References

[1] G. Alexits, *Konvergenzprobleme der Orthogonalreihen*, Budapest 1960.

[2] S. Kaczmarz and H. Steinhaus, *Theorie der Orthogonalreihen*, Warszawa-Lwów 1935.

[3] G. G. Lorentz, *Riesz method of summation and orthogonal series*, Trans. Roy. Soc. Canada 45 (1951), p. 19-32.

[4] J. Meder, *On the summability almost everywhere of orthonormal series by the method of first logarithmic means*, Rozprawy Matem. 17, Warszawa 1959.

[5] K. Tandori, *Über die orthogonalen Funktionen*, Acta Sci. Math., II Mitt. 18 (1957), p. 149-168; IV Mitt. 19 (1958), p. 18-25; VI Mitt. 20 (1959), p. 14-18.

[6] A. Zygmund, *Sur la sommation des séries de fonctions orthogonales*, Bull. Acad. Pol., Cracovie 1927, p. 295-308.

[7] — *Sur l'application de la première moyenne arithmétique dans la théorie des séries de fonctions orthogonales*, Fund. Math. 10 (1927), p. 356-362.

Reçu par la Rédaction le 15. 7. 1960

STUDIA MATHEMATICA, T. XX. (1961)

Convolution of functions of several variables

by

J. MIKUSIŃSKI (Warszawa)

Introduction. In this paper we give a new proof of a theorem on convolution which is an extension to several variables of the well known Titchmarsh theorem [5]. The first proof, due to Lions [1], has been based on the Fourier Transform. Another proof, due to Mikusiński and Ryll-Nardzewski [4], has been based upon a geometrical method. The proof of this paper is based on the concept of Banach algebra.

We give several equivalent formulations of the theorem (Theorems VIII-VIII d).

1. Let $\mathcal{A}$ be a commutative Banach algebra over the field of complex numbers, and let $\mathcal{A}_1$ be its least extension with unity.

Let $E(t)$ $(t \geqslant 0)$ be an *exponential operator*, i. e. an operator such that $E(t)x \in \mathcal{A}$ for $x \in \mathcal{A}$, $x \neq 0$, and moreover

1° $E(0) = 1$;

2° $E(t)(xy) = \big(E(t)x\big)y$;

3° For every $x \in \mathcal{A}$, the function $E(t)x$ is continuous;

4° There exists an element $l \in \mathcal{A}_1$, non divisor of zero, such that

$$\frac{d}{dt}E(t)l = E(t)\ (^1).$$

Letting $y = 0$ in 2° we obtain

$$E(t)0 = 0.$$

It is also easy to verify that, by 2°,

$$E(t_1)x \cdot E(t_2)y = E(t_1)y \cdot E(t_2)x = E(t_1)E(t_2)xy.$$

We shall prove that

$$E(t_1)E(t_2) = E(t_1+t_2). \tag{1}$$

(1) This means that $\frac{d}{dt}E(t)lx = E(t)x$ for every $x \in \mathcal{A}$.

In fact, for an arbitrarily fixed positive number u, put

$$F(t) = E(t)l^2\cdot E(u-t)l^2.$$

Then

$$\frac{d}{dt}F(t) = E(t)l\cdot E(u-t)l^2 - E(t)l^2\cdot E(u-t)l = 0.$$

Thus $F(t)$ does not depend on t and we have $F(t) = F(0)$, i. e.

$$E(t)E(u-t) = E(u),$$

since l is not a divisor of zero. If $t_1, t_2 \geqslant 0$, we can substitute $t = t_1$, $u = t_1+t_2$, which reduces the last equality to (1).

We can also consider exponential operators, defined for all real t or for all complex t. Then equality (1) holds for all real or complex t_1, t_2.

2. Let $f(t), g(t), \ldots$ denote $\mathscr{A}$-valued functions, integrable in $[0, T]$; equalities between these functions will mean equalities almost everywhere.

Write generally

$$f(t)*g(t) = \int_0^t f(t-\tau)g(\tau)d\tau,$$

$$\tilde{f} = \int_0^T E(t)f(t)dt.$$

Theorem I. *If*

$$h(t) = f(t)*g(t), \qquad k(T-t) = f(T-t)*g(T-t),$$

then

$$\tilde{f}\tilde{g} = \tilde{h}+E(T)\tilde{k}.$$

Proof. We have

$$\tilde{f}\tilde{g} = \iint_{\square} E(u+v)f(u)g(v)\,du\,dv,$$

where the domain of integration is the square

$$\square = (0\leqslant u\leqslant T,\ 0\leqslant v\leqslant T).$$

That square can be decomposed into two triangles

$$\triangle = (0\leqslant u,\ 0\leqslant v,\ u+v\leqslant T),$$
$$\nabla = (T < u+v,\ u\leqslant T,\ v\leqslant T).$$

