Operational calculus in linear spaces

by

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1. Introduction. There are papers of Heaviside, Volterra [12], Curry [4], Plessner [10], Mikusiński [5-8], Słowikowski [11], Bellert [1-2], Niculescu [9] treating the operational calculus by abstract methods, without using Laplace transformation.

Bellert in his paper [2] has given a uniform theory for the operational calculus with different interpretations in ordinary linear differential, difference and differential-difference equations with constant coefficients and in Euler equations. He has defined an endomorphism T of a space X linear over the field Γ of complex numbers, satisfying the condition

$$(1)\qquad \sum_{n=0}^{N} a_n T^n x \neq 0 \quad \text{for} \quad a_N \neq 0 \text{ and } x \neq 0,$$

$$a_n \in \Gamma,\ x \in X.$$

Then the ring of endomorphisms $\sum_{n=0}^{N} a_n T^n$ can be extended to the field of elements

$$\frac{a_0 + a_1 T + \ldots + a_n T^n}{\beta_0 + \beta_1 T + \ldots + \beta_n T^n},$$

which contains in particular the operator $p = 1/T$, so that we can obtain Heaviside's method.

Słowikowski in his paper [11] proves that the operational calculus may be applied to differential equations

$$x^{(n)}(t) + A_{n-1} x^{(n-1)}(t) + \ldots + A_0 x(t) = f(t)$$

where coefficients $A_{n-1}, \ldots, A_0$ are endomorphisms of the space X. He considers also as an example the wave equation, which could not be treated in paper [2] as a differential equation.

This paper treats of an operational calculus in linear spaces. It contains a direct generalisation of papers [2] and [11], and is in some degree connected with papers [7], [8], [9].

Algebra of operators in linear spaces

2. Derivative, integral, constants. Suppose we are given two linear spaces C^0 and C^1 over a field Γ, and a linear operation S from C^1 onto C^0, i. e.

$$S(\alpha x_1+\beta x_2) = \alpha Sx_1+\beta Sx_2 \quad \text{for} \quad x_1, x_2 \in C^1,\ \alpha, \beta \in \Gamma. \tag{2}$$

In the following $C^1 \subset C^0$.

The general solution of the equation $Sx = f$, where $x \in C^1$, $f \in C^0$, is of the form $x = Tf+c$, where $Sc = 0$ and T is a linear operation with the properties

$$\begin{cases} STf = f, & T(C^0) \subset C^1 \subset C^0, \\ \text{if } Tf = 0, & \text{then} \quad f = 0 \text{ for } f \in C^0. \end{cases} \tag{3}$$

Operation S will be called a *derivative*, operation T will be called an *integral*. Elements c satisfying $Sc = 0$ will be called *constants*. In what follows we suppose that there exist constants not equal 0.

Let C^n be the domain of operation S^n; of course $C^{n+1} \subset C^n$. Let $C^\infty = \bigcap_{n=1}^{\infty} C^n$. For $x \in C^n$ we have the identity

$$x = (x-TSx)+T(Sx-TS^2x)+\ldots+T^{n-1}(S^{n-1}x-TS^nx)+T^nS^nx,$$

which implies the Taylor Theorem:

$$x = c_0+Tc_1+\ldots+T^{m-1}c_{m-1}+T^mS^mx \quad \text{for} \quad m = 1, 2, \ldots, n, \tag{4}$$

where $c_i = S^ix-TS^{i+1}x$, $i = 0, 1, \ldots, n-1$.

The development (4) will be called the *Taylor development of order m of x*.

The operation $sx = x-TSx$ from C^1 into the set of constants, called the *limit condition*, is linear.

Evidently we have

$$sTf = 0 \quad \text{for} \quad f \in C^0, \tag{5}$$

and, from (4), $c_i = sS^ix$, $i = 0, 1, \ldots, n-1$.

3. Linear derivative equations. Consider a linear differential equation

$$Lx \equiv S^nx+A_{n-1}S^{n-1}x+\ldots+A_0x = f, \quad x \in C^n,\ f \in C^0, \tag{6}$$

where $A_{n-1}, \ldots, A_0$ are endomorphisms of C^0 and C^1, linear and commutative with T on C^0, and with S on C^1.

The assumption $x \in C^n$ implies the Taylor development of x of order n: $x = c_0+Tc_1+\ldots+T^{n-1}c_{n-1}+T^nS^nx$, where the constants $c_0, c_1, \ldots, c_{n-1}$ are uniquely determined by element x and endomorphism T. If constants $c_0, c_1, \ldots, c_{n-1}$ do not define the solution x uniquely, then the difference $x_\omega = x_1-x_2$ of two solutions of equation (5) with the same Taylor development of order n satisfies the equation

$$Lx_\omega \equiv S^nx_\omega+A_{n-1}S^{n-1}x_\omega+\ldots+A_0x_\omega = 0 \tag{7}$$

with the limit conditions $sx_\omega = \ldots = sS^{n-1}x_\omega = 0$.

We shall prove the following theorem:

Theorem 1. *If any two solutions of equation (7) have the same Taylor development of order n, then their difference x_ω satisfies the condition*

$$x_\omega = T^nS^nx_\omega, \quad n = 1, 2, \ldots \tag{8}$$

Proof. From (7) we have

$$S^nx_\omega = -A_{n-1}S^{n-1}x_\omega-\ldots-A_0x_\omega;$$

then

$$-A_{n-1}S^nx_\omega-\ldots-A_0Sx_\omega = S(-A_{n-1}S^{n-1}x_\omega-\ldots-A_0x_\omega) = S^{n+1}x_\omega$$

because endomorphisms $A_{n-1}, \ldots, A_0$ are commutative with S on C^1. We also have $x_\omega \in C^{n+1}$, and by induction $x_\omega \in C^\infty$. Then x_ω has a Taylor development of any order, $x_\omega = c_0^\omega+Tc_1^\omega+\ldots+T^{m-1}c_{m-1}^\omega+T^mS^mx_\omega$, $m = 1, 2, \ldots$, and for $m > n$ we have

$$\begin{aligned} Lx_\omega = {} & (c_n^\omega+A_{n-1}c_{n-1}^\omega+\ldots+A_0c_0^\omega)+ \\ & +T(c_{n+1}^\omega+A_{n-1}c_n^\omega+\ldots+A_0c_1^\omega)+ \\ & +T^2(c_{n+2}^\omega+A_{n-1}c_{n+1}^\omega+\ldots+A_0c_2^\omega)+\ldots \\ & +T^{m-1}(c_{m+n-1}^\omega+A_{n-1}c_{m+n-2}^\omega+\ldots+A_0c_{m-1}^\omega)+ \\ & +T^{m-n}S^mx_\omega+T^{m-n+1}A_{n-1}S^mx_\omega+\ldots+T^mA_0S^mx_\omega. \end{aligned}$$

