Let S be a set, $\overline{\overline{S}} = m \geqslant \aleph_0$, $F(x)$ a set-mapping defined on S such that $\overline{\overline{F(x)}} < n < m$ for every $x \in S$, for an $n < m$. Let further $\{S_\nu\}_{\nu<\varphi}$ be a system of disjoint subsets of S satisfying the conditions: $\overline{\overline{S_\nu}} = m$ for every $\nu < \varphi$ and $\overline{\overline{\varphi}} < m$. Then there exists a free subset $S' \subset S$ such that $\overline{\overline{S' \cap S_\nu}} = m$ for every $\nu < \varphi$.

The proof given in [7] makes use of the generalized continuum hypothesis in the case when m is singular.

It is easy to see that using the idea of the proof of our Theorem 1 this generalization of the Ruziewicz conjecture can also be proved without using the generalized continuum hypothesis.

On the other hand in his paper [8] G. Fodor states the following generalization of the Ruziewicz conjecture.

Let S be a set, $\overline{\overline{S}} = m \geqslant \aleph_0$, and $F(x)$ a set-mapping defined on S, satisfying the condition $\overline{\overline{F(x)}} < n < m$ for every $x \in S$ for some $n < m$. Let further $\Pi(S')$ denote the set $\bigcup\limits_{x \neq y, x, y \in S'} \big(F(x) \cap F(y)\big)$ for every $S' \subset S$. Then there exists a subset $S' \subset S$, $\overline{\overline{S'}} = m$ such that $\overline{\overline{\Pi(S')}} < m$.

Fodor proves this theorem for singular m using the generalized continuum hypothesis; our method does not enable us to prove this theorem without using this hypothesis. The simplest unsolved problem here is: Is it possible to prove Fodor's theorem without using this hypotheses for $m = \aleph_{\omega_2}$ or for $m = \aleph_{\omega_1}$?

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A new analytic approach to hyperbolic geometry

by

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Introduction

Hilbert was the first who constructed in plane hyperbolic geometry without the axiom of continuity a commutative ordered field $\overline{\overline{\mathfrak{E}}} = \langle \overline{\overline{E}}, +, \cdot, < \rangle$ and founded an analytic geometry over it (see [3] or [2], Appendix III). The field $\overline{\overline{\mathfrak{E}}}$ is known in the literature as the *end-calculus* since the class $\overline{\overline{E}}$ consists of pencils of parallel half-lines, which Hilbert refers to as *ends*. The analytic geometry over $\overline{\overline{\mathfrak{E}}}$ is based upon a coordinate system for straight lines.

In this paper a new commutative ordered field $\overline{\overline{\mathfrak{S}}} = \langle \overline{\overline{S}}, +, \cdot, < \rangle$ is constructed in the same system of geometry. This field seems to be conceptually simpler and more adequate for the foundation of analytic geometry than $\overline{\overline{\mathfrak{E}}}$. It is generated by a *hyperbolic calculus of segments*, more precisely by an algebraic system $\mathfrak{S} = \langle S, +, \cdot, < \rangle$ in which the class S consists of the segments. The operations $+$ and $\cdot$ of $\mathfrak{S}$ are defined in terms of such simple notions as the Lambert quadrangle and the right triangle and are not relativized to any fixed geometrical objects, while the relation $<$ coincides with the usual less-than relation for the segments. Finally a rectangular coordinate system over $\overline{\overline{\mathfrak{S}}}$ can be constructed (the two coordinates of a point being elements of $\overline{\overline{S}}$), and moreover the analytic geometry based on it is identical with that of the two-dimensional Klein space the absolute of which coincides with the unit circle.

Chapter I is algebraic. We introduce there the notion of a *unit interval algebra* and reduce the problem of constructing a commutative ordered field to that of constructing a unit interval algebra.

Chapter II is geometrical. In Section 1 we describe the axiomatic theory $\mathcal{H}'$ of the hyperbolic geometry in which the field $\overline{\overline{\mathfrak{S}}}$ is to be constructed. In Sections 2-13 we construct the system $\mathfrak{S}$, furthermore we prove it to be a unit interval algebra, and consequently, using the result of Chapter I, we obtain the ordered field $\overline{\overline{\mathfrak{S}}}$. In Sections 14-18 we outline the foundations of the analytic geometry over $\overline{\overline{\mathfrak{S}}}$.

Chapter III is also geometrical. In Section 1 we clarify the relationship between the end-calculus and the hyperbolic calculus of segments. In Section 2 we study the field $\overline{\overline{\mathfrak{S}}}$ in the full hyperbolic geometry with the axiom of continuity.

The present paper is closely connected with the author's article [11]. That rather metamathematical article includes, in particular, a construction of the hyperbolic calculus of segments in the space geometry. The present paper gives the first presentation of the plane construction of $\mathfrak{S}$. We also point out the author's note [10], where an absolute calculus of segments, closely related to the hyperbolic one, has been constructed.

I. The unit interval algebra

1. Postulates. Consider an algebraic system $\mathfrak{S} = \langle S, +, \cdot, < \rangle$ with two binary operations $+$ and $\cdot$ and a binary relation $<$ in a non-empty set S, the two operations being not assumed to be always performable. To express the fact that $x+y$ does or does not exist we shall write $x+y \in S$, $x+y \notin S$, respectively. We shall use the same notation for the operation $\cdot$. We refer to the system $\mathfrak{S} = \langle S, +, \cdot, < \rangle$ as a *unit interval algebra* if and only if it satisfies the following postulates:

(i) *If* $x \in S$, *then* x *non*-$<x$.

(ii) *If* $x, y \in S$, *then* $x = y$ *or* $x < y$ *or else* $y < x$.

(iii) *If* $x, y, x+y \in S$, *then* $x+y = y+x$.

(iv) *If* $x, y, x+y, (x+y)+z \in S$, *then* $y+z \in S$ *and* $(x+y)+z = x+(y+z)$.

(v) *If* $x, z \in S$, *then* $x < z$ *iff* (1) $x+y = z$ *for some* $y \in S$.

(vi) *If* $x, y \in S$, *then* $x \cdot y \in S$ *and* $x \cdot y = y \cdot x$.

(vii) *If* $x, y, z \in S$, *then* $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(viii) *If* $x, z \in S$, *then* $z < x$ *iff* $z = x \cdot y$ *for some* $y \in S$.

(ix) *If* $x, y, z, x+y \in S$, *then* $(x+y) \cdot z = x \cdot z + y \cdot z$.

It is seen at once that if $\mathfrak{F}$ is an arbitrary commutative ordered field with the zero element 0 and the unit element 1, then the open interval $(0, 1)$ of $\mathfrak{F}$ is a unit interval algebra. The aim of the subsequent discussion is to show that, on the contrary, every unit interval algebra can be extended to a commutative ordered field.

We start with some simple consequences of the postulates (i)-(ix). In all the formulas the variables x, y, z, t, u, v are assumed to range over the set S.

(1) We use *iff* for *if and only if*.

(1.1) *If* $x < y$ *and* $y < z$, *then* $x < z$.

(1.2) *If* $x+y \in S$ *and* $z < y$, *then* $x+z \in S$.

(1.3) *If* $x+y \in S$ *and* $x+y = x+z$, *then* $y = z$.

(1.4) *If* $x+y \in S$, $x+y = z+t$, *and* $x < z$, *then* $t < y$.

(1.5) *If* $x \cdot y = x \cdot z$, *then* $y = z$.

(1.6) *If* $x \cdot y = z \cdot t$ *and* $x < z$, *then* $t < y$.

(1.7) *If* $x \cdot x = y \cdot y$, *then* $x = y$.

(1.8) $x < y$ *iff* $x \cdot z < y \cdot z$.

(1.9) $x < y$ *iff* $x \cdot x < y \cdot y$.

(1.10) *If* $x \cdot z + y \cdot z = t \cdot z$, *then* $x+y = t$.

The statements (1.1)-(1.4) are easily derivable from the postulates (i)-(v); the statements (1.5)-(1.9) are easily derivable from the postulates (i), (ii) and (vi)-(viii). To prove (1.10) let us notice that from $x \cdot z + y \cdot z = t \cdot z$ it follows by (v) and (1.8) that $x+u = t$ for some u in S, which by (ix) implies $x \cdot z + u \cdot z = t \cdot z$. Consequently by (1.3), (1.5), and (vi) we get $u = y$, and thus $x+y = t$.

2. Complement of x. Given an element t in S, there is for every x in S a unique element x'_t such that

$$x \cdot t + x'_t \cdot t = t \tag{2.1}$$

(by (v), (vi), (viii), (1.3), and (1.5)). The element x'_t does not depend on the parameter t. For, given two different parameters t and u, say $u < t$, we have $u = t \cdot v$ for some v in S, and with the help of (ix) and (vii) we derive from (2.1) the identity $x \cdot u + x'_t \cdot u = u$, which implies $x'_t = x'_u$. Consequently, *for every x in S there is a unique element x' in S such that*

$$x \cdot t + x' \cdot t = t \quad \textit{for every} \quad t \in S. \tag{2.2}$$

We shall refer to x' as the *complement* of x.

On the other hand,

$$\textit{if } y+z = t, \textit{ then } y = x \cdot t \textit{ and } z = x' \cdot t \textit{ for some } x \in S. \tag{2.3}$$

In fact, by (iii), (v), and (viii) we have $y = x \cdot t$ and $z = u \cdot t$ for some x and u in S. Hence $x \cdot t + u \cdot t = t$, and with the help of (2.2) we get $u = x'$.

From (2.2) and (iii) we derive at once

$$x'' = x. \tag{2.4}$$

Moreover

$$\textit{if} \quad x < y, \quad \textit{then} \quad y' < x', \tag{2.5}$$

which follows with the help of (1.8) and (1.4) from $x \cdot t + x' \cdot t = y \cdot t + y' \cdot t = t$.

Furthermore we have

(2.6) $$\text{if } x+y=z, \text{ then } x+z'=y',$$

since from $x+y=z$ and $t\cdot z+t\cdot z'=t\cdot y+t\cdot y'=t$ we deduce $t\cdot x+t\cdot z'=t\cdot y'$, which by (1.10) gives us $x+z'=y'$.

Finally, let us notice that

(2.7) $$x+x'\notin S \text{ and } x+y\in S \text{ for every } y<x'.$$

In fact, were $x+x'\in S$, then by (2.2) and (ix) for any t we would have $(x+x')\cdot t=t$, which by (viii) would imply $t<t$, contrary to (i). On the other hand, if $y<x'$, then $y+z=x'$ for some z in S, which with the help of (2.6) and (2.4) implies at once $x+y=z'$. Thus $x+y\in S$.