Thus

$$\iint_{\square} = \iint_{\triangle} + \iint_{\nabla}.$$

On substituting in the first integral on the right side $u+v = t$, $v = \tau$, we obtain

$$\iint_{\triangle} = \int_0^T E(t)dt\int_0^t f(t-\tau)g(\tau)d\tau = \tilde{h}.$$

Similarly, on substituting in the second integral on the right side $u+v = T+t$, $v = T-\tau$, we obtain

$$\iint_{\triangle} = E(T)\int_0^T E(t)dt\int_0^{T-t} f(t+\tau)g(T-\tau)d\tau = E(T)\tilde{k}.$$

This proves Theorem I.

In this Section we needed very few properties of Banach spaces; we need only some properties of integrals. Therefore the result holds in any space where those properties hold.

3. In this Section we admit the following hypothesis on the norm $|x|$ of $x\in\mathscr{A}$:

(2) $$|x|^2 = |x^2|$$

for every $x\in\mathscr{A}$.

Using that hypothesis we shall prove

Theorem II. *If* $f(t)*f(t) = 0$ *in* $[0, T]$, *then* $f(t) = 0$ *in* $[0, \frac{1}{2}T]$.

Proof. Let $h(t) = f(t)*f(t)$. Then by Theorem I

$$\tilde{f}^2 = E(T)\tilde{k},$$

where

$$k(T-t) = f(T-t)*f(T-t).$$

Put in particular $E(t) = e^{-st}$ (s any fixed complex number). Then

(3) $$|\tilde{k}| \leqslant \int_0^T |k(t)|dt = M \qquad (\mathrm{Re}\, s\geqslant 0)$$

and

(4) $$e^{\mathrm{Re}\, sT/2}|\tilde{f}| \leqslant \sqrt{M}.$$

Hence

$$e^{\mathrm{Re}\, sT/2}\left|\int_0^{T/2} e^{-st}f(t)dt\right| \leqslant \sqrt{M} + \int_{T/2}^T e^{-\mathrm{Re}\, s(t-T/2)}|f(t)|dt \leqslant L = \sqrt{M}+\int_{T/2}^T |f(t)|dt.$$

Replacing t by $\frac{1}{2}T-t$, we obtain

$$|F(s)|\leqslant L \quad \text{for} \quad \mathrm{Re}\, s\geqslant 0,$$

where $F(s)$ is an $\mathscr{A}$-valued entire function, defined by the integral

(5) $$F(s) = \int_0^{T/2} e^{st}f(\tfrac{1}{2}T-t)dt.$$

We proved that $F(s)$ is bounded for $\mathrm{Re}\, s\geqslant 0$. Evidently it is also bounded for $\mathrm{Re}\, s < 0$. Thus it is bounded in the whole complex plane and, consequently, it is a constant function. But, $F(s)$ tends to 0, as

$s \to -\infty$, which is easily seen from (5). Thus $F(s) = 0$ identically. Hence $f(\frac{1}{2}T - t) = 0$ and $f(t) = 0$ in $[0, \frac{1}{2}T]$.

The only step, where we needed the hypothesis (2), was in the inclusion from (3) to (4).

4. In this section we do not need hypothesis (2). Let

$$\tilde{f}(s) = \int_0^T E(st) f(t)\,dt,$$

where $E(t)$ denotes an arbitrary exponential operator, defined for positive t.

THEOREM III. *If $\tilde{f}(s) = 0$ in an interval $0 \leqslant s_1 \leqslant s \leqslant s_2$, then*

$$E(st) f(t) = 0 \tag{6}$$

for $0 \leqslant t \leqslant T$, $s_1 \leqslant s \leqslant s_2$.

Proof. Since

$$l\tilde{f}(s) = \int_0^T E(st)\, lf(t)\,dt = 0,$$

we find, on differentiating in s,

$$\int_0^T tE(st) f(t)\,dt = 0.$$

After n similar steps

$$\int_0^T t^n E(st) f(t)\,dt = 0 \qquad (n = 1, 2, \ldots).$$

This implies (6), by the moment theorem.

Suppose, now, that the exponential operator satisfies the following conditions:

5° $E(T) = 0$,

6° $x^2 = 0$ $(x \in \mathcal{A})$ implies $E(t_0)\, x = 0$.