From conditions $s^iLx_\omega = 0$, $i = 0, 1, \ldots$, taking $m > n+1$ we obtain an infinite set of equations,

$$c_n^\omega+A_{n-1}c_{n-1}^\omega+\ldots+A_0c_0^\omega = 0, \ldots,$$

$$c_{n+1}^\omega+A_{n-1}c_{n+i-1}^\omega+\ldots+A_0c_i^\omega = 0, \ldots \quad (i = 0, 1, \ldots),$$

and because $c_0^\infty = \ldots = c_{n-1}^\infty = 0$ we have $c_k^\infty = 0$, $k = 0, 1, \ldots$ We see that $x_\infty = T^n S^n x_\infty$, $n = 1, 2, \ldots$, which follows from Taylor's formula (4).

The solution $x_\infty \neq 0$ of equation (7) satisfying (8) will be called a *singular solution*. We see that the number of linear independent solutions of the homogeneous equation

$$(9) \qquad S^n x + A_{n-1} S^{n-1} x + \ldots + A_0 x = 0$$

is not greater than $d = nd_0 + \varrho$, where d_0 is the dimension of the linear space of constants and ϱ is the dimension of the space of singular solutions.

The example of equation $\Delta z(x, y) = 0$, where $\Delta z = \partial^2 z/\partial x^2 + \partial^2 z/\partial y^2$, proves that the number of linear independent solutions can be greater than the order of equation. The example of equation $\Delta z + \alpha z = 0$, α being a real number, $\alpha > 0$, with limit condition $z|_{\partial\Omega} = 0$, $\partial\Omega$ being the contour of Ω, proves that singular solutions exist.

We can also consider systems of linear derivative equations

$$(10) \qquad Sx_i = A_{i1} x_1 + \ldots + A_{in} x_n + f_i, \qquad x_i \in C^1,\ f_i \in C^0$$

with limit conditions

$$(11) \qquad sx_i = c_{i0}, \qquad i = 1, \ldots, n.$$

If the endomorphisms $A_{11}, \ldots, A_{nn}$ of C^0 and C^1 are commutative with T on C^0, and with S on C^1, and if two solutions $x_1^{(1)}, \ldots, x_n^{(1)}$ and $x_1^{(2)}, \ldots, x_n^{(2)}$ of system (10) have the same Taylor development of order 1, that is

$$sx_i^{(1)} = c_{i0} = sx_i^{(2)}, \qquad i = 1, \ldots, n,$$

then their differences $x_{i\infty} = x_i^{(1)} - x_i^{(2)}$, $i = 1, \ldots, n$, satisfy condition (8).

If equations (10) are homogeneous ($f_i = 0$), then the number of linear independent solutions is again not greater than $d = nd_0 + \varrho$.

4. Operational calculus in linear spaces. In the following considerations we shall suppose that endomorphisms $A_{n-1}, \ldots, A_0$ of C^0, and C^1 have the following property:

(12) if $S^n x + A_{n-1} S^{n-1} x + \ldots + A_0 x = 0$, $s(S^i x) = 0$, $i = 0, 1, \ldots, n-1$, then $x = 0$.

Besides we shall assume that endomorphisms $A_{n-1}, \ldots, A_0$ commute on C^0, and commute with T on C^0, and S on C^1.

Multiplying the equation

$$13) \qquad S^n x + A_{n-1} S^{n-1} x + \ldots + A_0 x = f, \qquad x \in C^n,\ f \in C^0,$$

with limit conditions

$$(14) \qquad sS^i x = c_i, \qquad i = 0, 1, \ldots, n-1,$$

by T^n we see from Taylor's formula (4) that (12) may be written in the form

$$(15) \qquad (I + A_{n-1} T + \ldots + A_0 T^n) x = C_{n-1}(x) + A_{n-1} T C_{n-2}(x) + \ldots + A_1 T^{n-1} C_0(x) + T^n f,$$

where

$$C_k(x) = c_0 + Tc_1 + \ldots + T^k c_k, \qquad k = 0, 1, \ldots, n-1,$$

$Ig = g$ for $g \in C^0$; constants $c_0, \ldots, c_k$ are defined by Taylor's formula and by (14).

We shall prove the following theorem:

THEOREM 2. *If*

$$(16) \qquad (I + A_{n-1} T + \ldots + A_0 T^n) x = 0,$$

hen $x = 0$.

Proof. From supposition (16) we have $x \in C^n$, so that from Taylor's formula $x = c_0 + Tc_1 + \ldots + T^{n-1} c_{n-1} + T^n S^n x$, and $sS^i x = c_i$, $i = 0, 1, \ldots, n-1$.

Now we have from (13)-(16) the equality

$$(17) \qquad c_0 + Tc_1 + \ldots + T^{n-1} c_{n-1} + A_{n-1} T(c_0 + Tc_1 + \ldots + T^{n-2} c_{n-2}) + \ldots + A_1 T^{n-1} c_0 + T^n f = 0.$$

Multiplying (17) by s we have $c_0 = 0$. Substituting $c_0 = 0$ in (17) and multiplying by sS we have $c_1 = 0$, and by induction $c_2 = \ldots = c_{n-2} = c_{n-1} = 0$. We then have $T^n f = 0$, whence $f = 0$ from (3). We thus have the assumption of (12), and $x = 0$. We see also that conditions (12) and (16) are equivalent.