3. One-half element. Consider the equation

$$x=x'.$$

It follows at once from (2.5) that it has at most one solution in S. Actually there is an element in S satisfying the equation. To find it we pick an arbitrary y in S. If $y=y'$, we put $x=y$. If $y<y'$, then $y+y\in S$ by (2.7); let $y+y=t$. Then by (2.3) we have $x\cdot t=y=x'\cdot t$ for some x in S. Clearly $x=x'$. If $y'<y$, we repeat the argument for y' instead of y. Consequently, *the condition $x=x'$ is satisfied by a unique element in S*; we denote this element by $\frac{1}{2}$. Hence

(3.1) $$\tfrac{1}{2}=(\tfrac{1}{2})'.$$

By (2.7),

(3.2) $$\text{if } x\in S, \text{ then } \tfrac{1}{2}\cdot x+\tfrac{1}{2}\in S,$$

since $\frac{1}{2}\cdot x<\frac{1}{2}=(\frac{1}{2})'$. From (3.2) with the help of (1.2) we derive

(3.3) $$\text{if } x,y\in S, \text{ then } \tfrac{1}{2}\cdot x+\tfrac{1}{2}\cdot y\in S.$$

4. Euclidean unit interval algebra. From (3.1) it follows in particular that $\frac{1}{2}\cdot x+\frac{1}{2}\cdot x=x$ for every x in S; consequently

(4.1) $$\text{if } x\in S, \text{ then } x=y+y \text{ for some } y\in S.$$

The analogous statement

(x) $$\text{If } x\in S, \text{ then } x=y\cdot y \text{ for some } y\in S$$

for the operation $\cdot$ by no means follows from the postulates (i)-(ix). On the other hand an unit interval algebra S may obviously satisfy the postulate (x); in this case it is said to be *Euclidean.*

Let $\mathfrak{S}=\langle S,+,\cdot,<\rangle$ be a Euclidean unit interval algebra. By (x) and (1.7) *the equation $x=y\cdot y$ has for a given x a unique solution for y*; we denote it as usual by $\sqrt{x}$. Thus

(4.2) $$x=\sqrt{x}\cdot\sqrt{x}.$$

From (4.2) with the help of (1.7) and (1.9) we deduce

(4.3) $$x\cdot y=z \quad \text{iff} \quad \sqrt{x}\cdot\sqrt{y}=\sqrt{z}$$

and

(4.4) $$x<y \quad \text{iff} \quad \sqrt{x}<\sqrt{y}.$$

Let us introduce a new operation $+'$ by putting

(4.5) $$x+'y=z \quad \text{iff} \quad \sqrt{x}+\sqrt{y}=\sqrt{z}.$$

We shall refer to the algebraic system $\mathfrak{S}'=\langle S,+',\cdot,<\rangle$ as the square root derivative of $\mathfrak{S}$. From (4.2)-(4.5) it follows at once that the function f, $f(x)=\sqrt{x}$, maps the system $\mathfrak{S}$ isomorphically onto the system $\mathfrak{S}'$. In consequence we get

THEOREM 4.6. *The square root derivative $\mathfrak{S}'$ of a Euclidean unit interval algebra $\mathfrak{S}$ is again a Euclidean unit interval algebra.*

5. Embedding theorem.

THEOREM 5.1. *Any unit interval algebra $\mathfrak{S}=\langle S,+,\cdot,<\rangle$ can be imbedded in a commutative ordered field $\overline{\overline{\mathfrak{S}}}=\langle\overline{\overline{S}},+,\cdot,<\rangle$ in such a way that S consists of all those elements $\overline{\overline{x}}\in\overline{\overline{S}}$ for which $0<\overline{\overline{x}}<1$, provided 0 is the zero element and 1 is the unit element of the field $\overline{\overline{\mathfrak{S}}}$. In fact, $\overline{\overline{\mathfrak{S}}}$ is up to isomorphism uniquely determined by $\mathfrak{S}$ in the following sense: if $\overline{\overline{\mathfrak{S}}}_1$ and $\overline{\overline{\mathfrak{S}}}_2$ are two ordered fields generated by $\mathfrak{S}$, then there is an isomorphic mapping of $\overline{\overline{\mathfrak{S}}}_1$ onto $\overline{\overline{\mathfrak{S}}}_2$ which leaves all elements of S unchanged. In addition, if $\mathfrak{S}$ is Euclidean, then $\overline{\overline{\mathfrak{S}}}$ is also Euclidean.*

Proof. We construct the algebraic system $\overline{\mathfrak{S}}_1=\langle\overline{S}_1,+_1,\cdot_1,<_1\rangle$ defined as follows. $\overline{S}_1$ is the Cartesian square $S\times S$ of S, and for any two elements $\langle x_1,x_2\rangle$ and $\langle y_1,y_2\rangle$ of $\overline{S}_1$

$$\langle x_1,x_2\rangle+_1\langle y_1,y_2\rangle=\langle\tfrac{1}{2}\cdot(x_1\cdot y_2)+\tfrac{1}{2}\cdot(x_2\cdot y_1),\ \tfrac{1}{2}\cdot(x_2\cdot y_2)\rangle,$$

$$\langle x_1,x_2\rangle\cdot_1\langle y_1,y_2\rangle=\langle x_1\cdot y_1,x_2\cdot y_2\rangle,$$

$$\langle x_1,x_2\rangle<_1\langle y_1,y_2\rangle \quad \text{iff} \quad x_1\cdot y_2<x_2\cdot y_1.$$

Let us notice that in the definition of $+_1$ we made an essential use of statement (3.3). Furthermore, we introduce the relation $\equiv$,

$$\langle x_1,x_2\rangle\equiv\langle y_1,y_2\rangle \quad \text{iff} \quad x_1\cdot y_2=x_2\cdot y_1$$

between elements of $\overline{S}_1$. It is easy to check that $\equiv$ is a congruence relation in $\overline{\mathfrak{S}}_1$ and that the quotient algebra $\overline{\mathfrak{S}}_1/{\equiv}$ is a commutative semi-field. More precisely, the algebra $\langle\overline{S}_1,+_1,<_1\rangle/{\equiv}$ is a commutative semi-group ordered in the natural way, the algebra $\langle\overline{S}_1,\cdot_1\rangle/{\equiv}$ is a commutative group, and the operations $+_1/{\equiv}$ and $\cdot_1/{\equiv}$ are connected by the distributivity law. Clearly, the unit element 1 of the group

$\langle \bar{S}_1, \cdot_1 \rangle/{\equiv}$ coincides with the coset consisting of all elements $\langle x_1, x_2 \rangle$ with $x_1 = x_2$. Let S_1 be the subclass of $\bar{S}_1$ consisting of all couples $\langle x_1, x_2 \rangle$ with $x_1 < x_2$. By (1.6) the class S_1 is closed under the relation $\equiv$. The algebraic subsystem $\langle S_1, +_1, \cdot_1, <_1 \rangle/{\equiv}$ of $\bar{\mathfrak{S}}_1/{\equiv}$ easily proves to be isomorphic with the unit interval algebra $\mathfrak{S}$. In consequence, by applying the well-known exchange theorem (compare e.g. [13], p. 46) we modify $\bar{\mathfrak{S}}_1/{\equiv}$ to an isomorphic system $\bar{\mathfrak{S}} = \langle \bar{S}, +, \cdot, < \rangle$ which is an extension of $\mathfrak{S} = \langle S, +, \cdot, < \rangle$. The class S consists then of all those elements $\bar{x}$ of $\bar{S}$ for which $\bar{x} < 1$.

By an analogous procedure we construct a commutative ordered field $\bar{\bar{\mathfrak{S}}} = \langle \bar{\bar{S}}, +, \cdot, < \rangle$ which is an extension of $\bar{\mathfrak{S}} = \langle \bar{S}, +, \cdot, < \rangle$ such that $\bar{S}$ consists of the positive elements in $\bar{\bar{S}}$. This time both the operations $+$ and $\cdot$ are always performable (in $\bar{\mathfrak{S}}$) which makes the construction still simpler.

In conclusion, the ordered field $\bar{\bar{\mathfrak{S}}}$ is an extension of $\mathfrak{S}$ such that for every $\bar{\bar{x}}$ in $\bar{\bar{S}}$

$$(1) \qquad \bar{\bar{x}} \in S \quad \text{iff} \quad 0 < \bar{\bar{x}} < 1,$$

where 0 is the zero element and 1 is the unit element of $\bar{\bar{\mathfrak{S}}}$.

It is easily seen that, if two ordered fields $\bar{\bar{\mathfrak{S}}}_1$ and $\bar{\bar{\mathfrak{S}}}_2$ are extensions of $\mathfrak{S}$, both satisfying the condition (1), then the identity function on S can be extended to an isomorphic mapping of $\bar{\bar{\mathfrak{S}}}_1$ onto $\bar{\bar{\mathfrak{S}}}_2$.

In case the unit interval algebra $\mathfrak{S}$ is Euclidean, postulate (x) immediately implies

$$(2) \qquad \text{if } \bar{x} \in \bar{S}, \text{ then } \bar{x} = \bar{y} \cdot \bar{y} \text{ for some } \bar{y} \in \bar{S},$$

and since $\bar{S}$ is the set of the positive elements in $\bar{\bar{S}}$ this means that the ordered field $\bar{\bar{\mathfrak{S}}}$ is Euclidean.

II. Hyperbolic calculus of segments and the related inner coordinatization of the plane

1. The theory $\mathcal{H}'$. Hilbert constructed the end-calculus in the subsystem $\mathcal{H}$ of the plane hyperbolic geometry based on the first three groups of his absolute axioms (i.e. on the plane axioms of incidence, of order, and of congruence) and on his hyperbolic axiom on the intersecting and non-intersecting lines (see [2], pp. 162). It is convenient to put the last axiom in the form of two separate axioms:

Ax 1. *For every line L and for every point a outside of L there are at least two distinct lines K_1 and K_2 both passing through a and neither intersecting L.*

Ax 2. *For every point a and for every half-line H there is a half-line G with the origin a and parallel to H.*

In [8] Szász showed that Ax 1 can be equivalently replaced by a weaker axiom

Ax 1′. *For some line L and for some point a outside of L there are at least two distinct lines K_1 and K_2 both passing through a and neither intersecting L.*

Ax 1′ is just the negation of the Euclid's axiom as formulated by Hilbert in [2], p. 28. Ax 2 is a special consequence of the continuity axiom.

In what follows we shall refer to the axiomatic theory described above (with Ax 1 replaced by Ax 1′) as $\mathcal{H}$. It is understood here in a slightly modified way by assuming the lines to be point sets (the incidence relation thus coinciding with the membership relation), not individuals distinct from points, as Hilbert does.

The hyperbolic calculus of segments may be constructed in the theory $\mathcal{H}$. It is more convienient, however, to deal with an axiomatic theory $\mathcal{H}'$ which is equivalent to $\mathcal{H}$ and moreover satisfies the following two conditions:

1. All the primitive notions coincide with relations among points.

2. All the axioms are formulated in the elementary language, i.e. by using only variables ranging over points.

On the other hand, $\mathcal{H}'$, as well as $\mathcal{H}$, is assumed to be provided with some set-theoretical basis; thus in $\mathcal{H}'$ besides the variables ranging over points there are also variables ranging over arbitrary point sets, over arbitrary classes of point sets, etc., and the membership symbol ϵ is included among the logical constants of the theory.