THEOREM IV. *If $E(t)\big(f(t) * f(t)\big) = 0$ in $[0, T]$, then $E(t_0 + t) f(t) = 0$ in $[0, T]$.*

Proof. From the hypothesis it follows that $E(st)\big(f(t) * f(t)\big) = 0$ in $[0, T]$, as $s \geqslant 1$. Thus $\tilde{h}(s) = 0$ and by Theorem I we find

$$\tilde{f}(s)^2 = 0 + 0 = 0 \qquad (s \geqslant 1).$$

This implies $E(t_0)\tilde{f}(s) = 0$, i. e.

$$\int_0^T E(st) E(t_0) f(t)\,dt = 0$$

and, by theorem III, $E(st) E(t_0) f(t) = 0$ in $[0, T]$. For $s = 1$, the last equality is equivalent to the assertion.

5. Let $\mathcal{B}$ be the space of all $\mathcal{A}$-valued functions, integrable in $[0, T]$. We consider $\mathcal{B}$ as an algebra with ordinary addition and with convolution as multiplication. We shall denote, in this section, the convolution of two functions f and g from $\mathcal{B}$ by fg, as an ordinary product. Taking

$$|f| = \int_0^T |f(t)|\,dt \tag{7}$$

as norm of f we make $\mathcal{B}$ a Banach algebra. We define $E(u)$ as the translation operator

$$E(u) f = \begin{cases} f\left(t - \dfrac{T}{U} u\right) & \text{for} \quad \dfrac{t}{T} \geqslant \dfrac{u}{U}, \\ 0 \text{ elsewhere}, \end{cases}$$

U being a given positive number. It is easy to verify that this operator is in $\mathcal{B}$ an exponential operator defined for $u \geqslant 0$ such that l is the function assuming everywhere the value 1. Moreover, $E(U) = 0$. If we suppose that condition (2) is satisfied in $\mathcal{A}$, then Theorem II can be written in the form

$$x^2 = 0 \ (x \in \mathcal{B}) \text{ implies } E(\tfrac{1}{2}U) x = 0.$$

We can therefore apply Theorem IV to $\mathcal{B}$-valued functions $p(u)$, integrable in the interval $[0, U]$. In this way we obtain

THEOREM IV'. *If $E(u)\big(p(u) * p(u)\big) = 0$ in $[0, U]$, then $E(\frac{1}{2}U + u) = 0$ in $[0, U]$.*

If instead of (7) we take the norm

$$|f| = \max_{0 \leqslant t \leqslant T} \left| \int_0^t f(t)\,dt \right|,$$

then the space $\mathcal{B}$ is not complete. On completing it in the usual way we obtain a space of distributions (measures). In that case the translation $E(u)$ is simply the Dirac distribution $\delta(t - u)$. This interpretation can also be used to obtain all the results presented below, even in a slightly stronger form. An advantage of this interpretation is that one does not need to introduce at the beginning (Section 1) special exponential operators, but it suffices to restrict considerations to $\mathcal{A}$-valued functions. On the other hand, the theory of distributions seems to be a tool which is to strong. In fact, it introduces, besides the translation operator, many elements which are not concerned with our purposes. Moreover, in the theory of distributions would require some supplements because of the exceptional role of end points of the interval $[0, T]$.

$\mathcal{B}$-valued functions $p(u)$ of one variable u can also be interpreted as $\mathcal{A}$-valued functions

$$p(t, u)$$

of two arguments t and u. Then the convolution $p(u)*q(u)$ is the convolution

$$p(t,u)*q(t,u) = \int_0^t \int_0^u p(t-\tau, u-\omega)\, q(\tau,\omega)\, d\tau d\omega,$$

and the equality $E(u)p(u) = 0$ in $[0,U]$ means that

$$p(t,u) = 0$$

in the triangle

$$\triangle(1):\quad 0 \leqslant t,\quad u \leqslant 0,\quad \frac{t}{T}+\frac{u}{U} \leqslant 1;$$

more generally, the equality $E(u_0+u)p(u) = 0$ in $[0,U]$, where $0 \leqslant u_0 \leqslant U$, means that $p(t,u) = 0$ in the triangle

$$\triangle\left(1-\frac{u_0}{U}\right):\quad 0 \leqslant t,\ 0 \leqslant u,\ \frac{t}{T}+\frac{u}{U} \leqslant 1-\frac{u_0}{U}.$$

Thus Theorem IV′ can be written in the form of

THEOREM IV″. *If* $p(t,u)*p(t,u) = 0$ *in* $\triangle(1)$, *then* $p(t,u) = 0$ *in* $\triangle(\frac{1}{2})$.