An operation of the form

$$(18) \qquad W(T) = aT^{k_1}(I + B_1 T + \ldots + B_n T^n)^{k_2}, \qquad a \in \Gamma,\ a \neq 0$$

$$(k_1,\ k_2 \text{ integers},\ k_1 \geqslant 0,\ k_2 \geqslant 0)$$

will be called a *polynomial*.

Let Π be a commutative semigroup of polynomials $W_1(T), W_2(T), \ldots$ which satisfy the condition

$$(19) \qquad \text{if } W(T) x = 0, \text{ then } x = 0.$$

Pairs $[f, W(T)]$, where $f, g, \ldots, \in C^0$, $W_1(T), W_2(T), \ldots \in \Pi$, with equality

(20) $\quad [f, W_1(T)] = [g, W_2(T)]$ if and only if $W_1(T)g = W_2(T)f$

and operations

$$[f, W_1(T)] + [g, W_2(T)] = [W_2(T)f + W_1(T)g,\ W_1(T)W_2(T)],$$
$$\alpha[f, W_1(T)] = [\alpha f, W_1(T)] \tag{21}$$

will be called *results*. Equality (20) and operations (21) are well defined, because we have (19) and

$$W(T)(C^0) \subset C^0 \quad \text{for} \quad W(T) \in \Pi.$$

Instead of $[f, W_1(T)]$ we shall write $\frac{f}{W_1(T)}$. We then have

$$\begin{cases} \dfrac{f}{W_1(T)} = \dfrac{g}{W_2(T)} \text{ if and only if } W_2(T)f = W_1(T)g, \\ \dfrac{f}{W_1(T)} + \dfrac{g}{W_2(T)} = \dfrac{W_2(T)f + W_2(T)g}{W_1(T)W_2(T)}, \\ \alpha \dfrac{f}{W_1(T)} = \dfrac{\alpha f}{W_1(T)}. \end{cases} \tag{22}$$

We shall also write $\frac{W_1(T)}{W_2(T)} \frac{f}{W_3(T)}$ instead of $\frac{W_1(T)f}{W_2(T)W_3(T)}$ if $W_1(T)$, $W_2(T)$, $W_3(T) \in \Pi$.

The set $C^0(\Pi)$ of all results is a linear space. The operators $\frac{W_1(T)}{W_2(T)}x$, where $x \in C^0(\Pi)$, $W_1(T), W_2(T) \in \Pi$, are endomorphisms of that space. Just as a group of operators $\frac{W_1(T)}{W_2(T)}$ has a semigroup Π of polynomials $\left(\frac{W(T)}{I} = W(T)\right)$ isomorphically contained in it, so the space of results $C^0(\Pi)$ has the space C^0 isomorphically contained in it $\left(x = \frac{I}{I}x\right)$.

By means of the space of results it is possible to solve (13) with limit conditions (14). We take a commutative semigroup containing the polynomial $I + A_{n-1}T + \ldots + A_0T^m$ and we obtain

$$x = \frac{I}{I + A_{n-1}T + \ldots + A_0T^m}\left(\sum_{i=0}^{n-1} V_i(T)c_i + T^n f\right), \quad x \in C^0(\Pi), \tag{23}$$

where the polynomials $V_i(T)$ are of degrees lower than n (in general $V_i(T) \notin \Pi$).

Polynomial $I + A_{n-1}T + \ldots + A_0T^m$ is decomposable into the product of polynomials of lower degrees if and only if (13) is equivalent to the system of two equations of lower orders.

In particular we can have in semigroup Π a canonical decomposition of a polynomial:

(24) $\quad I + A_{n-1}T + \ldots + A_0T^m = (I - R_1T)^{i_1}(I - R_2T)^{i_2}\ldots(I - R_mT)^{i_m}$

where endomorphisms $R_1, \ldots, R_m$ commute, and a canonical form of (13):

$$(S - R_1)^{i_1}(S - R_2)^{i_2}\ldots(S - R_m)^{i_m}x = f,$$
$$sS^ix = c_i, \quad i = 0, 1, \ldots, n-1, \text{ where } i_1 + \ldots + i_m = n. \tag{25}$$

Now we shall prove two fundamental theorems of the Algebra of Operators:

THEOREM 3. *If* (13), *where the endomorphisms* $A_{n-1}, \ldots, A_0$ *commute, is reducible to a canonical form* (25) *and if it has a unique solution, then that solution can be obtained by Heaviside's algorithm consisting in a decomposition of operators into simple fractions*:

$$\frac{I}{(I - TR_\nu)^i}, \quad i = 0, 1, \ldots, i.$$

Proof. From (24) we have

$$x = \sum_{i=0}^{n-1} \frac{I}{W(T)} V_i(T)c_i + \frac{T^n}{W(T)}f, \tag{26}$$
$$W(T) = (I - TR_1)^{i_1}\ldots(I - TR_m)^{i_m},$$

where the degrees of polynomials $V_i(T)$ are lower than the degree of $W(T)$. If we decompose the operators $\frac{I}{W(T)}$, $\frac{T^n}{W(T)}$ into simple fractions we obtain results of the form $\frac{I}{(I - TR_\nu)^i}g$, $i = 0, 1, \ldots, i_\nu$, $g \in C^0$.

THEOREM 4. *If equation* (10) *with limit conditions* (11) *has the unique solution, if the endomorphisms* $A_{11}, \ldots, A_{nn}$ *commute on* C^0, *and if the polynomial*

$$W(T) = \begin{vmatrix} I - A_{11}T, & -A_{12}T, & \ldots, & -A_{1n}T \\ \cdot & \cdot & \cdot & \cdot \\ -A_{n1}T, & -A_{n2}T, & \ldots, & I - A_{nn}T \end{vmatrix} \tag{27}$$

has a canonical decomposition in a semigroup Π *containing* $W(T)$, *then the solution* x *can be obtained by Heaviside's algorithm.*

Proof. We multiply the equations by T, and solve by simple fractions equations with unknowns $x_1, x_2, \ldots, x_n$.