We do not specify the primitive notions and axioms of the theory $\mathcal{H}'$. It is known that in $\mathcal{H}'$ the ternary relation $\boldsymbol{L}$ of *collinearity* may serve as the only primitive notion (see [7], Corollary on p. 93 [2]), hence the ternary relation $\boldsymbol{B}$ of *betweenness* can serve this purpose as well. Among other relations with this property let us mention the quaternary relation $\boldsymbol{E}$ of *equidistance* (see [6], Section 3). Clearly, one may assume a larger system of primitive relations, e.g. consisting of all the relations $\boldsymbol{L}$, $\boldsymbol{B}$, and $\boldsymbol{E}$. In the latter case an axiom system for $\mathcal{H}'$ could easily be obtained from that for $\mathcal{H}$ under approriate definitions of the notions involved.

At any rate, each of the relations $\boldsymbol{L}$, $\boldsymbol{B}$, and $\boldsymbol{E}$ is a primitive or defined notion of $\mathcal{H}'$, and each of the axioms of $\mathcal{H}$ is an axiom or a theorem of $\mathcal{H}'$. In particular the sentences Ax 1′ and Ax 2, and consequently Ax 1, are theorems of $\mathcal{H}'$.

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(2) Royden does not give the proof of Corollary in all details. At any rate his proof works without any doubts for $\mathcal{H}'$ instead of $\mathcal{H}^*$ described in [7].

2. Hjelmslev's theorem. The first fundamental theorem on which our construction is based is

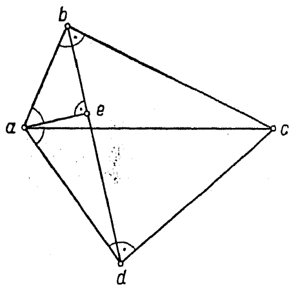

Fig. 1

THEOREM 2.1. *Given a convex quadrangle abcd with right angles at vertices b and d* (Fig. 1), *if e is the perpendicular projection of the vertex a upon the diagonal bd, then the angles bae and cad are congruent.*

This theorem had been well known as a theorem of Euclidean geometry. Hjelmslev was the first who proved it in absolute geometry (see [5]), and thus in a subtheory of H′.

3. The class $\boldsymbol{S}$ of free segments. By a *segment* we understand here any non ordered pair pq of distinct points p and q. We shall write $pq \cong p_1q_1$ to express that the segments pq and p_1q_1 are congruent. The set of all segments congruent to a given segment pq is called the *free segment* determined by pq and is denoted by $[pq]$. Free segments will be represented by variables $A, B, C, X, Y, Z, P, Q, R, \ldots$, possibly with subscripts, and the set of all free segments will be denoted by $\boldsymbol{S}$. Obviously,

(3.1) $$\boldsymbol{S} \neq 0\,.$$

A definition of the usual *less-than* relation for free segments follows.

(3.2) $$X < Y \ \textit{iff}\ \ \boldsymbol{B}(pqr)\,,\ X = [pq],\ \textit{and}\ \ Y = [pr] \quad \textit{for some distinct points}\ p, q, r\,.$$

Then

(3.3) $$X\ \textit{non-}{<}\ X;$$

(3.4) $$X = Y\ \textit{or}\ X < Y\ \textit{or else}\ Y < X;$$

(3.5) $$\textit{if}\ X < Y\ \textit{and}\ Y < Z\,,\ \textit{then}\ X < Z\,.$$

As usual the symbol $>$ will denote the relation converse to $<$.

The operation of the *usual addition* for the free segments will play in this work a preliminary role only. We give its definition and list its fundamental properties.

(3.6) $$X+Y = Z\ \ \textit{iff}\ \ \boldsymbol{B}(pqr)\,,\ X = [pq]\,,\ Y = [qr]\,,\ \textit{and}\ Z = [pr] \quad \textit{for some distinct points}\ p, q, r\,.$$

Then

(3.7) $$X+Y = Y+X;$$

(3.8) $$(X+Y)+Z = X+(Y+Z);$$

(3.9) $$X < Z\ \ \textit{iff}\ \ X+Y = Z\ \ \textit{for some}\ \ Y \in \boldsymbol{S};$$

(3.10) $$X = Y+Y\ \ \textit{for some}\ \ Y \in \boldsymbol{S}\,.$$

From (3.3), (3.4), and (3.7)-(3.9) we easily derive

(3.11) $$\textit{if}\ \ X+X = Y+Y\,,\ \ \textit{then}\ \ X = Y;$$

(3.12) $$\textit{if}\ \ X+Y_1 = X+Y_2\,,\ \ \textit{then}\ \ Y_1 = Y_2;$$

(3.13) $$\textit{if}\ X_1+Y_1 = X_2+Y_2\ \textit{and}\ X_1 < X_2\,,\ \textit{then}\ Y_1 > Y_2\,.$$

We put

(3.14) $$X - Y = Z\ \ \textit{iff}\ \ X = Y+Z\,.$$

4. Free angles. By an *angle* we understand here any non-ordered pair GH of half-lines G and H which are supposed to be non-collinear and to have a common origin. We shall write $GH \cong G_1H_1$ to express that the angles GH and G_1H_1 are congruent. The set of all angles congruent to a given angle GH is called the *free angle* determined by GH and is denoted by $[GH]$. Free angles will be represented by variables $\alpha, \beta, \gamma, \xi, \eta, \zeta$, possibly with subscripts. All right angles form a free angle; we refer to it as the right free angle and denote it by ϱ. All other free angles consist of only acute, or else of only obtuse angles. Therefore one can speak about the *acute* (or *obtuse*) free angles.

We introduce the relation $<$ and the operation $+$ for free angles in a complete analogy to the way in which we have done it for the free segments, only the betweenness relation $\boldsymbol{B}$ for three points on a line should be now replaced by the betweenness relation $\boldsymbol{B}$ for three half-lines in a half-pencil (see e.g. [1], pp. 47 and 50). The addition of free angles is then not always performable; at any rate the sum of two acute free angles always exists. Except for performability all the properties of $<$ and $+$ stated for free segments apply to free angles. In particular, for every free angle ξ there is a unique free angle η such that $\xi = \eta+\eta$. We denote by $\varrho/2$ the angle η such that $\varrho = \eta+\eta$. The symbol $>$ will denote the relation converse to $<$, and the formulas $\xi-\eta = \zeta$ and $\xi = \eta+\zeta$ will be used interchangeably.

5. The Lobachevskian function Π. Given a free segment X, by $\Pi(X)$ we understand the free angle defined as follows. Take an oriented line L and a point p not on L such that $X = [pq]$, where q is the perpendicular projection of p upon L (Fig. 2). Let G be the unique half-line which has the origin p and is parallel to L (see Ax 2), and let H coincide with the half-line pq. We then put $\Pi(X) = [GH]$. In other words $\Pi(X)$ is the *angle of parallelism* for p with respect to L. It is well known that this definition does not depend on the line L and the point p, but only on the free segment X itself.

With the help of Ax 1 one can easily prove that

(5.1) $\quad$ *if* $X < Y$, *then* $\Pi(X) > \Pi(Y)$.

Moreover,

(5.2) $\quad$ *the range of* Π *consists of all acute free angles.*

In fact, (5.2) is equivalent (in absolute geometry) to the statement on the existence of a line parallel to both sides of a given angle, the latter statement being proved by Hilbert in the theory $\mathfrak{H}$ (see [2], Theorem 3

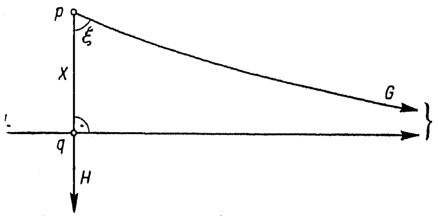

Fig. 2

on p. 165). By (5.1) and (5.2) the function Π establishes a one-to-one correspondence between the free segments and the acute free angles. We make the following convention: A free segment and the corresponding free angle are denoted by the corresponding letters of the Latin and Greek alphabets ([3]), with the same subscript or superscript, if any; e.g.

$$\Pi(A) = \alpha, \quad \Pi(X) = \xi, \quad \Pi(Y) = \eta, \quad \Pi(X') = \xi'.$$

In this notation the statement (5.1) takes the form

(5.3) $\quad$ *if* $X < Y$, *then* $\xi > \eta$.

6. The auxiliary operation ′. For every free segment X the condition

(6.1) $$\Pi(X) + \Pi(X') = \varrho$$

defines a unique free segment X' which can be referred to as the *complement* of X. Clearly

(6.2) $$X'' = X,$$

and by (5.1)

(6.3) $\quad$ *if* $X < Y$ *then* $X' > Y'$.

By the convention of Section 5 the operation ′ on free segments induces an operation ′ on free angles. In fact, the formula (6.1) is equivalent to the formula

$$\xi + \xi' = \varrho$$

([3]) This notation is adopted from [6].

and hence

(6.4) $$\xi' = \varrho - \xi$$

for every acute free angle ξ.

7. The free right triangle. We say that an ordered quintupel $X\alpha Z\beta Y$, formed by three free segments X, Z, Y and two free angles α, β, is a *free right triangle*—symbolically, $\boldsymbol{T}(X\alpha Z\beta Y)$—if and only if there is a right triangle abc with a right angle at the vertex c (Fig. 3) such that

$$X = [ac], \quad \alpha = [\sphericalangle a], \quad Z = [ab], \quad \beta = [\sphericalangle b], \quad Y = [bc].$$

We then say that abc is a *representative* of $X\alpha Z\beta Y$. We refer to X, α, Z, β, Y as the *terms* of $X\alpha Z\beta Y$.

Obviously

(7.1) $$\boldsymbol{T}(X\alpha Z\beta Y) \quad \textit{iff} \quad \boldsymbol{T}(Y\beta Z\alpha X);$$

(7.2) *for every* X *and* Y *there are* α, Z, β *such that* $\boldsymbol{T}(X\alpha Z\beta Y)$;

(7.3) *for every* Z *and* β *there are* X, α, Y *such that* $\boldsymbol{T}(X\alpha Z\beta Y)$;

(7.4) *if* $B > Y$, *then there are* X, α, Z *such that* $\boldsymbol{T}(X\alpha Z\beta Y)$;

(7.5) *if* $\boldsymbol{T}(X\alpha Z\beta Y)$, *then any two of the five terms* X, α, Z, β, Y *determine uniquely the remaining three.*

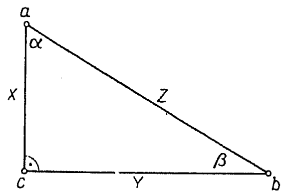

Fig. 3

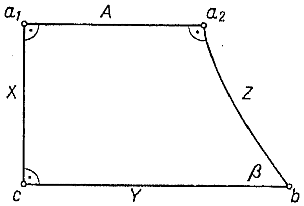

Fig. 4

8. The free Lambert quadrangle. We say that an ordered quintupel $XAZ\beta Y$, formed by four free segments X, A, Z, Y and a free angle β, is a *free Lambert quadrangle*—symbolically, $\boldsymbol{Q}(XAZ\beta Y)$—if and only if there is a Lambert quadrangle a_1a_2bc with right angles at vertices a_1, a_2, c (Fig. 4) such that

$$X = [a_1c], \quad A = [a_1a_2], \quad Z = [a_2b], \quad \beta = [\sphericalangle b], \quad Y = [bc].$$

We then say that a_1a_2bc is a *representative* of $XAZ\beta Y$. We refer to X, A, Z, β, Y as the *terms* of $XAZ\beta Y$.