6. The last result can be generalised to an arbitrary number of variables. Let $F(t_1,\ldots,t_n)$, $G(t_1,\ldots,t_n)$, ... denote $\mathcal{A}$-valued functions defined in the simplex

$$S(1):\quad 0 \leqslant t_1,\ldots, 0 \leqslant t_n,\ \frac{t_1}{T_1}+\ldots+\frac{t_n}{T_n} \leqslant 1.$$

Denote more generally by $S(\theta)$ $(0 \leqslant \theta \leqslant 1)$ the simplex

$$S(\theta):\quad 0 \leqslant t_1,\ldots, 0 \leqslant t_n,\ \frac{t_1}{T_1}+\ldots+\frac{t_n}{T_n} \leqslant \theta.$$

We suppose further on that the Banach algebra $\mathcal{A}$ satisfies the condition (2). By a convolution we understand

$$F*G = \int_0^{t_1}\ldots\int_0^{t_n} F(t_1-\tau_1,\ldots,t_n-\tau_n)\, G(\tau_1,\ldots,\tau_n)\, d\tau_1\ldots d\tau_n. \tag{8}$$

THEOREM V. *If* $F*F = 0$ *in* $S(1)$, *then* $F = 0$ *in* $S(\frac{1}{2})$.

In the case $n = 1$ Theorem V is reduced to Theorem II. In case $n = 2$ Theorem V is reduced to Theorem IV″. Generally, we can prove Theorem V by induction.

Proof. We suppose that Theorem V is true for some number $n-1 \geqslant 1$ of variables. Then we conclude that it is also true for n variables. The argument is the same as in the foregoing section.

Let $\mathcal{B}_{n-1}$ be the space of all $\mathcal{A}$-valued functions of $n-1$ variables $t_1,\ldots,t_{n-1}$, integrable in the simplex

$$S_{n-1}(1):\quad 0 \leqslant t_1,\ldots, 0 \leqslant t_{n-1},\ \frac{t_1}{T_1}+\ldots+\frac{t_{n-1}}{T_{n-1}} \leqslant 1.$$

The convolution of two functions F_{n-1}, G_{n-1} from $\mathcal{B}_{n-1}$ will be denoted by $F_{n-1}G_{n-1}$. The space $\mathcal{B}_{n-1}$ is an algebra with ordinary addition and with convolution as multiplication. As norm in $\mathcal{B}_{n-1}$ we take

$$|F_{n-1}| = \int_{S_{n-1}}\!\!\ldots\int |F_{n-1}(t_1,\ldots,t_{n-1})|\, dt_1\ldots dt_{n-1}.$$

With this norm $\mathcal{B}_{n-1}$ is a Banach algebra. We define $E(t_n)$ as the translation operator

$$E(t_n)F_{n-1} = \begin{cases} F\left(t_1-\frac{T_1}{T_n}t_n,\ldots,t_{n-1}-\frac{T_{n-1}}{T_n}t_n\right) & \text{for } \frac{t_1}{T_1}+\ldots+\frac{t_{n-1}}{T_{n-1}} \leqslant \frac{t}{T},\\ 0 \text{ elsewhere.} & \end{cases}$$

Then $E(T_n) = 0$. By induction hypothesis the following implication holds:

$$X^2 = 0\ (X \in \mathcal{B}_{n-1})\ \textit{implies}\ E(\tfrac{1}{2}T_n)X = 0.$$

Thus we can apply Theorem IV to $\mathcal{B}_{n-1}$-valued functions $F_n(t_n)$, integrable in $[0,T_n]$. So we obtain the implication:

If $E(t_n)\big(F_n(t_n)*F_n(t_n)\big) = 0$ *in* $[0,T_n]$, *then* $E(\frac{1}{2}T_n+t_n)F(t_n) = 0$ *in* $[0,T_n]$.

The $\mathcal{B}_{n-1}$-valued functions $F_n(t_n)$ of one variable can be interpreted as $\mathcal{A}$-valued functions $F(t_1,\ldots,t_n)$ of n arguments $t_1,\ldots,t_n$. Then the convolution $F_n(t_n)*G_n(t_n)$ is the convolution (8) and the equality $E(u_0+t_n)F_n(t_n) = 0$ in $[0,T_n]$ $(0 \leqslant u_0 \leqslant T_n)$ means that $F(t_1,\ldots,t_n) = 0$ in the simplex

$$S\left(1-\frac{u_0}{T_n}\right):\quad 0 \leqslant t_1,\ldots, 0 \leqslant t_n,\ \frac{t_1}{T_1}+\ldots+\frac{t_n}{T_n} \leqslant 1-\frac{u_0}{T}.$$

This proves that the last implication is equivalent to Theorem V.

7. Let $\mathcal{A}$ be a Banach algebra satisfying condition (2) and let $\mathcal{B}_0$ be the set of all $\mathcal{A}$-valued functions $F(t)$, $t = (t_1,\ldots,t_n)$, defined in the whole n-dimensional Euclidean space R^n, locally integrable in it, and vanishing outside the region $0 < t_1,\ldots, 0 < t_n$. Let $\mathcal{B}$ be the set of all functions $F(t) = G(t-u)$, where $u \in R^n$ and $G(t) \in \mathcal{B}_0$.