These results hold in particular for linear equations with numerical coefficients considered in paper [2], where different interpretations of space C^0 and integral T can be found.

Analysis of operators in linear topological, locally convex spaces

5. The uniqueness of solution. Let C^0 be a linear topological, locally convex space, sequentially complete, with topology defined by pseudonorms $|x|_\lambda$ $(\lambda \in \Lambda)$, i. e.

$$(28)\qquad \begin{cases} x_n \Rightarrow x \text{ if and only if } |x_n - x|_\lambda \to 0 \text{ for } \lambda \in \Lambda,\ n \to \infty, \\ x = 0 \text{ if and only if } |x|_\lambda = 0 \text{ for } \lambda \in \Lambda. \end{cases}$$

We shall discuss problems of uniqueness of solution of a linear derivative equation and we shall try to determine its form.

An endomorphism R of C^0 will be called a *strongly bounded endomorphism* if there exist positive numbers M_λ, such that

$$(29)\qquad |Rx|_\lambda \leqslant M_\lambda |x|_\lambda .$$

For fixed λ the least upper bound of such M_λ will be called the *λ-th pseudonorm* $|R|_\lambda$ of a strongly bounded endomorphism R.

The set $\boldsymbol{R}$ of strongly bounded endomorphisms forms an algebra with superposition as multiplication.

THEOREM 5 (see [11]). *If* $\sqrt[p]{|T^p|_\lambda} \to 0$ *for* $p \to \infty$ *and* $\lambda \in \Lambda$, *and if the endomorphisms* $A_{n-1}, \ldots, A_0 \in \boldsymbol{R}$ *are commutative with* S *and* $T \in \boldsymbol{R}$, *then* (12).

Proof. If equation (12) is satisfied then $(I + TA_{n-1} + \ldots + T^n A_0)x = 0$ and $x = (-1)^p T^p (A_{n-1} + \ldots + T^{n-1} A_0)^p x$, $p = 1, 2, \ldots$, so that

$$(30)\qquad |x|_\lambda \leqslant |T^p|_\lambda |A_{n-1} + \ldots + T^{n-1} A_0|_\lambda^p |x|_\lambda .$$

But we have

$$\sqrt[p]{|T^p|_\lambda}\, |A_{n-1} + \ldots + T^{n-1} A_0|_\lambda \sqrt[p]{|x|_\lambda} \leqslant \sqrt[p]{|T^p|_\lambda}\, \overline{M}_\lambda \sqrt[p]{|x|_\lambda},$$

where $\overline{M}_p^\lambda = |A_{n-1}|_\lambda + \ldots + |T|_\lambda^{n-1} |A_0|_\lambda$ and $\sqrt[p]{|T^p|_\lambda}\, \overline{M}_\lambda \sqrt[p]{|x|_\lambda} \to 0$ as $p \to \infty$ because $\sqrt{|T^p|_\lambda} \to 0$. Thus the series

$$\sum_{p=1}^{\infty} |T^p|_\lambda |A_{n-1} + \ldots + T^{n-1} A_0|_\lambda^p |x|_\lambda$$

is convergent, and $|T^p|_\lambda |A_{n-1} + \ldots + T^{n-1} A_0|_\lambda^p |x|_\lambda \to 0$, whence, by (30), $|x|_\lambda = 0$ for every $\lambda \in \Lambda$, and $x = 0$.

THEOREM 6. *If* $T, R \in \boldsymbol{R}$ *and* $|T^p|_\lambda = O_\lambda(q_\lambda^p)$, $0 < q_\lambda < 1$, $|R^p|_\lambda = O_\lambda(p^{\alpha_\lambda})$ *for* $\lambda \in \Lambda$, *then* $Sx - Rx = 0$, $sx = 0$ *implies* $x = 0$.

Proof. We have $x = TRx = (TR)^p x$ for $p = 1, 2, \ldots$; then $|x|_\lambda \leqslant |T^p|_\lambda |R^p|_\lambda |x|_\lambda$. But $|T^p|_\lambda |R^p|_\lambda |x|_\lambda \to 0$ as $p \to \infty$ for every $\lambda \in \Lambda$. The proof is just like the last part of the proof of theorem 5. Hence $x = 0$.

6. Operational convergence and analytic elements. Let an integral be a continuous endomorphism of C^0, and let Π^c be a subsemigroup of Π composed of all continuous endomorphisms contained in Π. In the space of results $C^0(\Pi^c)$ we introduce a convergence called *operational convergence* (see [5]), defined by the formula

$$(31)\qquad \xi_n \to \xi \text{ if and only if there exists a polynomial } W(T) \in \Pi^c \text{ such that } W(T)\xi_n \Rightarrow W(T)\xi .$$

It can be proved that there exists at most one limit for any sequence of results. If elements $x_n \in C^0$ form a convergent sequence, then that sequence tends to the same limit also in the operational sense, since $x_n = Ix_n$, $x = Ix$. The space of results $C^0(\Pi^c)$ becomes a linear space with convergence (may be non-topological); addition of results and multiplication by a number are sequentially continuous.

With convergence (31) every operator $\dfrac{W_1(T)}{W_2(T)}$, where $W_1(T)$, $W_2(T) \in \Pi^c$ is continuous. If as in (26)

$$x_m = \sum_{k=0}^{n-1} \frac{I}{W(T)} V_i(T) c_{i,m} + \frac{T^m}{W(T)} f_m$$

and if $c_{i,m} \Rightarrow c_i$, $f_m \Rightarrow f$ as $m \to \infty$, then

$$x_m \to \sum_{k=0}^{n-1} \frac{I}{W(T)} V_i(T) c_i + \frac{T^m}{W(T)} f,$$

so that sequential continuity of limit conditions $c_{i,m}$ and of elements f_m implies operational sequential continuity of solutions of (13).