Obviously

(8.1) $$\boldsymbol{Q}(XAZ\beta Y) \quad \textit{iff} \quad \boldsymbol{Q}(AXY\beta Z),$$

and it is easy to check that

(8.2) *if* $\boldsymbol{Q}(XAZ\beta Y)$, *then any two of the five terms* X, A, Z, β, Y *determine uniquely the remaining three.*

9. Liebmann's theorem. The last theorem fundamental for our construction can be expressed in terms of the free right triangle and the free Lambert quadrangle as the equivalence

(9.1) $\qquad \boldsymbol{T}(X\alpha Z\beta Y) \quad \textit{iff} \quad \boldsymbol{Q}(XA'Z\eta B)\,.$

This equivalence had been known only as a theorem of 3-dimensional hyperbolic geometry until Liebmann proved it in the theory $\mathfrak{H}$ (see [6], p. 189). For the convenience of the reader the Liebmann proof of (9.1) is repeated, in a somewhat different form, in Appendix, p. 155.

By (8.1) and (9.1) we have

$$\boldsymbol{Q}(XA'Z\eta B) \quad \text{iff} \quad \boldsymbol{Q}(A'XB\eta Z) \quad \text{iff} \quad \boldsymbol{T}(A'\xi'B\zeta Y);$$

hence

(9.2) $\qquad \boldsymbol{T}(X\alpha Z\beta Y) \quad \textit{iff} \quad \boldsymbol{T}(A'\xi'B\zeta Y)\,.$

Furthermore, by (7.1) and (9.2) we have

$$\boldsymbol{T}(A'\xi'B\zeta Y) \quad \text{iff} \quad \boldsymbol{T}(Y\zeta B\xi'A') \quad \text{iff} \quad \boldsymbol{T}(Z'\eta'X'\beta A') \quad \text{iff} \quad \boldsymbol{T}(A'\beta X'\eta'Z');$$

hence

(9.3) $\qquad \boldsymbol{T}(X\alpha Z\beta Y) \quad \textit{iff} \quad \boldsymbol{T}(A'\beta X'\eta'Z')$

(cf. [5], p. 191, where this equivalence is proved in a much more involved way.)

From (7.3), (7.4), and (7.1) with the help of (9.2) and (9.3) we deduce what follows:

(9.4) *for every X and β there are α, Z, Y such that $\boldsymbol{T}(X\alpha Z\beta Y)$;*

(9.5) *if $Z > Y$, then there are X, α, β such that $\boldsymbol{T}(X\alpha Z\beta Y)$;*

(9.6) *if $\alpha+\beta < \varrho$, then there are X, Y, Z such that $\boldsymbol{T}(X\alpha Z\beta Y)$.*

When reducing (9.6) to (7.4) we use also (5.3) and (6.4).

The statements (7.2), (7.3), (7.4), (9.4), (9.5), and (9.6) form together the full system of the existential statements for the free right triangle (see (7.1)). An analogous system can be established for the free Lambert quadrangle, however to do this is superfluous for our discussion.

10. The auxiliary operation $\odot$. Given three free segments X, Y and Z, we put $X \odot Y = Z$ if and only if there is a right triangle abc with a right angle at the vertex c (Fig. 5) such that $X = [ac]$, $Y = [bc]$, $Z = [ab]$ [4]. In other words,

(10.1) $\quad X \odot Y = Z \;\; \textit{iff} \;\; \boldsymbol{T}(X\alpha Z\beta Y)$ *for some α and β.*

---

[4] This operation was used in [4].

Thus for any given X and Y there is a unique Z such that $X \odot Y = Z$. Let us notice that

(10.2) *if $X \in \boldsymbol{S}$, then $X = Y \odot Y$ for a unique $Y \in \boldsymbol{S}$.*

In fact, putting $X = X_1 + X_1$ we have $X = Y \odot Y$ if and only if

$$\boldsymbol{T}\Big(X_1\alpha Y \tfrac{\varrho}{2} Z\Big) \quad \text{for some} \quad \alpha \text{ and } Z$$

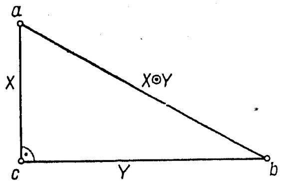

Fig. 5

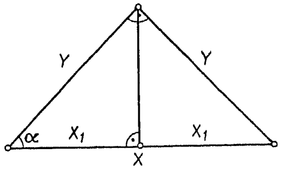

Fig. 6

(Fig. 6). By (3.10), (3.11), (9.4), and (7.5) the latter statement determines uniquely Y for the given X.

11. The operation $\bullet$. Given three free segments X, Y and Z, we put $X \bullet Y = Z$ if and only there is a right triangle abc with a right angle at the vertex c (Fig. 7) such that $X = [ab]$, $\eta = [\sphericalangle b]$, $Z = [bc]$. In other words,

(11.1) $\qquad X \bullet Y = Z \;\; \textit{iff} \;\; \boldsymbol{T}(A\beta X\eta Z)$ *for some A and β.*

It follows at once from the definition that for any given X and Y there is a unique Z such that $X \bullet Y = Z$, in other words

(11.2) *if $X, Y \in \boldsymbol{S}$, then $X \bullet Y \in \boldsymbol{S}$,*

that

(11.3) $\quad X \bullet Y = X \bullet Y_1 \quad \textit{implies} \quad Y = Y_1\,,$

and that

(11.4) $\quad X \bullet Y = Z \quad \textit{implies} \quad X > Z\,.$

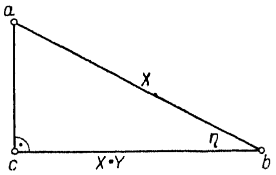

Fig. 7

Moreover, with the help of (11.1) and (9.5) we get

(11.5) $\qquad$ *if $X > Z$, then $X \bullet Y = Z$ for some $Y \in \boldsymbol{S}$.*

By (9.2) we have $\boldsymbol{T}(A\beta X\eta Z)$ if and only if $\boldsymbol{T}(B'\alpha'Y\xi Z)$ and consequently from (11.1) we deduce

(11.6) $\qquad X \bullet Y = Y \bullet X\,.$

Thus the operation $\bullet$ is commutative. We shall prove now the associativity law

(11.7) $\qquad (X \bullet Y) \bullet Z = X \bullet (Y \bullet Z)\,.$

Let

$$X\cdot Y = A\,,\qquad A\cdot Z = B\,,\qquad Y\cdot Z = C\,,\qquad X\cdot C = D\,.$$

We have to show that $B = D$. To this aim we pick five half-lines H_1, H_2, H_3, H_4, H_5, with a common origin p (Fig. 8), satisfying the conditions

(1) $\qquad \boldsymbol{B}(H_{i-1}H_iH_{i+1}) \quad \text{for} \quad i = 2, 3, 4$

and

(2) $\qquad [H_1H_2] = [H_3H_4] = \zeta \quad \text{and} \quad [H_2H_3] = [H_4H_5] = \xi\,,$

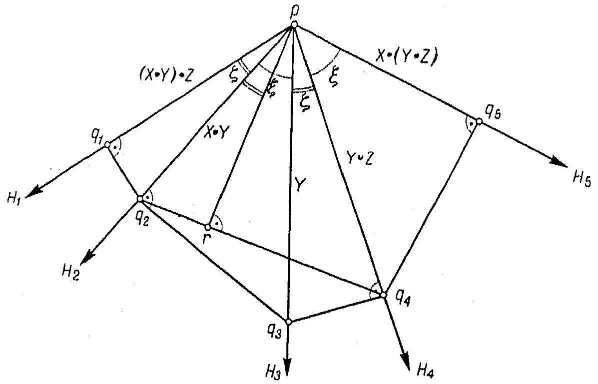

Fig. 8

and we take the point $q_3 \in H_3$ such that $[pq_3] = Y$. Let the points q_2, q_1, q_4, q_5 be the perpendicular projections of q_3 upon H_2, q_2 upon H_1, q_3 upon H_4, and q_4 upon H_5, respectively. Then, partly with the help of (11.6), we get

$$[pq_2] = A\,,\qquad [pq_1] = B\,,\qquad [pq_4] = C\,,\qquad [pq_5] = D\,.$$

Let r be the perpendicular projection of p upon the line q_2q_4. By applying Theorem 2.1 to the quadrangle $pq_2q_3q_4$ we get

$$\sphericalangle\, q_2pr \cong \sphericalangle\, q_3pq_4\,,$$

and hence by (2)

$$\sphericalangle\, q_1pq_2 \cong \sphericalangle\, q_2pr \quad \text{and} \quad \sphericalangle\, rpq_4 \cong \sphericalangle\, q_4pq_5\,,$$

which implies $pq_1 \cong pr \cong pq_5$. Consequently, $B = D$ which was to be proved

In conclusion let us notice that by (9 3), (10.1) and (11.1) we have

(11.8) $\qquad X \odot Y = Z \quad \text{iff} \quad X'\cdot Y' = Z'\,,$

which by (6.2) implies at once the dual formula

(11.9) $\qquad X\cdot Y = Z \quad \text{iff} \quad X' \odot Y' = Z'\,.$

Therefore the systems $\langle \boldsymbol{S}, \cdot\rangle$ and $\langle \boldsymbol{S}, \odot\rangle$ are isomorphic. In particular, by (10.2),

(11.10) $\qquad$ *if* $X \in \boldsymbol{S}$, *then* $X = Y\cdot Y$ *for a unique* $Y \in \boldsymbol{S}$.

12. The auxiliary operation $\oplus$. Given three free segments X, Y, Z, we put $X \oplus Y = Z$ if and only if there is a Lambert quadrangle $abcd$ with right angles at vertices a, b, c (Fig. 9) such that $X = [ba]$, $Y = [bc]$, $Z = [bd]$. In other words,

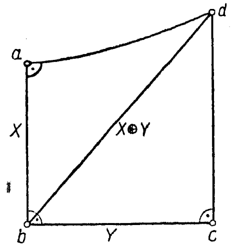

Fig. 9

(12.1) $\qquad X \oplus Y = Z$ *iff* $X = A\cdot Z$ *and* $Y = A'\cdot Z$ *for some* $A \in \boldsymbol{S}$.

