The lattice Z in problem B is not separable but methods used in Section 6 should be sufficient to overcome this difficulty. However when one tries to adapt methods of Section 4 to problem B (and to problems C and D as well) one is faced with the difficulty that not only the set $\{(x, y): x \leqslant y\}$ but also the set $\{(x, y): x \text{non} > y\}$ should be closed in $Z \times Z$. No reasonable topology seems to satisfy this condition and this is the chief reason why it is an open question as to whether or not methods similar to those of Section 4 are applicable to our problems.

We limited ourselves chiefly to the study of quantifiers whose interpretations were the l.u.b. and the g.l.b. operations. It is easy to construct examples showing that for an infinite Z, e.g. for $Z = \{\xi: \xi \leqslant \omega\}$, another choice of quantifiers may lead to a "functional calculus", in which the set of valid formulas is not recursively enumerable. It would be interesting to solve the following problem:

E. What is the general characterization of quantifiers which lead to functional calculi with recursively enumerable sets of valid formulas?

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The family of dendrites $\mathfrak{R}$-ordered similarly to the segment

by

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1. Introduction. The continuous mapping f of the topological space X onto the space Y is called the $\mathfrak{R}$-*mapping* if there exists a continuous mapping g: $Y \to X$ such that $fg =$ identity ([1] and [2]). It is easy to show ([1]) that the $\mathfrak{R}$-mappings are the same as the mappings of the form hr, where r is a retraction and h a homeomorphism.

If there exists an $\mathfrak{R}$-mapping f: $X \to Y$ then we shall write $Y \underset{\mathfrak{R}}{\leqslant} X$. If $Y \underset{\mathfrak{R}}{\leqslant} X$ and $X \underset{\mathfrak{R}}{\leqslant} Y$ then we shall write $X \underset{\mathfrak{R}}{=} Y$. If $X \underset{\mathfrak{R}}{\leqslant} Y$ but $X \underset{\mathfrak{R}}{\neq} Y$ then we shall write $X \underset{\mathfrak{R}}{<} Y$. The relation $\underset{\mathfrak{R}}{<}$ establishes the partial order of every class of spaces.

2. The family of dendrites (1) ordered by the relation $\underset{\mathfrak{R}}{<}$ similarly to the segment. At the end of the paper [2] K. Borsuk raised the following questions:

(i) Does there exist an uncountable family of spaces ordered by the relation $\underset{\mathfrak{R}}{<}$?

(ii) Does there exist a family of spaces ordered by the relation $\underset{\mathfrak{R}}{<}$ in a dense manner?

(iii) Does there exist a family of spaces ordered by the relation $\underset{\mathfrak{R}}{<}$ similarly to the set of all real numbers?

In the present paper we shall construct the family of dendrites ordered by the relation $\underset{\mathfrak{R}}{<}$ similarly to the segment. It solves the three mentioned problems even in the stronger formulation concerning compact 1-dimensional AR-sets.

(1) A dendrite is a locally connected continuum containing no simple closed curve. Dendrites are the same as compact 1-dimensional AR-sets. See for example [3], p. 224 and p. 290.

In the paper [4], p. 333, there is the example of the sequence of $\mathfrak{R}$-uncomparable dendrites $\{C_n\}$ $(n = 1, 2, \ldots)$ (it means that for $m \neq n$ neither $C_n \underset{\mathfrak{R}}{\leqslant} C_m$ nor $C_m \underset{\mathfrak{R}}{\leqslant} C_n$ holds).

Let $q_n = (1, 0) \in C_n$, $I_n = \overline{(0,0),(1,0)} \subset C_n$ $(n = 1, 2, \ldots)$. It follows from the construction of C_n ([4], p. 331-334) that

(1) *I_n contains the dense subset of ramification points of C_n $(n = 1, 2, \ldots)$.*

Let C denote the Cantor's discontinuum placed on the segment $\langle 0, 1\rangle$ of the real line. For this purpose we divide the segment $\langle 0, 1\rangle$ into three equal segments and remove the interior of the middle one. We proceed analogously with the remaining segments and so on. Let us suppose that in the k-th step of this construction we obtain the set $E_k = \bigcup_{l=1}^{2^{k-1}} L_{k,l}$ where $L_{k,l}$ is the closed segment. Then $C = \bigcap_{k=1}^{\infty} E_k$. Let $a_{k,l}$ denote the centre of the segment $L_{k,l}$ $(k = 1, 2, \ldots; l = 1, 2, \ldots, 2^{k-1})$ and let $p_{k,l}$ denote the point $(a_{k,l}, 1/k)$ in the plane.