We denote by $\boldsymbol{A}$ the subspace of c^0 composed of all elements x of the form (4) such that $T^m S^m x \to 0$ as $m \to \infty$. Elements x will be called *analytic elements* (see [9]).

THEOREM 7. *If R is a continuous endomorphism commutative with integral T, and if $x \in A$, then for every positive integer n we have $T^n x$, $S^n x$, $Rx \in A$,* and

$$\left\{\begin{aligned} T^n\Big(\sum_{m=0}^{\infty} T^m c_m\Big) &= \sum_{m=0}^{\infty} T^{n+m} c_m,\\ S^n\Big(\sum_{m=0}^{\infty} T^m c_m\Big) &= \sum_{m=0}^{\infty} T^{m-n} c_m,\\ R\Big(\sum_{m=0}^{\infty} T^m c_m\Big) &= \sum_{m=0}^{\infty} T^m (Rc_m).\end{aligned}\right. \tag{32}$$

Proof. If $x = c_0 + \ldots + T^{m-1} c_{m-1} + T^m S^m x$ for $m = 1, 2, \ldots$, then

$$T^n x = T^n c_0 + \ldots + T^{n+m-1} c_{m-1} + T^n (T^m S^m x),$$
$$Rx = Rc_0 + \ldots + T^{m-1} Rc_{m-1} + RT^m S^m x,$$

But $T^m S^m x \to 0$ as $m \to \infty$ implies, by the definition of operational convergence, $T^{n+m} S^m x \to 0$, $RT^m S^m x \to 0$ because T and R are continuous endomorphisms and R is commutative with T.

We thus obtain the first and the third of the equalities (32). We also have $x = \sum_{m=0}^{\infty} T^m c_m$, so that

$$S^n x = \frac{x - c_0 - Tc_1 - \ldots - T^{n-1} c_{n-1}}{T^n} = \sum_{m=n}^{\infty} T^{m-n} c_m.$$

THEOREM 8. *If $x = c_0 + \ldots + T^{m-1} c_{m-1} + T^m S^m x$ for $m = 1, 2, \ldots$ and if $|T^p|_\lambda = O_\lambda(q_\lambda^p)$, $0 < q_\lambda < 1$, $|S^p x|_\lambda = O_\lambda(p^{\alpha_\lambda})$, $\lambda \in \Lambda$, then $\sum_{m=0}^{p} T^m c_m \Rightarrow x$, $\sum_{m=k}^{p} T^{m-k} c_m \Rightarrow S^k x$ as $p \to \infty$.*

Proof. We have $|T^p S^p x|_\lambda \leqslant |T^p|_\lambda |S^p x|_\lambda \leqslant M_1 q_\lambda^p M_2 p^{\alpha_\lambda}$ and $|T^p S^{p+k} x|_\lambda \leqslant |T^p|_\lambda |S^{p+k} x|_\lambda \leqslant M_1 q_\lambda^p M_2 (p+k)^{\alpha_\lambda}$ for great p. The proof that $M_1 M_2 q_\lambda^p p^{\alpha_\lambda} \to 0$ and $M_1 M_2 q_\lambda^p (p+k)^{\alpha_\lambda} \to 0$ as $p \to \infty$ is just like the last part of the proof of theorem 5. Hence $T^p S^{p+k} x \Rightarrow 0$ as $p \to \infty$, for $m = 1, 2, \ldots$, $k = 0, 1, \ldots$

7. The multiplier τx in the space of analytic elements. Let us consider in the space A the operation

$$\tau\Big(\sum_{n=0}^{\infty} T^n c_n\Big) = \sum_{n=0}^{\infty} (n+1) T^{n+1} c_n. \tag{33}$$

The operation τ will be called a *multiplier*.

It is easy to see that

$$S\tau x = x + \tau S x, \qquad \sum_{n=0}^{\infty} T^n c_n = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} c_n. \tag{34}$$

It can be proved by induction that

$$S\tau^n x = n\tau^{n-1} x + \tau^n S x.$$

Hence we have

$$S^m \tau^n c_0 = \frac{d^m[\tau^n]}{d\tau^m} c_0 \tag{35}$$

for constant c_0.

THEOREM 9. *If $|T^p|_\lambda = O_\lambda(q_\lambda^p)$, $0 < q_\lambda < 1$, $|c_p|_\lambda = O_\lambda(p^{\alpha_\lambda})$, $\lambda \in \Lambda$, then $\sum_{n=0}^{\infty} T^n c_n \in D(\tau^m)$* (1).

Proof. We have

$$\Big|\sum_{p=\pi}^{\infty} (p+1)\ldots(p+m) T^{p+m} c_p\Big|_\lambda \leqslant \sum_{p=\pi}^{\infty} (p+1)\ldots(p+m) |T^{p+m}|_\lambda |c_p|_\lambda$$

and the last series tends to zero as $\pi \to \infty$, because it is the rest of a convergent series.

8. The exponential operation in the space A. Let us consider in the space A an exponential operation

$$e^{R\tau}\Big(\sum_{n=0}^{\infty} T^n c_n\Big) = \sum_{n=0}^{\infty} T^n \Big[\sum_{i=0}^{n} \binom{n}{i} R^i c_{n-i}\Big] \tag{36}$$

defined for every endomorphism R continuous and commutative with T and S.

We have

$$e^{R\tau}\Big(\sum_{n=0}^{\infty} T^n c_n\Big) = \sum_{n=0}^{\infty} \tau^n \Big[\sum_{i=0}^{\infty} \frac{R^i}{i!} \frac{c_{n-i}}{(n-i)!}\Big],$$

(1) $D(R)$ denotes the domain of operation R.

so that

(37) $$e^{R_1\tau}e^{R_2\tau} = e^{(R_1+R_2)\tau} \text{ for } R_1 \text{ commutative with } R_2,$$

(38) $$e^{O\tau} = I, \quad (e^{R\tau})^{-1} = e^{-R\tau},$$

(39) $$Se^{R\tau}x = e^{R\tau}\cdot Sx + e^{R\tau}Rx, \quad se^{R\tau}x = sx.$$

THEOREM 10. *If $f \in D(e^{-R\tau})$, then the equation*

$$Sx - Rx = f, \quad sx = c_0$$

has the solution

(40) $$x = e^{R\tau}(Te^{-R\tau}f + c_0).$$

Proof. We multiply the equation by $e^{-R\tau}$, and obtain $S(e^{-R_\tau}x) = e^{-R\tau}f$, $s(e^{-R\tau}x) = c_0$, whence (40).

