values 0 and 1 in it, there is not much possibility of choice as to the period of the interpolating function: the virtual periods are in a zero-set. Further, given a denumerable set, we can choose 0-1 values at a_n so that no continuous periodic function with period from this set satisfies the requirements. This can be generalized to

THEOREM 2. *If Λ is a countable set of real numbers and a_n are arbitrary reals, then there is a sequence $a_n = 0$ or 1 such that no Bohr function with exponents in Λ satisfies $f(a_n) = a_n$ $(n = 1, 2, \ldots)$.*

To prove this observe that the group of reals can be continuously and isomorphically imbedded into a metric compact group K in such a way that all Bohr functions with exponents in Λ admit a continuous extension over K (see e. g. [1], p. 186). Thus, the sequence a_n must contain a subsequence a_{n_k} which is convergent in K. Putting $a_{n_k} = 0$ or 1 alternatively for $k = 1, 2, \ldots$ we get the desired effect, irrespectively of the values of the remaining a_n.

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[2] Jan Mycielski, *On a problem of interpolation by periodic functions*, this volume, p. 95-97.

[3] H. Weyl, *Über die Gleichverteilung von Zahlen mod. Eins*, Mathematische Annalen 77 (1916), p. 313-352.

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