Clearly, not for every X and Y there is a Z such that $X \oplus Y = Z$. However, if Z exists it is uniquely determined by X and Y. To express that $X \oplus Y$ does or does not exist, we shall write $X \oplus Y \in \boldsymbol{S}$, $X \oplus Y \notin \boldsymbol{S}$, respectively. It follows immediately from the definition that

(12.2) $\qquad$ *if* $X \oplus Y \in \boldsymbol{S}$ *and* $Z < Y$, *then* $X \oplus Z \in \boldsymbol{S}$;

(12.3) $\qquad X \oplus X' \notin \boldsymbol{S}$, *and* $X \oplus Y \in \boldsymbol{S}$ *for every* $Y < X'$.

Thus X' is the smallest segment which cannot be added to X.

The formula (12.1) implies at once

(12.4) $\qquad A\cdot Z \oplus A'\cdot Z = Z\,.$

This may be treated as a particular case of the distributivity law

(12.5) $\qquad$ *if* $X \oplus Y \in \boldsymbol{S}$, *then* $X\cdot Z \oplus Y\cdot Z = (X \oplus Y)\cdot Z\,,$

which, with the help of (12.1), reduces to the associativity law for $\cdot$. In fact, if $X \oplus Y = B$, then

$$X = A\cdot B \text{ and } Y = A'\cdot B \text{ for some } A \in \boldsymbol{S}\,,$$

and

$$\begin{aligned}(X \oplus Y)\cdot Z = B\cdot Z &= A\cdot(B\cdot Z) \oplus A'\cdot(B\cdot Z)\\ &= (A\cdot B)\cdot Z \oplus (A'\cdot B)\cdot Z = X\cdot Z \oplus Y\cdot Z\,.\end{aligned}$$

If follows at once from the definition that

(12.6) $\qquad$ *if* $X \oplus Y \in \boldsymbol{S}$, *then* $X \oplus Y = Y \oplus X\,,$

and that

(12.7) $\qquad X \oplus Y = Z \quad$ *implies* $\quad X < Z\,.$

From (12.1) and (11.5) we deduce the converse formula

(12.8) *if* $X < Z$, *then* $X \oplus Y = Z$ *for some* $Y \in S$.

Finally, we pass to the associativity law. We start with the preliminary formula

(12.9) $$X \cdot Y \oplus Y' = X \odot Y'.$$

By (11.4) and (11.6) we have $X \cdot Y < Y$, from which it follows by (12.3), (12.6), and (6.2) that $X \cdot Y \oplus Y' \in S$; consequently there is a Lambert quadrangle $abcd$ with right angles at vertices a, b, c (Fig. 10) such that

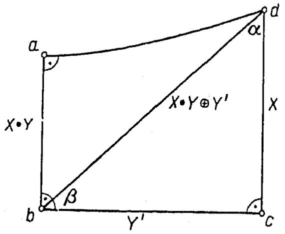

Fig. 10

$$[ba] = X \cdot Y, \quad [bc] = Y', \quad [bd] = X \cdot Y \oplus Y'.$$

Hence

(1) $$Q\big((X \cdot Y)\, Y' [cd] [\sphericalangle\, d] [da]\big)$$

and

(2) $$T\big([cd]\, \alpha (X \cdot Y \oplus Y') \beta Y'\big)$$

for some α and β.

Moreover, we have $T\big(A_1 \beta_1 X \eta (X \cdot Y)\big)$ for some A_1 and β_1, which implies, by (7.1) and (9.1),

(3) $$Q\big((X \cdot Y)\, Y' X \alpha_1 B_1\big).$$

With the help of (8.2) it follows from (1) and (3) that $[cd] = X$, and consequently (2) implies at once (12.9).

The associativity law

(12.10) *if* $X \oplus Y, (X \oplus Y) \oplus Z \in S$, *then* $Y \oplus Z \in S$ *and* $(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z)$

is proved as follows. By (12.6) and (12.7) we have $Y < X \oplus Y$, from which it follows, by (12.2) and (12.6), that $Y \oplus Z \in S$. Let

$$X \oplus Y = A, \quad A \oplus Z = B, \quad Y \oplus Z = C.$$

Then $X < A$ and $A < B$ by (12.7), and therefore by (3.5)

$$X < B.$$

By (12.1) and (11.5), for some P, Q, R, S we now have

$$X = P \cdot A, \quad Y = P' \cdot A, \quad A = Q \cdot B, \quad Z = Q' \cdot B,$$
$$Y = R \cdot C, \quad Z = R' \cdot C, \quad X = S \cdot B.$$

Then

$$P \cdot A = S \cdot B, \quad P' \cdot A = R \cdot C, \quad Q' \cdot B = R' \cdot C, \quad A = Q \cdot B,$$

from which with the help of the associativity law for $\cdot$ we derive

$$(P \cdot Q) \cdot B = S \cdot B, \quad (P' \cdot Q) \cdot B = R \cdot C, \quad Q' \cdot B = R' \cdot C.$$

From the second and the third equalities we get by means of (12.4), (12.5), (12.9), and (11.8)

$$C = (P' \cdot Q) \cdot B \oplus Q' \cdot B = (P' \cdot Q \oplus Q') \cdot B = (P' \odot Q') \cdot B = (P \cdot Q)' \cdot B;$$

then by (11.3) and (11.6) from the first identity we get $P \cdot Q = S$, and thus $C = S' \cdot B$. Since, moreover, $X = S \cdot B$, by (12.4) we get $X \oplus C = B$, which concludes the proof.

13. The field $\overline{\overline{\mathfrak{S}}} = \langle \overline{\overline{S}}, +, \cdot, < \rangle$. Formulas (3.1), (3.3)-(3.5), (12.6), (12.10), (12.7), (12.8), (11.2), (11.6), (11.7), (11.4), (11.5), (12.5), and (11.10) clearly imply the following theorem.

THEOREM 13.1. *The system* $\mathfrak{S}^\circ = \langle S, \oplus, \cdot, < \rangle$ *is a Euclidean unit interval algebra.*

We modify now the operation of addition by putting

(13.2) $$X + Y = Z \quad \textit{iff} \quad \sqrt{X} \oplus \sqrt{Y} = \sqrt{Z},$$

the square root being defined as in Section 4 of Chapter I, Then the system $\mathfrak{S} = \langle S, +, \cdot, < \rangle$ is the square root derivative of $\mathfrak{S}^\circ$, and using Theorem 4.6 of Chapter I we derive from Theorem 13.1

THEOREM 13.3. *The system* $\mathfrak{S} = \langle S, +, \cdot, < \rangle$ *is a Euclidean unit interval algebra.*

In consequence, by Theorem 5.1 of Chapter I, we have

THEOREM 13.4. *The system* $\mathfrak{S} = \langle S, +, \cdot, < \rangle$ *can be imbedded in a commutative Euclidean field* $\overline{\overline{\mathfrak{S}}} = \langle \overline{\overline{S}}, +, \cdot, < \rangle$, *with the zero element* **0** *and the unit element* **1**, *in such a way that*

(1) *for every* X *in* $\overline{\overline{S}}$ *we have* $X \in S$ *iff* $\mathbf{0} < X < \mathbf{1}$.

The field $\overline{\overline{\mathfrak{S}}}$ *is uniquely determined up to isomorphism by* $\mathfrak{S}$ *in the sense of Theorem* 5.1.

From now on we assume that this field has been fixed and we apply to it the familiar field-theoretical notation. In particular, the operations $+$ and $\cdot$ and the relation $<$ are now understood to be defined for arbitrary elements of the field and not only for free segments. Also the variables A, B, C, X, Y, Z are now understood to range over all elements of $\overline{\overline{S}}$ and not only over free segments (this stipulation being already applied in the formula (1)).

On the elements of the Cartesian product $\bar{\bar{S}}\times\bar{\bar{S}}$ we introduce in the usual way the operations of addition, subtraction, and scalar product. Let $\mathfrak{X}=\langle X_1, X_2\rangle$ and $\mathfrak{Y}=\langle Y_1, Y_2\rangle$ be in $\bar{\bar{S}}\times\bar{\bar{S}}$, then

$$\mathfrak{X}+\mathfrak{Y}=\langle X_1+Y_1, X_2+Y_2\rangle\,,$$
$$\mathfrak{X}-\mathfrak{Y}=\langle X_1-Y_1, X_2-Y_2\rangle\,,$$
$$\mathfrak{X}\cdot\mathfrak{Y}=X_1\cdot Y_1+X_2\cdot Y_2\,.$$

The properties of these operations are assumed to be known. We put $\mathfrak{X}^2=\mathfrak{X}\cdot\mathfrak{X}$.

14. The characterization of preliminary operations in terms of the field operations. In this section we shall deal with free segments. From (13.2) we get at once

(14.1) $$X\oplus Y=Z \quad \textit{iff} \quad X^2+Y^2=Z^2\,.$$

It seems natural to refer to (14.1) as the *Pythagorian formula.* When applying it in particular to (12.4) we get $(A\cdot Z)^2+(A'\cdot Z)^2=Z^2$ for any arbitrary A and Z; consequently $A^2+(A')^2=\mathbf{1}$, i.e.

(14.2) $$A'=\sqrt{\mathbf{1}-A^2}\,.$$

From (11.8), (14.2), and (6.2) we obtain by a simple calculation

(14.3) $$X\odot Y=\sqrt{X^2+Y^2-X^2\cdot Y^2}\,.$$

From (14.1) and (14.3) we derive

(14.4) $$\textit{if } Q(XAZ\beta Y),\textit{ then } X=\sqrt{\mathbf{1}-A^2}\cdot Z\,,$$

since $Q(XAZ\beta Y)$ implies $X\oplus A=A\odot Z$ (see Fig. 4).