In the particular case where an integral T is a continuous endomorphism of C^0, and $\sup_{\lambda\in\Lambda}|T|_\lambda = \|T\| < +\infty$, and if the function $F(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic for $|z| \geqslant \|T\|$, then

(41) $$TF(T)x = \frac{1}{2\pi i}\oint_C \frac{F(z)}{z} e^{\tau/z}(Te^{-\tau/z}x)\,dz \text{ for } x \in D(e^{-\tau/z}),$$

where the contour C is counterclockwise and encloses the circle $|z| \leqslant \|T\|$ Formula (41) is a generalisation of the Laplace formula.

The proof of this remark is the following. Condition $|T|_\lambda \leqslant \|T\|$ implies that for $|z| \geqslant \|T\|$ the operation $I - T/z$ has the inverse

$$\frac{I}{I - \frac{T}{z}} = \sum_{n=0}^{\infty} \frac{T^n}{z},$$

so that

$$TF(T)x = \frac{1}{2\pi i}\oint_C F(z) \frac{\frac{T}{z}}{I - \frac{T}{z}}x\,dz = \frac{1}{2\pi i}\oint_C \frac{F(z)}{z}\,\frac{zT}{z-T}x\,dz.$$

The result $\dfrac{zT}{z-T}x$ is the solution of the equation

$$Sy - \frac{1}{z}y = x, \quad sy = 0.$$

THEOREM 11. *If the domain of the operation $\tau^{m-1}e^{R\tau}$ contains all constants, then the equation $(S-R)^m x = 0$ has a solution of the form*

$$x = \sum_{i=1}^{m-1} \tau^i e^{R\tau} c_i,$$

where c_i are constants.

Proof. From (36) we have

$$e^{R\tau}c = \sum_{n=0}^{\infty} T^n R^n c = \sum_{n=0}^{\infty} \frac{\tau^n R^n}{n!} c \quad \text{for } c \text{ constant},$$

whence

$$\tau^i e^{R\tau} c = \sum_{n=0}^{\infty} \frac{\tau^{n+i} R^n}{n!} c.$$

By formula (35) $e^{R\tau}c, \tau e^{R\tau}c, \ldots, \tau^{m-1}e^{R\tau}c$ are solutions of the equation $(S-R)^m x = 0$. Then the equation $(S-R)^m x = 0$, $sS^i x = c_i$, $i = 0, 1, \ldots, m-1$, has the solution

$$x = \sum_{i=0}^{m-1} \frac{1}{i!} \tau^i e^{R\tau} c_i.$$

THEOREM 12. *If $\sqrt[p]{|T^p|_\lambda} \to 0$, $|c_p|_\lambda = O_\lambda(m_\lambda^p)$, $\lambda \in \Lambda$, then*

$$x = \sum_{n=0}^{\infty} T^n c_n \in D(e^{R\tau}).$$

Proof. We have

$$\left|\sum_{p=\pi}^{\infty} T^p\left[\sum_{i=0}^{p}\binom{p}{i}R^i c_{p-i}\right]\right|_\lambda \leqslant \sum_{p=\pi}^{\infty}|T^p|_\lambda \sum_{i=0}^{p}\binom{p}{i}|R|_\lambda^i M_\lambda m_\lambda^{p-i}$$
$$\leqslant \sum_{p=\pi}^{\infty}|T^p|_\lambda M_\lambda(|R|_\lambda + m_\lambda)^p,$$

and the last series tends to zero as $\pi \to \infty$ because it is the rest of a convergent series.

THEOREM 13. *If $|T^p|_\lambda = O_\lambda(q_\lambda^p)$, $0 < q_\lambda < 1$, $|R^p|_\lambda = O_\lambda(p^{a_\lambda})$, then the domain of the operation $\tau^m e^{R\tau}$ contains all constants.*

Proof. We have

$$\left|\sum_{p=\pi}^{\infty} T^p R^p c_0\right| \leqslant \sum_{p=\pi}^{\infty}|T^p|_\lambda |R^p|_\lambda |c_0|_\lambda$$

and the last series tends to zero just as in the proof of theorem 12. Hence the series $\sum_{p=0}^{\infty} T^p R^p c_0$ is convergent. But

$$|R^p c_0|_\lambda \leqslant |R^p|_\lambda |c_0|_\lambda = O_\lambda(p^{a_\lambda})$$

and theorem 9 implies $e^{R\tau} c_0 \in D(\tau^m)$.

9. Relations between an integral and convolution and multiplication. Suppose that the set of constants is a commutative algebra with unit l. In the space $\boldsymbol{A}$ we can define a new operation $a*b$, called *convolution*:

$$\sum_{n=0}^{\infty} T^n a_n * \sum_{n=0}^{\infty} T^n b_n = \sum_{n=0}^{\infty} T^{n+1} \Big(\sum_{k=0}^{n} a_k b_{n-k} \Big). \tag{42}$$

If convolution $a*b$ is defined for every pair of analytic elements a, b, then the space $\boldsymbol{A}$ with operation $*$ forms a commutative algebra. We have

$$l*a = Ta, \qquad Rl*a = TRa \quad \text{for} \quad a \in \boldsymbol{A} \tag{43}$$

and for every continuous endomorphism R on C^0 which commutes with T.

THEOREM 14. Duhamel's formula. *If R is a continuous endomorphism which commutes on C^0 with T, and if $h = Rl$, $k = Rf$ where $h, f, k \in \boldsymbol{A}$, then*

$$k = S(h*f). \tag{44}$$

Proof. From relations (43) we have $Rl*f = TRf$, so that $h*f = Tk$ and $k = S(h*f)$.