For each segment $L_{k,l}$ there exist exactly two segments $L_{k+1,l'} \subset L_{k,l}$, $L_{k+1,l''} \subset L_{k,l}$. Let us join the points: $p_{k,l}$ with $p_{k+1,l'}$ and $p_{k,l}$ with $p_{k+1,l''}$ by the segments. The sum of all such segments with the set C together consists the dendrite D.

Let us number the vertices $p_{k,l}$ with the aid of the one index getting the sequence $\{q'_n\}$ $(n = 1, 2, \ldots)$. Let us assume that $q'_1 = p_{1,1}$. Let h_n be a homeomorphism mapping the dendrite C_n into the plane such that denoting $C'_n = h_n(C_n)$ we have

(i) $h_n(q_n) = q'_n$,

(ii) $C'_n \cap D = q'_n$,

(iii) $C'_n \cap C'_m = 0$ for $m \neq n$,

(iv) Diameter $C'_n < 1/n$ $(n = 1, 2, \ldots)$.

Let F denote the dendrite $D \cup \bigcup_{n=1}^{\infty} C'_n$.

For each $c \in C$ there exists exactly one arc M_c joining c with q'_1 in F. M_c cuts the square $Q = \{(x, y) |\ 0 \leqslant x, y \leqslant 1\}$ into two components Q^1_c, Q^2_c. Let $(0, 1) \in Q^1_c$ for each $c \in C$.

Let the dendrite F_c $(c \in C)$ be the sum of M_c, all dendrites C'_n where $q'_n \in M_c$, and $D \cap Q^1_c$. It is easy to see that

(2) *The set of ramification points of F_c is contained in $\bigcup_{n=1}^{\infty} C'_n$.*

Obviously for $c_1, c_2 \in C$, $c_1 < c_2$ the relation $F_{c_1} \underset{\mathfrak{R}}{\leqslant} F_{c_2}$ holds. We shall prove that $F_{c_1} \underset{\mathfrak{R}}{<} F_{c_2}$ holds, too.

For this purpose let us assume that $F_{c_1} \underset{\mathfrak{R}}{\geqslant} F_{c_2}$, it means that there exists a homeomorphism h: $F_{c_2} \to F_{c_1}$. Since $c_1 < c_2$, there exists $q'_n \in F$ $(n > 2)$ such that

$$C'_n \subset F_{c_2} - F_{c_1}. \tag{3}$$

By the properties (1) and (2) there exists $q'_m \in F_{c_1}$ such that putting $I'_n = h_n(I_n)$ we have

$$h(I'_n) \subset C'_m. \tag{4}$$

From (3) and (4) we conclude that $n \neq m$. Let

$$\nu = \begin{cases} 2 & \text{if} \quad m = 1, \\ 1 & \text{if} \quad m \geqslant 2. \end{cases}$$

There are two possibilities: If $h(q'_n)$ can be joined in F_{c_1} with q'_m by an arc disjoint with the interior of $h(I'_n)$ then we have $h(C'_n) \subset C'_m$. In the contrary case we have $h(C'_\nu) \subset C'_m$. In both the cases we have the contradiction with the $\mathfrak{R}$-uncomparability of C_n, C_m, and C_ν.

In such a way we obtained the family of dendrites $\{F_c\}$ $(c \in C)$ ordered by the relation $\underset{\mathfrak{R}}{<}$ similarly to the Cantor set C. On the other hand, the closed segment is similar to the subset C_0 of Cantor set C. Indeed, let C_0 origins from C by removal of the left ends of excluded segments. Then the well-known "stair-function" φ: C onto $\langle 0, 1\rangle$ is one-to-one and monotonic on C_0 and establishes the similarity. In such a way we have obtained the family of dendrites $\{F_t\}$ $(t \in \langle 0, 1\rangle)$ ordered by the relation $\underset{\mathfrak{R}}{<}$ similarly to the segment $\langle 0, 1\rangle$.

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