It remains to express the operations $+$ and $-$, i.e. usual addition and subtruction of free segments, in terms of $\boldsymbol{+}$ and $\cdot$. This problem is a little more involved. Let us assume that $X+Y=Z$. Then $X<Z$ by (3.9), therefore $\sqrt{X}<\sqrt{Z}$, and hence $\sqrt{X}=A\cdot\sqrt{Z}$ for some $A\in S$ by (11.5). Then

$$\sqrt{X\cdot Z}=A\cdot Z \quad\text{and}\quad X=A\cdot\sqrt{X\cdot Z}$$

and consequently there is a right triangle $a_1b_1c_1$ (Fig. 11) with the right angle at vertex c_1 ($\sphericalangle a_1$ representing the free angle α) such that

$$[a_1b_1]=Z\,,\quad [a_1c_1]=\sqrt{X\cdot Z}\,,\quad\text{and}\quad [a_1d_1]=X\,,$$

provided d_1 is the perpendicular projection of c_1 upon a_1b_1. Then $[b_1d_1]=Y$ by (3.12). By the same argument there is a right triangle $a_2b_2c_2$ with the right angle at the vertex c_2 such that

$$[a_2b_2]=Z\,,\quad [b_2c_2]=\sqrt{Y\cdot Z}\,,\quad\text{and}\quad [b_2d_2]=Y\,,$$

provided d_2 is the perpendicular projection of c_2 upon a_2b_2. Then $[a_2d_2]=X$. It is easily seen that $c_1d_1\cong c_2d_2$ and consequently $b_1c_1\cong b_2c_2$, i.e. $[b_1c_1]=\sqrt{Y\cdot Z}$. In conclusion we get

$$\sqrt{X\cdot Z}\odot\sqrt{Y\cdot Z}=Z$$

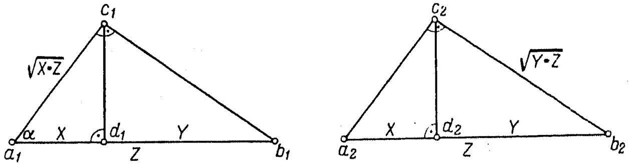

Fig. 11

which with the help of (14.3) proves to be equivalent to

$$X+Y\boldsymbol{-}X\cdot Y\cdot Z=Z\,.$$

From the last formula we get at once

(14.5) $$X+Y=\frac{X\boldsymbol{+}Y}{\mathbf{1}\boldsymbol{+}X\cdot Y}$$

and

(14.6) $$Z-X=\frac{Z\boldsymbol{-}X}{\mathbf{1}\boldsymbol{-}Z\cdot X}\quad\text{for}\quad X<Z\,.$$

It follows immediately from (14.5) that

(14.7) $$X+Y<X\boldsymbol{+}Y$$

for every two free segments X and Y.

15. The distance function. Given two arbitrary points p and q, we put

$$\delta(p,q)=\begin{cases}\mathbf{0} & \text{if}\quad p=q\,,\\ [pq] & \text{if}\quad p\neq q\,,\end{cases}$$

and refer to the element $\delta(p,q)$ as the *distance* between the points p and q; clearly $\delta(p,q)$ is always an element of $\bar{\bar{S}}$. This terminology is justified by the fact that the function δ satisfies (in $\bar{\bar{\mathfrak{S}}}$) the three distance axioms. For, by the formula (1) in Theorem 13.4,

(15.1) $$\delta(p,q)\geqslant\mathbf{0}\textit{ and }\delta(p,q)=\mathbf{0}\textit{ iff }p=q;$$

furthermore

(15.2) $$\delta(p,q)=\delta(q,p);$$

finally

(15.3) $$\delta(p,q)\boldsymbol{+}\delta(q,r)\geqslant\delta(p,r)\,,$$

the triangle inequality for $\dotplus$ resulting by (14.7) at once from the triangle inequality for $+$.

Let us notice that by the formula (1) in Theorem 13.4 we have $\delta(p, q) < \mathbf{1}$ for any two points p and q.

16. The rectangular coordinate system. We take two arbitrary perpendicular axes (oriented lines) L_1 and L_2 intersecting in a point a (Fig. 12). Given a point p, let p_1 and p_2 be the perpendicular projections of p upon L_1 and L_2, respectively. We put for $i = 1, 2$

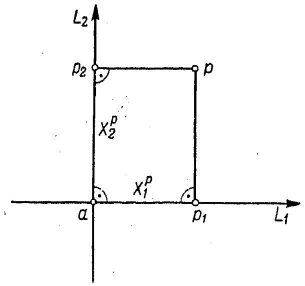

Fig. 12

$$X_i^p = \begin{cases} \mathbf{0} & \text{if} \quad p_i = a\,, \\ [ap_i] & \text{if} \quad a \prec p_i \text{ on } L_i\,, \\ -[ap_i] & \text{if} \quad p_i \prec a \text{ on } L_i\,, \end{cases}$$

provided $\prec$ is the order of points on the axis L_i. Thus X_1^p and X_2^p are elements of $\bar{\bar{S}}$. We shall refer to the function Φ defined for every point p by the formula

$$\Phi(p) = \langle X_1^p, X_2^p \rangle$$

as the *rectangular coordinate system* with the *axes* L_1 and L_2, and to X_1^p and X_2^p as the *first* and *second coordinates* of the point p in the system Φ. We shall refer to the point a as the *origin* of Φ.

With the help of the formula (1) in Theorem 13.4 and (14.1) we easily get

THEOREM 16.1. *Every rectangular coordinate system Φ establishes a one-to-one correspondence between the points p of the hyperbolic plane and the elements $\langle X_1, X_2 \rangle$ of the Cartesian product $\bar{\bar{S}} \times \bar{\bar{S}}$ satisfying the condition* $X_1^2 + X_2^2 < \mathbf{1}$,

In particular, for every point p

$$\Phi^2(p) < \mathbf{1}\,, \tag{16.2}$$

where $\Phi^2(p)$ stands for $\big(\Phi(p)\big)^2$.

From now on we assume the axes L_1 and L_2, and hence also the coordinate system Φ and its origin a, to be fixed.

By (16.2), given two points p and q, we have $\Phi^2(p) < \mathbf{1}$, $\Phi^2(q) < \mathbf{1}$, and $\big(\Phi(p)\cdot\Phi(q)\big)^2 \leqslant \Phi^2(p)\cdot\Phi^2(q) < \mathbf{1}$. Consequently the formula

$$F(p, q) = \frac{\big(\mathbf{1} - \Phi^2(p)\big)\cdot\big(\mathbf{1} - \Phi^2(q)\big)}{\big(\mathbf{1} - \Phi(p)\cdot\Phi(q)\big)^2} \tag{16.3}$$

defines a function F, which correlates with every two points p and q an element $F(p, q)$ of $\bar{\bar{S}}$. It is easy to check that

$$\mathbf{0} < F(p, q) \leqslant \mathbf{1} \tag{16.4}$$

for every two points p and q. Since $\Phi(a) = \langle \mathbf{0}, \mathbf{0} \rangle$, it follows from (16.3) that

$$F(a, p) = F(p, a) = \mathbf{1} - \Phi^2(p) \tag{16.5}$$

for every point p.

17. Analytic formulas for distance and perpendicularity. We take two arbitrary points p and q; let

$$\Phi(p) = \langle X_1^p, X_2^p \rangle \quad \text{and} \quad \Phi(q) = \langle X_1^q, X_2^q \rangle\,. \tag{1}$$

For simplicity let us assume that in addition

$$\mathbf{0} < X_1^p < X_1^q \quad \text{and} \quad \mathbf{0} < X_2^p < X_2^q\,. \tag{2}$$

We denote by p_i and q_i the perpendicular projections of the points p and q upon the axis L_i $(i = 1, 2)$, and by r the perpendicular projection

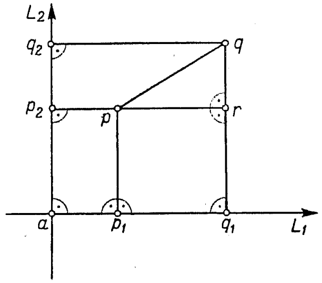

Fig. 13

of p upon the line qq_1 (Fig. 13). Then by means of (14.4), (14.6), (14.3) and (1) we get in turn

$$[pp_1] = \frac{X_2^p}{\sqrt{\mathbf{1} - (X_1^p)^2}} \quad \text{and} \quad [qq_1] = \frac{X_2^q}{\sqrt{\mathbf{1} - (X_1^q)^2}}\,, \tag{3}$$

$$[p_1q_1] = \frac{X_1^q - X_1^p}{\mathbf{1} - X_1^q \cdot X_1^p}\,, \tag{4}$$

$$[q_1r] = \sqrt{\mathbf{1} - [p_1q_1]^2}\cdot[pp_1]\,, \tag{5}$$

$$(6)\qquad [pr] = \frac{[p_1q_1]}{\sqrt{1-[q_1r]^2}},$$

$$(7)\qquad [qr] = \frac{[qq_1]-[q_1r]}{1-[qq_1]\cdot[q_1r]},$$

$$(8)\qquad [pq] = \sqrt{[pr]^2+[qr]^2-[pr]^2\cdot[qr]^2}\,.$$

A simple calculation leads from (3)-(8) to an analytic formula for distance

$$(17.1)\qquad \delta(p,q) = \sqrt{1-F(p,q)}\,,$$

the function F being defined by (16.3). The formula (17.1) remains valid in the general case, independent of the assumption (2).

With the help of (17.1) we easily derive from (14.3) the analytic formula for perpendicularity

$$(17.2)\qquad [\sphericalangle pqr] = \varrho \quad \textit{iff} \quad F(p,q)\cdot F(q,r) = F(p,r)\,.$$

Taking in (17.2) the point a (the origin of the coordinate system Φ) first for p and then for q, we get with the help of (16.3), (16.5), and (16.2),

$$(17.3)\qquad [\sphericalangle aqr] = \varrho \quad \textit{iff} \quad \Phi(q)\cdot\big((\Phi(r)-\Phi(q)\big) = 0$$

and

$$(17.4)\qquad [\sphericalangle par] = \varrho \quad \textit{iff} \quad \Phi(p)\cdot\Phi(r) = 0\,.$$

18. An analytic formula for collinearity. We consider an arbitrary straight line K. In case the origin a of the coordinate system Φ is not on K we denote by b the perpendicular projection of a upon K;

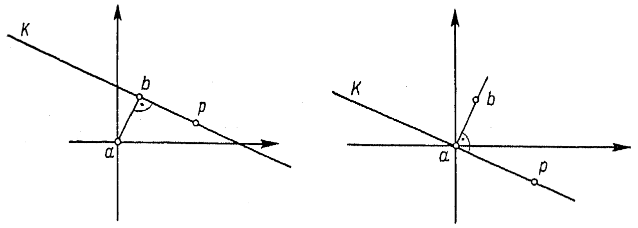

Fig. 14

in case a is on K we take as b an arbitrary point not on K (Fig. 14). Let $\Phi(b) = \langle B_1, B_2\rangle$. Then, for an arbitrary point p with $\Phi(p) = \langle X_1, X_2\rangle$, we have by (17.3) and (17.4)

$$p \in K \quad \text{iff} \quad \Phi(b)\cdot\big(\Phi(p)-\Phi(b)\big) = 0$$

in the first case; and

$$p \in K \quad \textit{iff} \quad \Phi(b)\cdot\Phi(p) = 0$$

in the second case. In both cases the straight line K has the linear equation

$$B_1\cdot X_1 + B_2\cdot X_2 + B_3 = 0$$

where B_3 coincides either with $-\Phi^2(b)$ or 0. In the first case, by applying (16.2), we get $B_1^2+B_2^2 = -B_3 > B_3^2$; and hence

$$B_1^2+B_2^2 > B_3^2\,.$$

Since $b \neq a$, this inequality remains valid in case $B_3 = 0$.