It is easy to show that if $F(t)$ and $G(t)$ belong to $\mathcal{B}$, then their convolution

$$F(t)*G(t) = \int_{R^n}\!\!\dots\!\int F(t_1-\tau_1, \dots, t_n-\tau_n)G(\tau_1, \dots, \tau_n)\,d\tau_1\dots d\tau_n$$

exists and belongs also to $\mathcal{B}$. In particular, if $F(t)$ and $G(t)$ belong to $\mathcal{B}_0$, then also $F(t)*G(t)$ belongs to $\mathcal{B}_0$.

The convolution has the following properties:

1° $F(t)*G(t) = F(t-u)*G(t+u)$ for every $u \in R^n$;

2° If $H(t) = F(t)*G(t)$, then $H(at) = a^n F(at)*G(at)$ for every positive number a.

Let c be a fixed vector in which all the coordinates $c_1, \dots, c_n$ are positive. We shall denote by $\mathcal{C}_\beta$ the set of all functions $F \in \mathcal{B}$ which vanish in the half-space

$$\frac{t_1}{c_1}+\dots+\frac{t_n}{c_n} \leqslant \beta. \tag{9}$$

If $F(t) \in \mathcal{C}_\beta$, $G(t) \in \mathcal{C}_\gamma$, then $F(t)*G(t) \in \mathcal{C}_{\beta+\gamma}$. If $F(t) \in \mathcal{C}_\beta$, then $F(t/a) \in \mathcal{C}_{a\beta}$ for every positive a. If $F(t) \in \mathcal{C}_\beta$ and $G(t) \in \mathcal{C}_\beta$, then $\mu F(t) + \nu G(t) \in \mathcal{C}_\beta$, μ and ν being arbitrary complex numbers.

THEOREM VI. *If $F \in \mathcal{B}$ and $F*F \in \mathcal{C}_{2\beta}$, then $F \in \mathcal{C}_\beta$.*

Proof. Let $\gamma > -\beta$ be a number such that $F(t-\gamma c) \in \mathcal{B}_0$. Then we have

$$F(t-\gamma c)*F(t-\gamma c) = 0 \tag{10}$$

for

$$\frac{t_1}{T_1}+\dots+\frac{t_n}{T_n} \leqslant 1 \qquad (T_i = 2(\beta+\gamma)c_i). \tag{11}$$

Of course, equality (10) holds also in simplex $S(1)$ defined by (11) and

$$0 \leqslant t_1, \ \dots, \ 0 \leqslant t_n. \tag{12}$$

By Theorem V we have

$$F(t-\gamma c) = 0 \tag{13}$$

in simplex $S(\frac{1}{2})$ defined by (12) and by

$$\frac{t_1}{T_1}+\dots+\frac{t_n}{T_n} \leqslant \frac{1}{2}. \tag{14}$$

Since $F(t-\gamma c) \in \mathcal{B}_0$, equality (13) holds also outside the region (12), thus it holds in the whole half-space (14), i. e. in

$$\frac{t_1}{c_1}+\dots+\frac{t_n}{c_n} \leqslant \beta+\gamma.$$

Consequently $F(t) = 0$ in (9), and this means that $F \in \mathcal{C}_\beta$.

8. All functions $F(t), G(t), \dots$ in this section are supposed to belong to $\mathcal{B}$.

THEOREM VII. *If $F*G \in \mathcal{C}_0$, then $F*t_iG \in \mathcal{C}_0$ $(i = 1, \dots, n)$.*

Let β be a number such that $\overline{F}(t) = F(t-\frac{1}{2}\beta c) \in \mathcal{C}_0$ and $\overline{G}(t) = G(t-\frac{1}{2}\beta c) \in \mathcal{C}_0$. Then the supposition $F*G \in \mathcal{C}_0$ is equivalent to $\overline{F}*\overline{G} \in \mathcal{C}_\beta$. It is also easy to verify that if $F*G \in \mathcal{C}_0$, then the assertions $F*t_iG \in \mathcal{C}_0$ and $\overline{F}*t_i\overline{G} \in \mathcal{C}_\beta$ are equivalent. Thus Theorem VII can be stated in the following equivalent form:

THEOREM VII′. *If $F \in \mathcal{C}_0$, $G \in \mathcal{C}_0$, $F*G \in \mathcal{C}_\beta$, then $F*t_iG \in \mathcal{C}_\beta$ $(i = 1, \dots, n)$.*