THEOREM 15. Borel's Formula. *If R_1, R_2 are continuous endomorphisms which commute on C^0 with T, then*

$$R_1 R_2 (f*g) = R_1 f * R_2 g,. \tag{45}$$

Proof. Let $h_1 = R_1 l$, $h_2 = R_2 l$. We have from Duhamel's formula

$$\begin{aligned} T^2(R_1 f * R_2 g) &= l*l*R_1 f * R_2 g = l*R_1 f * l * R_2 g \\ &= TR_1 f * TR_2 g = h_1 * f * h_2 * g = h_1 * \big(h_2 * (f*g)\big) \\ &= TR_1 TR_2 (f*g) = T^2 R_1 R_2 (f*g) \end{aligned}$$

and $R_1 f * R_2 g = R_1 R_2 (f*g)$.

We see that the convolution can be extended to the space $\boldsymbol{A}(\Pi^c)$ of results $\frac{W_1(T)}{W_2(T)} a$ in the following manner:

$$\frac{W_1(T)}{W_2(T)} a * \frac{W_3(T)}{W_4(T)} b = \frac{W_1(T) W_4(T)}{W_2(T) W_4(T)} (a*b), \tag{46}$$

where $a, b \in \boldsymbol{A}$, $W_1(T)$, $W_2(T)$, $W_3(T)$, $W_4(T) \in \Pi^0$.

The element $\varepsilon = l/T$ is the unit of this convolution algebra.

Suppose that the set of constants is a commutative algebra with unit l. In the space $\boldsymbol{A}$ we can define an operation $a \cdot b$ called *multiplication* (see [9]):

$$\Big(\sum_{n=0}^{\infty} T^n a_n \Big) \cdot \Big(\sum_{n=0}^{\infty} T^n b_n \Big) = \sum_{n=0}^{\infty} T^n \Big[\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \Big]. \tag{47}$$

If we denote by t the element Tl, we have

$$\sum_{n=0}^{\infty} T^n a_n = \sum_{n=0}^{\infty} \frac{t^n}{n!} a_n.$$

If multiplication $a \cdot b$ is defined for every pair of analytic elements, then the space $\boldsymbol{A}$ forms under multiplication a commutative algebra. We have formulas

$$\begin{cases} S(a\cdot b) = (Sa)\cdot b + a \cdot (Sb) & \text{for} \quad a, b \in \boldsymbol{A}, \\ \tau a = t \cdot a, \quad e^{R\tau} a = e^{Rt} \cdot a & \text{where} \quad e^{Rt} = \sum_{n=0}^{\infty} \frac{R^n t^n}{n!}. \end{cases} \tag{48}$$

10. Examples. A. *Differential equations with constant coefficients.* The space C^0 of continuous functions $\{x(t)\}$ of a real or complex variable, defined in a domain Ω, with values in a linear topologic locally convex space and with derivative $S\{x(t)\} = \{x'(t)\}$, is discussed by Słowikowski [11]. The limit condition has the form $s\{x(t)\} = \{x(t_0)\}$, where $t_0 \in \Omega$. The integral has the form

$$T\{x(t)\} = \Big\{ \int_{t_0}^{t} x(\tau)\, d\tau \Big\}, \quad \tau\{x(t)\} = \{(t - t_0) x(t)\},$$

$$e^{R\tau}\{x(t)\} = \{e^{R(t-t_0)} x(t)\}.$$

The element x is analytic if it is the sum of its Taylor series. The solutions of the equation $(S-R)^m x = 0$ are analytic for every strongly continuous endomorphism R. Słowikowski discusses in particular the wave equation $x''(t)+A^2x(t) = a(t)$, where A^2 is the operator $-\sum_{i=1}^{m}\frac{\partial^2}{\partial x_i^2}$ considered in a suitable B_0-space.

Mikusiński [5] considers the differential equations in the field of operators which is an extension of the convolution ring of the functions $\{x(t)\}$ without divisors of zero, $t \geqslant 0$.

B. *The iterated wave, harmonic and heat equations.* Let us consider the space C^0 of continuous real functions $\{u(x_1, x_2, x_3, t)\}$ defined in a four-dimensional space of points (x_1, x_2, x_3, t), with quasi-uniform convergence. We define the derivative

$$S\{u(x_1, x_2, x_3, t)\} = \left\{\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\frac{\partial^2 u}{\partial x_3^2}-\frac{\partial^2 u}{\partial t^2}\right\} = \Box u$$

and the integral

$$T\{f(x_1, x_2, x_3, t)\} = \left\{\frac{1}{4\pi}\iiint\limits_{r<t}\frac{f(\alpha_1, \alpha_2, \alpha_3, t-\tau)}{r}\right\}d\tau,$$

$$r = \sqrt{(x_1-\alpha_1)^2+(x_2-\alpha_2)^2+(x_3-\alpha_3)^2}$$

(*retarded potential*), thus obtaining the limit condition $su = u-T\Box u$.

The integral is a strongly bounded endomorphism and satisfies the assumptions of the uniqueness theorem, where the pseudonorms are

$$|u|_k = \max_{\substack{|x_i|\leqslant k\\ |t|\leqslant k}}|u(x_1, x_2, x_3, t)|.$$