On the other hand, every linear equation

$$(1)\qquad B_1\cdot X_1 + B_2\cdot X_2 + B_3 = 0 \quad \text{with} \quad B_1^2+B_2^2 > B_3^2$$

describes a line K. If $B_3 \neq 0$, then K passes through the point b, provided

$$\Phi(b) = \left\langle -\frac{B_1\cdot B_3}{B_1^2+B_2^2},\ -\frac{B_2\cdot B_3}{B_1^2+B_2^2}\right\rangle,$$

and is perpendicular to the line ab. If $B_3 = 0$, then K passes through the point a and is perpendicular to the line ab, provided $\Phi(b) = \langle B_1, B_2\rangle$ in case $B_1^2+B_2^2 < 1$ and

$$\Phi(b) = \left\langle \frac{B_1}{2\cdot(B_1^2+B_2^2)},\ \frac{B_2}{2\cdot(B_1^2+B_2^2)}\right\rangle$$

in case $B_1^2+B_2^2 > 1$.

Thus the linear equation (1) is the analytic representation of the line; consequently the analytic formula for the relation L of collinearity has the form

$$(18.1)\qquad L(pqr) \quad \textit{iff} \quad \begin{vmatrix} 1 & X_1^p & X_2^p \\ 1 & X_1^q & X_2^q \\ 1 & X_1^r & X_2^r \end{vmatrix} = 0$$

provided $\Phi(p) = \langle X_1^p, X_2^p\rangle$, $\Phi(q) = \langle X_1^q, X_2^q\rangle$, $\Phi(r) = \langle X_1^r, X_2^r\rangle$.

Since in the axiomatic system $\mathcal{H}'$ all the geometrical notions can be defined in terms of the relation L (see p. 135), the formula (18.1) is a base for the analytic geometry on the hyperbolic plane. Since the condition (18.1) of collinearty is the Cartesian one, then by Theorem 16.1 this analytic geometry coincides with that of two-dimensional Kleine space (based on the unit circle) over the field $\overline{\overline{\mathfrak{S}}}$.

III. Final remarks

1. The relation between the end-calculus and the hyperbolic calculus of segments. In this section we like to clear up the relation between the fields $\overline{\overline{\mathfrak{S}}} = \langle \overline{\overline{S}}, +, \cdot, <\rangle$ and $\overline{\overline{\mathfrak{E}}} = \langle E, +, \cdot, <\rangle$ (see Introduction) and between the coordinate systems based upon these two fields. Actually we shall consider two coordinate systems over $\overline{\overline{\mathfrak{E}}}$, one originated with Hilbert in [3] and another one introduced recently by Szász in [9]. When comparing the fields $\overline{\overline{\mathfrak{S}}}$ and $\overline{\overline{\mathfrak{E}}}$ we shall make full use of the results in [9]. We change only the notation. In particular, the ends, i.e. the elements of $\overline{\overline{E}}$, will be now denoted by German letters $\mathfrak{x}, \mathfrak{y}, \mathfrak{z}$. On the other hand, the symbols 0 and 1 will be used as in [9] to denote the zero element and the unit element of the field $\overline{\overline{\mathfrak{E}}}$. The arguments of Hilbert and Szász are carried through in the theory $\mathcal{H}$, and hence they can be repeated in $\mathcal{H}'$.

In [9] Szász introduces a one-to-one function $\mathfrak{e}$, which assigns to every free segment X an end

$$\mathfrak{e}(X) > 1 \tag{1.1}$$

in such a way that

$$\mathfrak{e}(X_1+X_2) = \mathfrak{e}(X_1)\cdot \mathfrak{e}(X_2), \tag{1.2}$$

the symbol $+$ denoting here the usual addition of free segments. By means of $\mathfrak{e}$ Szasz defines three function $\mathfrak{s}$, $\mathfrak{c}$, and $\mathfrak{t}$,

$$\mathfrak{s}(X) = \frac{\mathfrak{e}(X)-\mathfrak{e}(X)^{-1}}{2}, \quad \mathfrak{c}(X) = \frac{\mathfrak{e}(X)+\mathfrak{e}(X)^{-1}}{2}, \tag{1.3}$$
$$\mathfrak{t}(X) = \frac{\mathfrak{e}(X)-\mathfrak{e}(X)^{-1}}{\mathfrak{e}(X)+\mathfrak{e}(X)^{-1}}$$

where $\mathfrak{e}(X)^{-1} = 1/\mathfrak{e}(X)$ and $2 = 1+1$. The function $\mathfrak{t}$ maps in one-to-one way the class S of free segments onto the class E of all ends $\mathfrak{x}$ which satisfy the condition $0 < \mathfrak{x} < 1$. Moreover, with the help of the analytic formulas in [9] it can be shown that $\mathfrak{t}$ establishes an isomorphism between the systems $\mathfrak{S} = \langle S, +, \cdot, < \rangle$ and $\mathfrak{E} = \langle E, +, \cdot, < \rangle$. By formula (1) of Theorem 13.4 in Chapter II this isomorphism can be easily extended to the isomorphism of $\overline{\overline{\mathfrak{S}}}$ and $\overline{\overline{\mathfrak{E}}}$. Thus

THEOREM 1.4. *The fields $\overline{\overline{\mathfrak{S}}}$ and $\overline{\overline{\mathfrak{E}}}$ are isomorphic.*

Let us now turn to the coordinate systems. The class $\overline{\overline{E}}$ consists of all ends with the exception of one, which Hilbert denotes by ∞. In Hilbert's coordinate system two elements of $\overline{\overline{E}}$ are assigned as coordinates to every line the both ends of which differ from ∞. The remaining lines have no coordinates. In result, the linear equation of the point derived by Hilbert does not give the sufficient basis for the foundations of the analytic geometry.

The Szász coordinate system Ψ assigns three elements of $\overline{\overline{E}}$ as coordinates to every point. Szász derives the linear equation of a line (as well several other analytic formulas), and thereby founds an analytic geometry on the hyperbolic plane. The original description of the system Ψ

is rather involved (see [9], Section 2); we will give an equivalent one resulting at once from the concluding remarks in [9]. Let L_1 and L_2 be two perpendicular axes intersecting in a point a (Fig. 15) and such that the positive half-lines H_1 of L_1 and H_2 of L_2 represent the ends ∞ and 1, respectively. For simplicity we restrict curselve to points p lying in the inner domain of the angle H_1H_2 (for more ditails concerning Ψ see [9], p. 112). Let p_1, and p_2 be the perpendicular projections of p upon the axes L_1 and L_2 and let

$$X_1 = [ap_1], \quad X_2 = [ap_2],$$
$$A = [pp_2], \quad B = [pp_1], \quad C = [ap].$$

Then $\Psi(p) = \langle \mathfrak{x}_1, \mathfrak{x}_2, \mathfrak{x}_3 \rangle$, provided

$$\mathfrak{x}_1 = \mathfrak{s}(A), \quad \mathfrak{x}_2 = \mathfrak{s}(B), \quad \mathfrak{x}_3 = \mathfrak{c}(C).$$

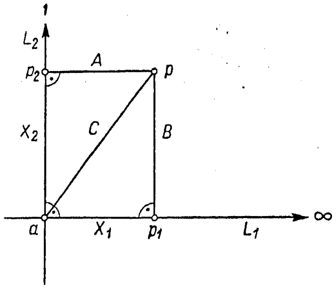

Fig. 15

Let us notice that the third coordinate is actually superfluoes since it is uniquely determined by the first two by means of the identity $x_3^2 = x_1^2 + x_2^2 + 1$ (see [9], Theorem on p. 104). On the other hand, with the help of analytic formulas given in [9] we derive

$$\mathfrak{t}(X_1) = \frac{\mathfrak{x}_1}{\mathfrak{x}_3} \quad \textit{and} \quad \mathfrak{t}(X_2) = \frac{\mathfrak{x}_2}{\mathfrak{x}_3}. \tag{1.5}$$

These identities establish a simple relation between the coordinates X_1 and X_2 of the point p in the rectangular coordinate system Φ with the axes L_1 and L_2, and the coordinates of the same point p in the system Ψ.

2. The field $\overline{\overline{\mathfrak{S}}}$ in the full hyperbolic geometry. If we enrich the axiom system $\mathcal{H}'$ by adding to it the non-elementary continuity axiom Co (e.g. as formulated in [12], p. 18) we get the axiom system of the *full hyperbolic geometry* $\overline{\mathcal{H}'}$. In conclusion we wish to make some remarks concerning the field $\overline{\overline{\mathfrak{S}}}$ in the theory $\mathcal{H}'$. With the help of Co we can prove in $\overline{\mathcal{H}'}$ that the field $\overline{\overline{\mathfrak{S}}}$ is continuously ordered and that every polynomial of an odd degree with coefficients in $\overline{\overline{S}}$ has a zero in $\overline{\overline{S}}$. Moreover, we showed in $\mathcal{H}'$ that the field $\overline{\overline{\mathfrak{S}}}$ is Euclidean. In consequence $\overline{\overline{\mathfrak{S}}}$ is a continuously ordered real closed field and therefore is isomorphic with the field $\mathfrak{R} = \langle R, +, \cdot, < \rangle$ of the real numbers.

It is easy to determine this isomorphism effectively. In fact, as is well known, in $\overline{\mathcal{H}'}$ we can correlate with every free segment X a real number $|X| > 0$, called the *natural measure* of X, in such a way that for every X and Y

$$|X+Y| = |X| + |Y|$$

and moreover

$$\left|\Pi^{-1}\left(\frac{\varrho}{2}\right)\right| = \ln\left(\sqrt{2}+1\right).$$

This correlation establishes a one-to-one correspondence between free segments and positive real numbers. Also, we can correlate with every free angle ξ a real number $|\xi| > 0$, called the *natural measure* of ξ, in such a way, that for every ξ and η

$$|\xi+\eta| = |\xi|+|\eta|\,, \quad \text{whenever} \quad \xi+\eta \quad \text{exists}\,,$$

and moreover $|\varrho| = \pi/2$.

Using the notion of the measure we can prove in $\overline{\mathcal{H}}'$ the *cosine* formula

(2.1) $\quad X \cdot Y = Z \quad$ *iff* $\quad \cos|\Pi(X)| \cdot \cos|\Pi(Y)| = \cos|\Pi(Z)|\,,$

and also the formula

(2.2) $\quad X \oplus Y = Z \quad$ *iff* $\quad \cos^2|\Pi(X)| \oplus \cos^2|\Pi(Y)| = \cos^2|\Pi(Z)|$

(see e.g. [1], p. 338). From (2.1) it follows at once that

(2.3) $$\cos^2|\Pi(X)| = \cos|\Pi(X^2)|\,.$$

With the help of (2.3) from (2.2) and the statement (13.2) of Chapter II we derive

(2.4) $\quad X+Y = Z \quad$ *iff* $\quad \cos|\Pi(X)| + \cos|\Pi(Y)| = \cos|\Pi(Z)|\,.$

Since Π in its whole domain and cos in the interval $(0, \pi/2)$ are two decreasing functions, we have also

(20.5) $\quad X < Y \quad$ *iff* $\quad \cos|\Pi(X)| < \cos|\Pi(Y)|\,.$

In addition we recall that in $\overline{\mathcal{H}}'$ we can prove the formula

(2.6) $$\cos|\Pi(X)| = \tanh|X| = \frac{e^{|X|}-e^{-|X|}}{e^{|X|}+e^{-|X|}}\,,$$

which is fundamental for the full hyperbolic geometry (⁵).