Proof. We shall prove the theorem in the form VII′. The index i will be arbitrarily fixed through out the proof. Let $\alpha(\beta)$ $(\beta > 0)$ be the greatest number such that, for every pair of functions F, G satisfying

$$F \in \mathcal{C}_0, \quad G \in \mathcal{C}_0, \quad F*G \in \mathcal{C}_\beta,$$

we have

$$F*t_iG \in \mathcal{C}_{\alpha(\beta)}.$$

Of course $t_iG \in \mathcal{C}_0$ and consequently $F*t_iG \in \mathcal{C}_0$. This implies that $0 \leqslant \alpha(\beta)$. On the other hand, if F and G are positive in the region $\frac{1}{2}\beta < t_1, \dots, \frac{1}{2}\beta < t_n$ and vanish outside it, then the convolution $F*G$ is positive in the region $\beta < t_1, \dots, \beta < t_n$. This implies that $\alpha(\beta) \leqslant \beta$. Thus $0 \leqslant \alpha(\beta) \leqslant \beta$.

Let x be a positive number and let $F_x(t) = F(t/x)$, $G_x(t) = G(t/x)$. Then $F_x \in \mathcal{C}_0$, $G_x \in \mathcal{C}_0$ and $F_x*G_x \in \mathcal{C}_{x\beta}$. Thus $F_x*t_iG_x \in \mathcal{C}_{\alpha(x\beta)}$. Hence we obtain $F*t_iG \in \mathcal{C}_{\alpha(x\beta)/x}$. From the definition of $\alpha(\beta)$ it follows that

$$\frac{1}{x}\alpha(x\beta) \leqslant \alpha(\beta).$$

Letting $\beta = 1$ we obtain $\dfrac{1}{x}\alpha(x) \leqslant \alpha(1)$, and replacing x by β

$$\alpha(\beta) \leqslant \beta\alpha(1).$$

On the other hand, letting $x = 1/\beta$ we obtain

$$\beta\alpha(1) \leqslant \alpha(\beta).$$

Both inequalities imply

$$\alpha(\beta) = \beta\gamma,$$

where $\gamma = \alpha(1)$, $0 \leqslant \gamma \leqslant 1$.

In the remaining part of the proof, it will be convenient to denote the convolution of F and G by FG and the product t_iF by F'. Then $(FG)' = F'G+FG'$.

Suppose that F and G are given and that

$$F\in\mathcal{C}_0,\quad G\in\mathcal{C}_0,\quad FG\in\mathcal{C}_\beta.$$

Then

$$FG'\in\mathcal{C}_{\beta\gamma}. \tag{15}$$

Since $G'\in\mathcal{C}_0$, we have similarly

$$F'G'\in\mathcal{C}_{\beta\gamma^2}, \tag{16}$$

for the convolution is commutative. On the other hand, the relation

$$FG\in\mathcal{C}_\beta \tag{17}$$

implies

$$F'G+FG'\in\mathcal{C}_\beta. \tag{18}$$

From (15) and (18) we get

$$FG\cdot F'G'+(FG')^2\in\mathcal{C}_{\beta+\beta\gamma}.$$

Since $\gamma^2\leqslant\gamma$, we have a fortiori

$$FG\cdot F'G'+(FG')^2\in\mathcal{C}_{\beta+\beta\gamma^2}.$$

From (16) and (17) we get

$$FG\cdot F'G'\in\mathcal{C}_{\beta+\beta\gamma^2}.$$

From the last two relations we obtain

$$(FG')^2\in\mathcal{C}_{\beta+\beta\gamma^2}$$

and by Theorem VI

$$FG'\in\mathcal{C}_{\beta\frac{1+\gamma^2}{2}}.$$

But γ is the largest number such that $FG\in\mathcal{C}_\beta$ implies $FG'\in\mathcal{C}_{\beta\gamma}$. Thus

$$\frac{1+\gamma^2}{2}\leqslant\gamma.$$

The only value of γ satisfying this inequality is 1. Thus $FG'\in\mathcal{C}_\beta$ and Theorem VII' as well as Theorem VII is proved.