Therefore the equation

$$a_n\Box^n u+a_{n-1}\Box^{n-1}u+\ldots+a_0 u = f,$$

$$su = u_0,\ s\Box u = u_1,\ \ldots,\ s\Box^{n-1}u = u_{n-1}$$

with constant coefficients has the unique solution. In particular the Klein-Gordon equation

$$\Box u+k^2 u = f,$$

$$u(x_1, x_2, x_3, 0) = \varphi(x_1, x_2, x_3),$$

$$u_t(x_1, x_2, x_3, 0) = \psi(x_1, x_2, x_3),$$

where $f(x_1, x_2, x_3, t)$, $\varphi(x_1, x_2, x_3)$, $\psi(x_1, x_2, x_3)$, are continuous functions, has the unique solution

$$u = \frac{u^0}{I+k^2T}+\frac{Tf}{I+k^2T} = \sum_{n=0}^{\infty}(-1)^n k^{2n}T^n u^0+\sum_{n=0}^{\infty}(-1)^n k^{2n}T^{n+1}f,$$

where

$$\Box u^0 = 0,\ u^0(x_1, x_2, x_3, 0) = \varphi(x_1, x_2, x_3),$$

$$u_t^0(x_1, x_2, x_3, 0) = \psi(x_1, x_2, x_3).$$

If space C^0 is the space of continuous real functions $u(x_1, x_2, x_3)$ defined in a closed domain $\bar{\Omega}$ with uniform convergence, then taking the derivative $S = \Delta$ (*Laplace operator*) and the integral defined by equations

$$\Delta Tu = u, \quad Tu|_{\partial\bar{\Omega}} = 0,$$

where $\partial\bar{\Omega}$ is the boundary of the domain $\bar{\Omega}$, we obtain harmonic equations.

For example, an equation $\Delta\Delta u+A_0 u = f$ where A_0 is a continuous endomorphism of C^0 with limit conditions $u|_{\partial\bar{\Omega}} = \varphi(x_1, x_2, x_3)$, $\Delta u|_{\partial\bar{\Omega}} = \psi(x_1, x_2, x_3)$ has a unique solution if the radius of the smallest sphere including is less than $\sqrt{6}/\sqrt[4]{\|A_0\|}$ (see [9]).

Similar remarks can be made for the iterated heat equation and for other partial differential equations.

C. *Difference equations.* The space C^0 of functions $\{x_n\}$ of an integer $n \geqslant 0$ with complex values and operations

$$S\{x_n\} = \{x_{n+1}-x_n\}, \quad s\{x_n\} = \{x_0\}$$

correspond to difference equations (see [2]). We have

$$T\{x_n\} = \left\{\sum_{k=0}^{n}x_k\right\}, \quad \tau\{x_n\} = \{nx_{n-1}\}, \quad e^{R\tau}\{x_n\} = \left\{\sum_{k=0}^{n}\binom{n}{k}R^k x_{n-k}\right\}.$$

The topology in C^0 is the convergence by each coordinate. All elements of C^0 are analytic.

D. *Another point of view for difference equations.* In the space C^0 from example C the operations $S\{x_n\} = \{x_{n+1}\}$, $s\{x_n\} = \{x_0, 0, 0, \ldots\}$ correspond to equations

$$a_m\{x_{n+m}\}+a_{m-1}\{x_{n+m-1}\}+\ldots+a_0\{x_n\} = \{f_n\}$$

discussed by Bellert [1]. We have $T\{x_n\} = \{x_{n-1}\}$, where

$$x_{-1} = 0, \quad \tau\{x_n\} = \{nx_{n-1}\}, \quad e^{R\tau}\{x_n\} = \left\{\sum_{k=0}^{n}\binom{n}{k}R^k x_{n-k}\right\}.$$

All elements are analytic.

E. *Euler's equations and differential-difference equations.* The Euler equations (see [2]) correspond to derivative $S\{x(t)\} = \{tx'(t)\}$ with limit condition $s\{x(t)\} = \{x(1)\}$. We have

$$T\{x(t)\} = \left\{\int_1^t \frac{x(\sigma)}{\sigma}\, d\sigma\right\}, \quad \tau\{x(t)\} = \{x(t)\ln t\},$$

$$e^{R\tau}\{x(t)\} = \{e^{R\ln t}x(t)\} = \{t^R x(t)\}.$$

We can also solve the differential-difference equations with derivation $S\{x(t)\} = \{x'(t+1)\}$ and limit condition

$$s\{x(t)\} = \begin{Bmatrix} c(t) & \text{for} & 0 \leqslant t \leqslant 1 \\ c(1) & \text{for} & t \geqslant 1 \end{Bmatrix}.$$

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Спектральная теория некоторых линейных операторов, мероморфно зависящих от параметра

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Различные вопросы теории линейных операторов, мероморфно зависящих от параметра рассматривались многими математиками. Исследованию спектра линейных интегральных уравнений с ядрами, зависящими мероморфно от параметра посвящены работы К. Миранда [1, 2], Р. Иглиша [3], Б. Манна [4] и автора [5, 6]. Резольвенту таких ядер в классе функций L^2 исследовал Я. Тамаркин [7]. Задача обращения для линейных мероморфно зависящих от параметра операторов в банаховом пространстве изучена автором [8]. Полученный в этой работе результат использован для исследования резольвенты мероморфных ядер интегральных уравнений в классе функций L^p, $p > 1$. В статье автора [9] исследуется спектр линейных мероморфно зависящих от параметра операторов в гильбертовом пространстве, обладающих конечным числом кратных вещественных полюсов.

В работе Х. Мюллера [10] исследованы линейные мероморфно зависящие от параметра операторы с конечным множеством простых вещественных полюсов.

В настоящей работе мы рассмотрим класс линейных мероморфно зависящих от параметра операторов с бесконечным множеством простых вещественных полюсов.

1. Постановка задачи. Пусть $\boldsymbol{X}$ — некоторое гильбертово пространство; A_0 и A_1 — линейные самосопряжённые операторы, действующие в $\boldsymbol{X}$ и имеющие конечные абсолютные нормы [11], кроме того $(A_0x, x) < (x, x)$ для любого $x \in \boldsymbol{X}$, $x \neq 0$. Положим

$$H_i x = \sum_{k=1}^{\sigma_i} \frac{(x, \varphi_k^{(i)})\varphi_k^{(i)}}{\varkappa_k^{(i)}}, \quad i = 1, 2, \ldots,$$

где $\{\varphi_{(i)}^k\}$ $(k = 1, \ldots, \sigma_i)$ — ортонормированная система элементов из $\boldsymbol{X}$, $\varkappa_k^{(i)}$ — некоторые вещественные числа; пусть абсолютные нормы и следы [11] конечномерных операторов H_i, обозначаемые соответ-