Let us denote by R the open interval $(0,1)$ of the real numbers. Then by (2.4), (2.1), (2.5), and (2.6) we arrive at

THEOREM 2.7. *The function* $\boldsymbol{t}$, *such that* $\boldsymbol{t}(X) = \tanh|X|$ *for every* X *in* S, *establishes an isomorphism between the algebraic systems*

$$\mathfrak{S} = \langle S, +, \cdot, < \rangle \quad \textit{and} \quad \mathfrak{R} = \langle R, +, \cdot, < \rangle\,.$$

(⁵) For arbitrary measure of free segments we would have $\cos|\Pi(X)| = \tanh(\varkappa\cdot|X|)$, but for the natural one the positive constant $\varkappa$ coincides with 1.

By formula (1) of Theorem 13.4 of Chapter II the isomorphism $\boldsymbol{t}$ between $\mathfrak{S}$ and $\mathfrak{R}$ can be uniquely extended to an isomorphism $\bar{\boldsymbol{t}}$ between the fields $\bar{\bar{\mathfrak{S}}}$ and $\bar{\bar{\mathfrak{R}}}$.

Theorem 2.7 clarifies the ideas underlying the construction of the hyperbolic calculus of segments.

APPENDIX

The proof of the Liebmann Theorem By the Liebmann Theorem we understand the statement (9.1) of Chapter II. The original proof of (9.1) in the system $\mathcal{H}$ due to Liebmann is to be found in [6]. We adopt the notation of the Chapter II. Moreover, we shall denote, if convenient, the half-line pq from the point p through the point q by $\boldsymbol{H}(pq)$, and the half-line complementary to $\boldsymbol{H}(pq)$ by $\boldsymbol{H}^*(pq)$.

We start with

THE LIEBMANN LEMMA. *If* $\boldsymbol{T}(X\alpha Z\beta Y)$ *or* $\boldsymbol{Q}(XA'Z\eta B)$, *then*

(I) $$\Pi(Z+A) = \eta-\beta\,,$$

(II) $$\Pi(Z-A) = \eta+\beta\,,$$

(III) $$\Pi(B+Y)+\Pi(A-X) = \frac{\varrho}{2}\,.$$

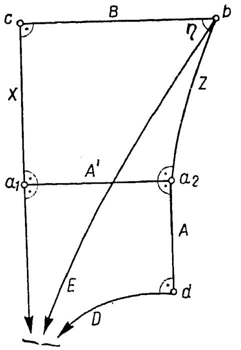

Fig. 16

Proof. First let us assume that $\boldsymbol{Q}(XA'Z\eta B)$ and let the Lambert quadrangle a_1a_2bc be a representative of the free Lambert quadrangle

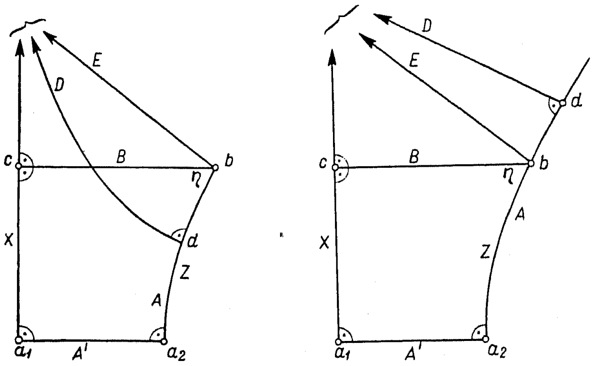

Fig. 17

$XA'Z\eta B$ (Fig. 16). We produce from the point b the half-line E parallel to the half-line ca_1, we take the point d on $\boldsymbol{H}^*(a_2b)$ such that $[a_2d] = A$, and from d we produce the half-line D parallel to the half-line ca_1. Then $D\|E$ and $DH(db)$ is a right angle; hence (I) holds. The proof of (II) differs from the proof of (I) only in that $\boldsymbol{H}(ca_1)$ and $\boldsymbol{H}^*(a_2b)$ are respectively

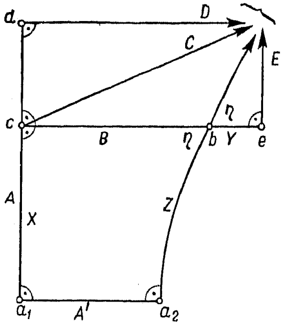

Fig. 18

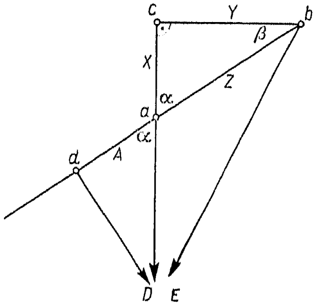

Fig. 19

replaced by $\boldsymbol{H}(a_1c)$ and $\boldsymbol{H}(a_2b)$ (Fig. 17). To prove (III) we take the points d and e such that

$$\boldsymbol{B}(a_1cd)\,, \quad \boldsymbol{B}(cbe)\,, \quad [a_1d] = A\,, \quad [be] = Y$$

(Fig. 18), and from c, d, e we produce the half-lines C, D, E parallel to the half-line a_2b. Then $C\|D$, $C\|E$, and moreover $DH(dc)$ and $EH(ec)$ are two right angles. Thus (III) holds.

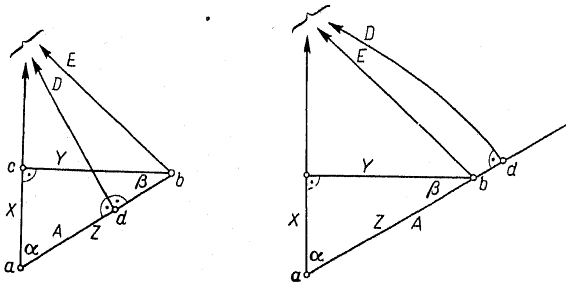

Fig. 20

Let us next assume that $\boldsymbol{T}(X\alpha Z\beta Y)$ and let the triangle abc be a representative of the free triangle $X\alpha Z\beta Y$. Then to obtain the formulas (I), (II), and (III) it is sufficient to put $a_1 = a_2 = a$ in the above proof concerning the quadrangle (Fig. 19, Fig. 20, Fig. 21). (6)

(6) Liebmann did not prove the part of Lemma which concerns the triangle since it was known as a theorem of $\mathfrak{H}$. The remark that we obtain the proof for the triangle from that for the quadrangle simply by identifying the points a_1 and a_2 is due to the author.

THE LIEBMANN THEOREM. *For any free segments X, Y, Z and any free angles α, β*

$$\boldsymbol{T}(X\alpha Z\beta Y) \quad \textit{iff} \quad \boldsymbol{Q}(XA'Z\eta B)\,.$$

Proof. Assume that $\boldsymbol{T}(X\alpha Z\beta Y)$. Obviously $\boldsymbol{Q}(X_1A'Z\eta_1B_1)$ for some X_1, η_1, B_1. Hence by Lemma

$$\Pi(Z+A) = \eta-\beta \quad \text{and} \quad \Pi(Z+A) = \eta_1-\beta_1\,,$$
$$\Pi(Z-A) = \eta+\beta \quad \text{and} \quad \Pi(Z-A) = \eta_1+\beta_1\,.$$

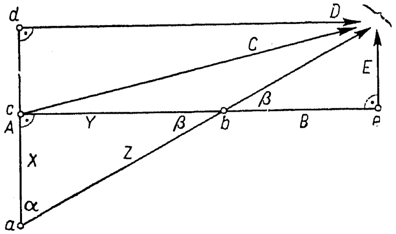

Fig. 21

Therefore $\beta_1 = \beta$ and $\eta_1 = \eta$, which implies $B_1 = B$ and $Y_1 = Y$. Hence, again by Lemma,

$$\Pi(B+Y)+\Pi(A-X) = \frac{\varrho}{2} \quad \text{and} \quad \Pi(B+Y)+\Pi(A-X_1) = \frac{\varrho}{2}.$$

Therefore $X_1 = X$. Hence $\boldsymbol{Q}(XA'Z\eta B)$. In conclusion,

$$\text{if} \quad \boldsymbol{T}(X\alpha Z\beta Y)\,, \quad \text{then} \quad \boldsymbol{Q}(XA'Z\eta B)\,.$$

The proof of the converse implication is quite similar. We assume that $\boldsymbol{Q}(XA'Z\eta B)$, obviously $\boldsymbol{T}(X_1\alpha Z\beta_1Y_1)$ for some X_1, β_1, Y_1, and by means of Lemma we get $X_1 = X$, $\beta_1 = \beta$, and $Y_1 = Y$.

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Characterization of the fixed point property for a class of set-valued mappings *

by

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1. Introduction. Let X be a space and $\mathcal{C}$ a class of set-valued mappings of X into itself. We say that X has the *fixed point property for* $\mathcal{C}$ if, for each $f \in \mathcal{C}$, there exists $x \in X$ such that $x \in f(x)$. In this paper our primary interest is to prove that if X is an arcwise connected compact metric space, then X has the fixed point property for the class of upper semi-continuous, continuum-valued mappings *if and only if* X is hereditarily unicoherent. Incidental to this result we note that such spaces also have the fixed point property for continuous mappings whose values are arbitrary closed sets, but we have not been able to characterize this fixed point property. It seems likely that, for arcwise connected compacta, this fixed point property is also characterized by hereditary unicoherence.

These results are all generalizations of what is usually called the Scherrer fixed point theorem, and they have a lengthy history. Four earlier papers are of especial interest in what follows, and we mention them here. In [4] A. D. Wallace proved that a dendrite has the fixed point property for upper semi-continuous, continuum-valued mappings, and his result was later generalized by the Eilenberg-Montgomery fixed point theorem ([1]). Our results include the Wallace theorem but they neither include nor are included by the Eilenberg-Montgomery theorem. R. L. Plunkett ([3]) proved that a dendrite has the fixed point property for continuous, closed set-valued mappings and, conversely, that if a Peano continuum has this fixed point property then it is a dendrite. As we shall observe, his proof of the converse proposition is equally valid for the mappings considered by Wallace. Finally, in [5] the author proved that a hereditarily unicoherent, arcwise connected continuum has the fixed

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