9. We assume, further on, that $F, G, \ldots$ are $\mathcal{A}$-valued functions from $\mathcal{B}$ and, moreover, that $\mathcal{A}$ has no divisors of zero and satisfies (2),

THEOREM VIII. *If $F*G\in\mathcal{C}_0$, then there exists a real number u such that $F\in\mathcal{C}_{-u}$ and $G\in\mathcal{C}_u$.*

Proof. By Theorem VII it follows from $F*G\in\mathcal{C}_0$ that

$$\int\limits_{R^n}\!\!\ldots\!\int \tau_i F(t_1-\tau_1,\ldots,t_n-\tau_n)G(\tau_1,\ldots,\tau_n)\,d\tau_1\ldots d\tau_n\in\mathcal{C}_0,$$

which means that the integral vanishes for $c_1t_1+\ldots+c_nt_n\leqslant 0$. Since i is arbitrary, we obtain by induction

$$\int\limits_{R^n}\!\!\ldots\!\int P(\tau_1,\ldots,\tau_n)F(t_1-\tau_1,\ldots,t_n-\tau_n)G(\tau_1,\ldots,\tau_n)\,d\tau_1\ldots d\tau_n=0$$

for $c_1t_1+\ldots+c_nt_n\leqslant 0$, where $P(\tau_1,\ldots,\tau_n)$ is an arbitrary polynomial. Since $F\in\mathcal{B}$ and $G\in\mathcal{B}$, the integral is, as a matter of fact, an integral on a bounded part of R^n. Thus, by the moment Theorem,

$$F(t_1-\tau_1,\ldots,t_n-\tau_n)G(\tau_1,\ldots,\tau_n)=0 \tag{19}$$

for $c_1t_1+\ldots+c_nt_n\leqslant 0$. Let u be the greatest number such that $G\in\mathcal{C}_u$, i. e. such that

$$G=0\quad\text{for}\quad c_1t_1+\ldots+c_nt_n\leqslant nu.$$

Then there is, for every positive number ε, a point $\tau_1,\ldots,\tau_n$, $c_1\tau_1+\ldots+c_n\tau_n\leqslant nu+\varepsilon$, at which G is essentially different from 0. Since $\mathcal{A}$ has no divisors of zero, we obtain from (19)

$$F(t_1-\tau_1,\ldots,t_n-\tau_n)=0$$

in the region $c_1t_1+\ldots+c_nt_n\leqslant 0$. Letting $v_i=t_i-\tau_i$, we have

$$F(v_1,\ldots,v_n)=0\quad\text{in}\quad c_1v_1+\ldots+c_nv_n\leqslant -c_1\tau_1-\ldots-c_n\tau_n,$$

and all the more

$$F(v_1,\ldots,v_n)=0\quad\text{in}\quad c_1v_1+\ldots+c_nv_n\leqslant -nu-\varepsilon.$$

Since ε is arbitrary, Theorem VIII follows.

Theorem VIII can also be stated in the following form:

THEOREM VIIIa. *If*

$$\int\limits_{R^n}\!\!\ldots\!\int F(t_1-\tau_1,\ldots,t_n-\tau_n)G(\tau_1,\ldots,\tau_n)\,d\tau_1\ldots d\tau_n=0 \tag{20}$$

in $c_1t_1+\ldots+c_nt_n\leqslant 0$ $(c_i>0)$, then there exists a real number u such that $F=0$ in $c_1t_1+\ldots+c_nt_n\leqslant -u$ and $G=0$ in $c_1t_1+\ldots+c_nt_n\leqslant u$.

By a simple change of variables we obtain a more general form of the theorem:

THEOREM VIIIb. *If (20) holds in $c_1t_1+\ldots+c_nt_n\leqslant T$ $(c_i>0$, T real$)$, then there exists a real number u such that $F=0$ in $c_1t_1+\ldots+c_nt_n\leqslant T-u$ and $c_1t_1+\ldots+c_nt_n\leqslant T+u$.*

10. We are going to give another form to Theorem VIII.

THEOREM VIIIc. *If*

$$\int\limits_0^{t_1}\!\!\ldots\!\int\limits_0^{t_n} F(t_1-\tau_1,\ldots,t_n-\tau_n)G(\tau_1,\ldots,\tau_n)\,d\tau_1\ldots d\tau_n=0 \tag{21}$$

in the simplex $0 \leqslant t_1, \ldots, 0 \leqslant t_n$, $c_1t_1+\ldots+c_nt_n \leqslant T$ $(c_i > 0,\ T > 0)$, *then there exist positive numbers* θ' *and* θ'' *such that* $\theta'+\theta'' \geqslant 1$ *and* $F = 0$ *in* $0 \leqslant t_1, \ldots, 0 \leqslant t_n$, $c_1t_1+\ldots+c_nt_n \leqslant T$, *and* $G = 0$ *in* $0 \leqslant t_1, \ldots, 0 \leqslant t_n$, $c_1t_1+\ldots+c_nt_n \leqslant \theta''T$.

We can suppose that $F = 0$ and $G = 0$ outside the region of integration and take then (20) instead of (21). By Theorem VIIIb there exists a real number u such that $F = 0$ in $c_1t_1+\ldots+c_nt_n \leqslant T-u = \theta'$ and $G = 0$ $c_1t_1+\ldots+c_nt_n \leqslant T+u = \theta''T$. Evidently $\theta'+\theta'' = 1$. If one of numbers θ' or θ'' is > 1, we can replace the second one by 0. Otherwise both numbers θ' and θ'' are non-negative.

As a particular case of Theorem VIIIc we obtain the well known Titchmarsh theorem on convolution [5], on admitting that F and G are complex valued functions and that $n = 1$.

11. Denote generally by C_F the smallest convex set outside of which F vanishes, and such that $(t_1, \ldots, t_n) \epsilon C_F$ implies $(u_1, \ldots, u_n) \epsilon C_F$ as $u_i \geqslant t_i$. Thus, if t is a real number sufficiently large, C_F contains the region $t_1 \geqslant t, \ldots, t_n \geqslant t$. If $F \epsilon \mathfrak{B}$, then C_F is also contained in such a region with properly chosen t.

The set C_F can be equivalently defined as follows. Let H_F^c, where $c = (c_1, \ldots, c_n)$ $(c_i > 0)$, denote the greatest half-space in which $F = 0$. Then C_F is the complement of $\bigcup\limits_c H_F^c$.

By vector sum C_F+C_G of two sets C_F and C_G we understand the set of points $(u_1+v_1, \ldots, u_n+v_n)$ such that the points $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ belong to C_F and C_G respectively.

THEOREM VIIId. $C_{F*G} = C_F+C_G$.

In fact, $H_{F*G}^c = H_F^c+H_G^c$ (vector sum), by Theorem VIIIb. This implies $\bigcup\limits_c H_{F*G}^c = \bigcup\limits_c H_F^c+\bigcup\limits_c H_G^c$, and on taking corresponding complements $C_{F*G} = C_F+C_G$.

References

[1] J. L. Lions, *Supports de produit de composition*, C. R. 232, (1951), p. 1622-1624.

[2] J. Mikusiński, *Une simple démonstration du théorème de Titchmarsh sur la convolution*, Bull. Ac. Pol. Sci. 7.12 (1959), p. 715-717.

[3] — *Le calcul opérationnel de l'intervalle fini*, Studia Math. 15 (1956), p. 225-251.

[4] J. Mikusiński et C. Ryll-Nardzewski, *Un théorème sur le produit de composition des fonctions de plusieurs variables*, *ibidem* 13 (1953), p. 62-68.

[5] E. C. Titchmarsh, *The zeros of certain integral functions*, Proceedings Lond. Math. Soc. 25 (1926), p. 283-302.

Reçu par la Rédaction le 22. 9. 1960

STUDIA MATHEMATICA, T. XX. (1961)

Über Niveaulinien fastperiodischer Funktionen

von

S. HARTMAN (Wrocław)

1. Die hier vorkommenden Begriffe und Hauptsätze aus der Theorie der fastperiodischen (fp.) Funktionen können z. B. in [1] gefunden werden.

Es sei $f(t)$ eine reelle B-fp. Funktion, d. h. eine fp. Funktion im Sinne von Besicovitch. Wird $c_a(y) = 1$ oder 0 gesetzt, je nachdem $y < a$ oder $y \geqslant a$ ist, so ist bekanntlich [7] die (monotone) Funktion

$$F_f(a) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} c_a(f(t))\,dt$$

für jedes reelle a mit Ausnahme einer höchstens abzählbaren Menge A wohlbestimmt und stetig. Der Integrand $c_a(f(t))$ für $a \notin A$ heiße *a-Niveaulinie* von f. Man weiß, daß die Niveaulinien der B-fp. Funktionen wieder B-fp. Funktionen sind ([7], S. 399). Setzt man über f mehr voraus, so kann man auch über c_a mehr behaupten; ist z. B. f ein trigonometrisches Polynom, so ist c_a eine S-fp. Funktion (d. h. fastperiodisch im Sinne von Stepanoff), und zwar für jedes a ([6], S. 210-211). Will man systematisch untersuchen, wie die Eigenschaften von f diejenigen von c_a beeinflussen, so führe man zweckmäßig den folgenden, von C. Ryll-Nardzewski stammenden Begriff ein:

Definition. Eine für alle t erklärte nach Lebesgue meßbare Funktion $f(t)$ heißt *R-fastperiodisch* wenn es für jedes $\varepsilon > 0$ zwei Bohrsche fp. Funktionen φ und ψ gibt, so daß

$$\varphi(t) \leqslant f(t) \leqslant \psi(t) \text{ überall,} \tag{1}$$

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [\psi(t)-\varphi(t)]\,dt < \varepsilon.$$

2. Bevor wir das Problem der Niveaulinien wieder aufnehmen, wollen wir die Eigenschaften der R-fp. Funktionen näher betrachten.